

As an example, the spectrum from a 20 kV accelerating potential has a cutoff wavelength of 0.062 nm and peaks at a wavelength of about 0.093 nm.

In addition to bremsstrahlung, real x-ray tubes emit characteristic x-rays specific to the target atoms. The characteristic x-rays show up as pronounced narrow spectral lines sitting on top of the bremsstrahlung continuum.

In practice, only a very small fraction of the electron beam energy shows up as x-ray radiation. Most of the energy actually goes towards heating the anode. Target materials like tungsten, having high atomic number, produce a higher yield of x-rays—but even then heating effects dominate. For this reason, the anode must be cooled and/or rotated to prevent melting.

1.6 Electromagnetic Waves in Matter

1.6.1 Speed of Light in a Dielectric Medium

In vacuum, all electromagnetic waves travel at the same speed, $c = 1/\sqrt{\epsilon_0\mu_0} = 3.00 \times 10^8$ m/sec (see Eq. 1.2). This follows from the wave equation (Eq. 1.1) developed by Maxwell. If one now considers Maxwell's equations applied to a region of space filled with a non-conducting, or *dielectric*, medium, the wave equation appears yet again, except that ϵ_0 and μ_0 , the electric permittivity and magnetic permeability of free space, are replaced by ϵ and μ , the permittivity and permeability of the medium. The result is that, like vacuum, homogeneous dielectric materials also support the propagation of EM waves, but the wave speed is now given by

$$v = \frac{1}{\sqrt{\epsilon\mu}}. \quad (1.48)$$

Basically, ϵ and μ are measures of a material's response to applied electric and magnetic fields. When an **E**-field is applied to a dielectric, the individual electric dipoles of the atoms or molecules become aligned. Similarly, an applied **B**-field has the effect of aligning the orientation of the individual electron current loops (or magnetic moments) of the atoms or molecules. These alignments have the effect of producing internal fields within the dielectric. As a result, the actual electric or magnetic field residing in the medium is the sum of the externally applied field and the internal field generated by the atoms or molecules of the material. The overall effect is to change the apparent coupling between the **E** and **B** fields in Maxwell's equations, which in turn, changes the propagation speed of the electromagnetic waves.

Ordinarily, the speed of an EM wave in various materials is quoted as an *index of refraction* (or simply *index*, for short). The index of refraction, n , of a medium is just the ratio of the wave speed in vacuum to the wave speed in the medium, i.e.,

$$\begin{aligned} n &\equiv \frac{c}{v} \\ &= \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}. \end{aligned} \quad (1.49)$$

<u>Substance</u>	<u>Index</u>
<u>Gases (0°C, 1 atm)</u>	
air	1.000293
carbon dioxide	1.00045
<u>Liquids (20°C)</u>	
benzene	1.501
ethyl alcohol	1.361
water	1.333
<u>Solids (20°C)</u>	
diamond	2.419
silica	1.458
crown glass	1.52
flint glass	1.66
polystyrene	1.49

Table 1.3 Indices of Refraction for Some Common Substances (using sodium light, vacuum wavelength of 589 nm)

For non-magnetic media, μ and μ_0 are essentially identical, so $n = \sqrt{\epsilon/\epsilon_0}$. The *dielectric constant*, κ , of a material is defined as its permittivity relative to that of free space:

$$\kappa \equiv \frac{\epsilon}{\epsilon_0}. \quad (1.50)$$

Therefore,

$$n = \sqrt{\kappa}. \quad (1.51)$$

The refractive indices of some common substances for yellow light are listed in Table 1.3. The larger the index of refraction, the slower the speed of the wave. For example, the speed of light in air differs from the speed in vacuum by only about 0.03 percent, whereas the speed is about 33 percent slower in water than in vacuum.

For a given frequency, the wavelength of the radiation depends on the speed of the wave, and hence the refractive index of the supporting medium, i.e.,

$$\begin{aligned} \lambda &= \frac{v}{\nu} = \frac{c/n}{\nu} = \frac{c/\nu}{n} \\ &= \frac{\lambda_0}{n}, \end{aligned} \quad (1.52)$$

where λ_0 is the wavelength in vacuum. In other words, compared to the case of vacuum, the wavelength is reduced by the factor n . Also, from Eq. 1.7, the wavenumber

k is larger than the wavenumber in vacuum, k_0 , by the factor n :

$$k = nk_0. \quad (1.53)$$

The precise value of the speed, and hence the refractive index, of a substance is dependent upon the frequency of the radiation, a general phenomenon known as *dispersion*. Notice that the values of n quoted in Table 1.3 are for yellow light having a vacuum wavelength of 589 nm. Generally, visible light towards the violet end of the spectrum will have a somewhat higher index and reddish light will have an index that is somewhat lower. Dispersion comes about because the speed of light is determined by the permittivity of the medium, and the permittivity is a function of frequency. A convincing example of this is that for microwave frequencies and lower, the relative permittivity (i.e., dielectric constant) κ for water is around 80. According to Eq. 1.51, the fact that yellow light has an index of 1.333 means that at optical frequencies, the permittivity of water must be drastically less. A more complete discussion of dispersion follows in the next section.

1.6.2 Dispersion in a Dielectric Medium

Many of the basic features underlying the frequency dependence of the speed of light in a dielectric are successfully explained by a relatively simple classical model of the interaction between an EM wave and the molecules of a material. The model considers each molecule, in effect, as being composed of two equal and opposite charges, q and $-q$, bound together. For simplicity, one of the charges is treated as being stationary (i.e., infinitely massive), while the other charge (mass m) is free to move. In a nonpolar molecule, for example, q and $-q$ correspond to the combined positive charge of the nuclei and the surrounding negatively-charged cloud of electrons. When the molecule is left unperturbed, the centers of the positive and negative charge distributions coincide. However, in the presence of an external electric field, the center of the electron cloud becomes displaced relative to the nuclei—in other words, the field produces an *induced dipole moment* in the molecule. There is now an asymmetry in the charge distribution, and the electron cloud experiences an electrostatic *restoring force* that tends to pull it back to equilibrium. In our simple model, this restoring force is represented by a linear spring (force constant K) connecting the two charges. The system has the properties of a simple harmonic oscillator, characterized by a natural, or resonant, vibration frequency $\omega_0 = \sqrt{K/m}$.

When visible light, or any other electromagnetic wave is present, the effect is to introduce a harmonically oscillating electric field at the location of the molecule. The field continually shakes the charge in the molecule back and forth at a frequency, ω , matching that of the wave. The oscillator (i.e., molecule) experiences both this harmonic *driving force* and the springlike restoring force discussed above. There is a third and final force, namely, a *damping force*, that tends to retard the motion and cause energy losses in the system. It comes about because of interactions between the

oscillator and other nearby molecules and because energy is carried away by radiation from the oscillating (and hence accelerating) charge. The latter effect is referred to as *radiation damping*. It can also be thought of as the reaction force of the radiation field back on the oscillator.

Each of the three forces identified above contributes a term to the oscillator's equation of motion. If the EM wave is assumed to be linearly polarized along the x -direction, the charge will displace along that direction, and the equation of motion is

$$m \frac{d^2 x}{dt^2} = -Kx - b \frac{dx}{dt} + qE_0 \cos \omega t. \quad (1.54)$$

The left-hand side is simply the product of the mass and acceleration of the oscillating charge, and the right-hand side is the sum of the three forces acting on it. The first term represents the restoring force, the second term is the damping force, and the third term corresponds to the driving force arising from the interaction of the charge with the oscillating field of the EM wave. Notice that the damping term (with damping constant b) resembles a drag force and is proportional to the speed of the oscillator at a given instant. The negative signs attached to the first two terms guarantee that the restoring force and the damping force are directed opposite to the oscillator's instantaneous displacement and velocity, respectively. Equation 1.54 can also be written as

$$\frac{d^2 x}{dt^2} + \frac{1}{\tau} \frac{dx}{dt} + \omega_0^2 x = \frac{q}{m} E_0 \cos \omega t, \quad (1.55)$$

where $\tau = m/b$ represents an effective time-constant associated with the damping, or energy dissipation, in the system. There are two important frequencies that appear—the resonant frequency, ω_0 , which is solely a property of the oscillator or type of molecule, and ω , which is the frequency of the driving wave.

We now seek the steady-state, or particular, solution to Eq. 1.55, which is an inhomogeneous, linear, second-order differential equation with constant coefficients. In cases like the one here, where the driving term is oscillatory, the process of solving the equation is simplified by replacing the factor $\cos \omega t$ with the complex exponential $\exp(-i\omega t)$, where i stands for the square-root of -1 . According to a mathematical identity known as *Euler's formula*,

$$e^{\pm i\omega t} = \cos \omega t \pm i \sin \omega t, \quad (1.56)$$

we see that $\cos \omega t$ is identical to the real part of $\exp(-i\omega t)$. This means that the solution to Eq. 1.55 is obtained by first determining the mathematical solution to the equation

$$\frac{d^2 \tilde{x}}{dt^2} + \frac{1}{\tau} \frac{d\tilde{x}}{dt} + \omega_0^2 \tilde{x} = \frac{q}{m} E_0 e^{-i\omega t}, \quad (1.57)$$

where \tilde{x} is complex, and then taking the real part of this result.

To solve Eq. 1.57, assume a solution of the form

$$\tilde{x}(t) = Ae^{-i\omega t}, \quad (1.58)$$

where A is some complex oscillation amplitude. Substitution into the equation of motion shows that the amplitude is a function of the driving frequency of the electromagnetic wave:

$$A(\omega) = \frac{(q/m) E_0}{\omega_0^2 - \omega^2 - i\frac{\omega}{\tau}}. \quad (1.59)$$

The meaning of a complex amplitude is as follows: As with any complex quantity, $A(\omega)$ can be re-expressed in the polar form

$$A(\omega) = |A(\omega)| e^{i\varphi}, \quad (1.60)$$

i.e., as the product of a magnitude and a complex phase factor (with phase angle φ). Substituting into the assumed form for the complex solution, Eq. 1.58, gives $\tilde{x}(t) = |A(\omega)| \exp[-i(\omega t - \varphi)]$. The actual physical solution is then obtained by taking the real part of this expression, which is just $x(t) = |A(\omega)| \cos(\omega t - \varphi)$. In other words, a complex amplitude means that there is some phase difference (φ) between the driving force and the resulting oscillation. The actual amplitude of the oscillation is given by the magnitude of $A(\omega)$, which is

$$|A(\omega)| = \frac{(q/m) E_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2}}}. \quad (1.61)$$

This function is sketched in Fig. 15, and represents the so-called *resonance curve* for a damped, driven oscillator. Naturally, the question at this point is how does finding the solution for the displacement $x(t)$ lead to an understanding of dispersion? The answer has to do with how different driving frequencies induce different degrees of charge separation in the molecules, and whether they result in different phase shifts between the perturbing wave and the oscillator motion.

The action of the EM field on a molecule is to produce an oscillating electric dipole, with (complex) dipole moment $p(t) = q \cdot \tilde{x}(t)$. Using Eqs. 1.58 and 1.59, one finds that the dipole moment can be reduced to the form

$$p(t) = \alpha(\omega) E(t), \quad (1.62)$$

where

$$\alpha(\omega) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\frac{\omega}{\tau}} \quad (1.63)$$

is called the *molecular polarizability*. This parameter is a measure of how easy it is, at frequency ω , for the oscillating electric field to induce a separation of the bound charges in the molecule. If now considers a region of dielectric containing N molecules per unit volume, one can speak of the local *electric polarization* P of the medium, which is the dipole moment per unit volume, i.e.,

$$P = Np = N\alpha E. \quad (1.64)$$

It is important not to confuse the polarization of the medium with the polarization associated with the EM wave. In a simple, linear dielectric, another measure of the degree to which a medium is polarizable is the value of the relative permittivity, κ , in excess of the value in vacuum (namely unity). More specifically,

$$P = (\kappa - 1) \epsilon_0 E. \quad (1.65)$$

Combining with Eq. 1.64 gives the result

$$\kappa = 1 + \frac{N}{\epsilon_0} \alpha. \quad (1.66)$$

The connection between our model and the refractive index, or speed of light in the medium, can now be made. Recall that $n = \sqrt{\kappa}$, where the refractive index n was a real quantity. Note that in the present treatment the molecular polarizability α is, in general, complex, and therefore, so is κ . We now introduce a *complex index of refraction*, \tilde{n} , given by

$$\tilde{n} = \sqrt{1 + \frac{N}{\epsilon_0} \alpha}. \quad (1.67)$$

At sufficiently low density, such as in a gas, one has $N\alpha/\epsilon_0 \ll 1$, and one can expand the right-hand side of Eq. 1.67 in a Taylor series. Keeping only the two leading terms, and plugging in Eq. 1.63 for α , gives

$$\begin{aligned} \tilde{n} &= 1 + \frac{N}{2\epsilon_0} \alpha \\ &= 1 + \frac{Nq^2}{2m\epsilon_0} \left(\frac{1}{\omega_0^2 - \omega^2 - i\frac{\omega}{\tau}} \right). \end{aligned} \quad (1.68)$$

This complex refractive index can be broken up into its real part and imaginary part, i.e., $\tilde{n} = n + ik$, with

$$n = 1 + \frac{1}{2} \omega_p^2 \left[\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2}} \right] \quad (1.69)$$

and

$$k = \frac{1}{2} \omega_p^2 \left[\frac{\omega/\tau}{(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2}} \right]. \quad (1.70)$$

We have introduced the parameter ω_p , where

$$\omega_p^2 = \frac{Nq^2}{m\epsilon_0}. \quad (1.71)$$

ω_p is known as the *plasma frequency* of the medium.

As a specific example, consider variations in the refractive index of crown glass for the visible region of the spectrum ($\nu = \omega/2\pi \approx 5 \times 10^{14}$ Hz). If we make the simplifying assumptions that the frequencies of interest are much lower than the natural vibration frequency of molecules in the glass (i.e., $\omega \ll \omega_0$, and the light is far from resonance) and that damping effects are small (i.e., τ is very large), then

$$n \approx 1 + \frac{1}{2}\omega_p^2 \left(\frac{1}{\omega_0^2 - \omega^2} \right) \quad (1.72)$$

and \mathbf{k} is essentially negligible. In other words, $\tilde{n} \approx n$, and the refractive index is purely real, as one might expect. Figure 16 displays data for the index of crown glass as a function of vacuum wavelength. The accompanying curve is the best fit to the data using the very simple model of Eq. 1.72. The resonant frequency can be extracted from the fit, with a result of $\nu_0 = \omega_0/2\pi = 2.95 \times 10^{15}$ Hz—i.e., the resonant frequency is in the ultraviolet. The characteristic shape of the curve shown here is typical of the dispersion curve for many transparent substances at frequencies far below resonance: At short wavelengths, the index of a given material is higher than at long wavelengths. This type of behavior, where $dn/d\omega > 0$ (or $dn/d\lambda < 0$), is referred to as *normal dispersion*.

Now consider the behavior of the refractive index in the vicinity of resonance, i.e., when $\omega \approx \omega_0$. We then have $\omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega_0(\omega_0 - \omega)$, and Eqs. 1.69 and 1.70 for the real and imaginary parts of the index reduce to

$$n = 1 + \frac{\omega_p^2 \left(\frac{\omega_0 - \omega}{4\omega_0} \right)}{(\omega_0 - \omega)^2 + \frac{1}{(2\tau)^2}} \quad (1.73)$$

and

$$\mathbf{k} = \frac{\omega_p^2/8\tau\omega_0}{(\omega_0 - \omega)^2 + \frac{1}{(2\tau)^2}}. \quad (1.74)$$

A general sketch of n and \mathbf{k} near resonance are shown in Fig. 17.

Consider the dispersion curve for n , the real part of the index. At $\omega = \omega_0$, the value of n is unity, and just below and above resonance, there appears a maximum and a minimum, respectively. For frequencies between the maximum and minimum, the slope of the curve is negative, i.e., $dn/d\omega < 0$, and one says that the medium exhibits *anomalous dispersion*. On either side of this region, the slope is positive ($dn/d\omega > 0$) and, as before, normal dispersion occurs.

A puzzling feature of Fig. 17 is that for frequencies above the resonant frequency, the real index, n , is less than unity. This means that c/n , the speed of the wave, is greater than c , the speed of light in vacuum. However, according to Einstein's

Theory of Special Relativity, it is impossible for a signal to propagate at speed greater than c . To resolve this enigma, one must make a distinction between the *phase velocity* and the *group velocity* of a wave. As mentioned early on, $v = c/n$ is the phase velocity, and corresponds to the speed of a perfectly harmonic, monochromatic wave, as in Eq. 1.4. Such a wave can, in fact, travel faster than c . But in order to carry any information, some sort of modulation, or variation in amplitude, must be impressed on the idealized wave. Even the process of simply switching the source of the wave on or off introduces some degree of modulation since the wave will no longer be infinite in extent and duration. Information, or the signal carried by the wave, is encoded in the modulation envelope. In general, however, the speed of the envelope may be different than the phase velocity, c/n . The rate at which the envelope propagates is known as the *group velocity* of the wave, and is given by

$$v_g = \frac{d\omega}{dk}. \quad (1.75)$$

Compare this to the expression for the phase velocity:

$$v = \frac{c}{n} = \frac{\omega/k_0}{n} = \frac{\omega}{nk_0} = \frac{\omega}{k}. \quad (1.76)$$

It is not too difficult to show that these relations lead to yet another expression for the group velocity:

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}}. \quad (1.77)$$

In the region of normal dispersion above the resonant frequency, even though the phase velocity c/n is greater than c , the slope $dn/d\omega$ is positive, and Eq. 1.77 guarantees that the group velocity will always be less. Hence, the signal (i.e., the envelope of the wave) propagates at a speed less than c . Notice, however, that in the region of anomalous dispersion, where $dn/d\omega$ is negative, Eq. 1.77 seems to indicate that $v_g > v$, and apparently the group velocity may exceed c . However, the idea of group velocity loses its significance in this region, and again the transfer of information never exceeds c [5].

1.6.3 Absorption Near Resonance

We have yet to discuss the meaning of the \mathbf{k} , the imaginary part of the index of refraction, and its behavior as a function of frequency, as displayed in Fig. 17. To understand the significance of \mathbf{k} , consider a simple EM plane wave propagating in the $+z$ -direction through a dielectric medium. In complex form, the wave is written as

$$\tilde{E}(z, t) = E_0 e^{i(kz - \omega t)}. \quad (1.78)$$

k represents the wavenumber in the medium, i.e., it is the product of the index of refraction and k_0 , the wavenumber in vacuum (see Eq. 1.53). We now know that, in

general, the medium has a complex index $\tilde{n} = n + i\mathbf{k}$, so we replace k by $\tilde{n}k_0$, and the field reduces to

$$\tilde{E}(z, t) = \mathbf{E}_0 e^{-\mathbf{k}\omega z/c} e^{i(n\omega z/c - \omega t)}. \quad (1.79)$$

The actual physical field is obtained by taking the real part of the above expression, so

$$E(z, t) = \mathbf{E}_0 e^{-z/\delta} \cos\left(\frac{n\omega}{c}z - \omega t\right). \quad (1.80)$$

The parameter

$$\delta = c/\mathbf{k}\omega \quad (1.81)$$

is called the *skin-depth* of the medium at frequency ω . Recall that I , the intensity of the wave, is proportional to the magnitude of the field squared (averaged over time), so

$$I = I_0 e^{-(2/\delta)z}, \quad (1.82)$$

where I_0 is the intensity at $z = 0$. For a purely real refractive index, $\mathbf{k} = 0$, which means that the skin-depth is infinite and the wave propagates without any attenuation or energy loss. However, if the index is complex, the skin-depth is finite and the wave is attenuated. The energy carried by the wave is absorbed exponentially along the direction of propagation. The quantity $2/\delta$ is called the *absorption or attenuation coefficient* of the medium. For a substance to be transparent to radiation of frequency ω , the skin-depth at that frequency must be much larger than the thickness of the material.

Return now to Fig. 17, and the sketch of \mathbf{k} as a function of frequency. It shows that the imaginary part of the index is very pronounced in the vicinity of resonance, signifying that absorption in this region is strong. The shape of the \mathbf{k} vs. ω curve is *Lorentzian*, and the two frequencies marking where the curve hits half the maximum height correspond to the edges of the anomalous dispersion region, also called the *absorption band*. The width of this region, or the curve's full-width at half-maximum (FWHM), is simply $1/\tau$.

1.6.4 Corrections to Dispersion Model

Identified here are two important corrections to the simple molecular dispersion model developed thus far [27]:

1. Up to this point, the model has assumed a low molecular density in the dielectric. However, in general, this assumption will only hold for dilute gases. It turns out that the effects of increased density can be properly accounted for by simply replacing the quantity $\kappa - 1$ with $3(\kappa - 1)/(\kappa + 2)$ in Eq. 1.66. The revised equation is known as the *Clausius-Mossotti relation*:

$$\frac{3\epsilon_0}{N} \left(\frac{\kappa - 1}{\kappa + 2} \right) = \alpha. \quad (1.83)$$

The correction to the original equation comes from the fact that in addition to the externally applied field, each molecule also experiences another field produced by the polarization of the surrounding molecules. The Clausius-Mossotti relation is particularly useful because it relates macroscopic quantities (left-hand side of Eq. 1.83) to α , which is a molecular quantity. Furthermore, since the molecular polarizability should only depend on the type of molecule (and frequency), the left-hand side of Eq. 1.83 remains constant, independent of density. Hence, if one knows κ for a molecular species at one density, the value can be computed for another density, and likewise for $\sqrt{\kappa}$, or the refractive index. As far as the dispersive behavior of a dielectric medium is concerned, it can be shown that the only effect of increased density is to simply shift the center of the absorption band downward from frequency ω_0 to a value of $\sqrt{\omega_0^2 - \omega_p^2/3}$.

- The model presented has assumed that a molecule has only a single resonance ω_0 . In general, there are resonances at a number of frequencies, i.e., ω_{01} , ω_{02} , ω_{03} , ..., etc. As before, resonances in the UV part of the spectrum are ordinarily associated with natural oscillations of electrons bound in the molecule. Resonances in other regions of the spectrum, however, usually originate from other oscillation modes in the system. For example, resonances in the IR are usually caused by interatomic vibrations in the molecule. Each resonance, ω_{0i} , has its own characteristic mass m_i and damping time τ_i . The result is that the molecular polarizability, α , now contains a contribution from each resonant frequency of the molecule, so that

$$\alpha(\omega) = \sum_i \alpha_i(\omega), \quad (1.84)$$

where

$$\alpha_i(\omega) = \frac{f_i \frac{q^2}{m_i}}{\omega_{0i}^2 - \omega^2 - i \frac{\omega}{\tau_i}}. \quad (1.85)$$

Each resonance has a weighting factor, or *oscillator strength*, denoted by f_i . As shown in Fig. 18, the overall effect of having multiple resonances is that each gives rise to anomalous dispersion and its own absorption band.

1.6.5 Wave Propagation in a Conducting Medium

Any medium capable of conducting an electric current contains free charges. Unlike the situation discussed for a dielectric, these charges are unbound and no restoring force acts on them when they are displaced by an electric field. In a metal, for example, outer (i.e., valence) electrons are released from the grip of individual atoms and, in the process, form a pool of so-called *conduction electrons*. These electrons are no longer localized near any particular atom. Instead, they are completely free to

move about within the solid, much like the motion of particles in an ideal gas. The metal is said to contain a *free electron gas*.

The basic behavior of EM waves in a conductor can be modelled by saying that existence of unbound charge is equivalent to setting $\omega_0 \rightarrow 0$. Applying this condition to Eq. 1.63 for the molecular polarizability α , and plugging the expression into Eq. 1.66 for κ , gives

$$\kappa = 1 - \frac{\omega_p^2}{\omega^2 + i\frac{\omega}{\tau}}. \quad (1.86)$$

But recall that $\kappa = \tilde{n}^2 = (n + ik)^2 = (n^2 - k^2) + 2ink$. Equating the real and imaginary parts of this expression with those from the right-hand side of Eq. 1.86 gives

$$n^2 - k^2 = 1 - \frac{\omega_p^2}{\omega^2 + \frac{1}{\tau^2}} \quad (1.87)$$

and

$$2nk = \frac{1}{\omega\tau} \left(\frac{\omega_p^2}{\omega^2 + \frac{1}{\tau^2}} \right). \quad (1.88)$$

The most common situation is one where the damping is small and $\tau^{-1} \ll \omega_p$. This allows for a straightforward determination of n and k at low, intermediate, and high frequencies, as discussed below.

In the limit of low frequency, where $\omega \ll \tau^{-1}$, the above equations reduce to

$$n^2 - k^2 \cong -\omega_p^2\tau^2 \quad \text{and} \quad 2nk \cong \frac{\omega_p^2\tau}{\omega}. \quad (1.89)$$

Solving for n and k then gives

$$n \simeq k \simeq \frac{\omega_p}{\sqrt{2\omega/\tau}} \gg 1. \quad (1.90)$$

This result is applicable, for example, to radio waves and microwaves in metals. Later on, it will be shown that at these wavelengths, most of the energy carried by the wave is generally reflected from the surface of a metal, but because of the large imaginary part of the index, the small part that enters the medium is strongly absorbed, causing heating of the conductor. From Eq. 1.81, the skin-depth of the medium is given by

$$\delta = \sqrt{\frac{2c^2}{\omega(\omega_p^2\tau)}}. \quad (1.91)$$

It turns out [6] that the quantity $\omega_p^2\tau$ is identical to η/ϵ_0 , where η is the DC (i.e., low-frequency) *electrical conductivity* of the medium. This means that the conductivity

is the only material property needed to determine the skin depth:

$$\delta = \sqrt{\frac{2c^2\epsilon_0}{\eta\omega}} = \sqrt{\frac{2}{\mu_0\eta\omega}}. \quad (1.92)$$

Most metals have conductivities on the order of $10^8 \Omega^{-1}\cdot\text{m}^{-1}$. So at a typical microwave frequency (say 10 GHz), one finds the skin-depth to be on the order of a micron or less. Compare this to the skin-depth of seawater at the same frequency (seawater conducts because the salt dissociates into ions, which are free charges). Seawater has a conductivity of about $4\text{--}5 \Omega^{-1}\cdot\text{m}^{-1}$, which gives a microwave skin-depth of a few millimeters. The penetration of the wave can be increased still further by going down to radio wave frequencies. At a frequency of 10 kHz, for example, the skin-depth of the wave in seawater is a few meters, showing that it can penetrate significantly.

At intermediate frequencies, where $\tau^{-1} \ll \omega \ll \omega_p$, we have

$$n^2 - \mathbb{k}^2 \cong -\frac{\omega_p^2}{\omega^2} \quad \text{and} \quad 2n\mathbb{k} \cong \frac{\omega_p^2}{\omega^3\tau}. \quad (1.93)$$

The real and complex index then become

$$n \cong \frac{\omega_p}{2\omega^2\tau} \quad \text{and} \quad \mathbb{k} \cong \frac{\omega_p}{\omega}. \quad (1.94)$$

In this frequency regime, \mathbb{k} is much larger than n , and the skin-depth is simply

$$\delta = \frac{c}{\omega_p}. \quad (1.95)$$

These results apply to metals, with light in the infrared.

Finally, consider the index at high frequencies, i.e., the case $\omega \gg \omega_p$. In this case,

$$n^2 - \mathbb{k}^2 \cong 1 \quad \text{and} \quad 2n\mathbb{k} \cong \frac{\omega_p^2}{\omega^3\tau}, \quad (1.96)$$

which leads to

$$n \cong 1 \quad \text{and} \quad \mathbb{k} \cong \frac{\omega_p^2}{2\omega^3\tau}. \quad (1.97)$$

Here, we have $\mathbb{k} \ll 1$, so the skin-depth is extremely large. The conductor is essentially transparent to the radiation. This explains, at least from a classical point of view, a number of important phenomena. For example, consider copper, which has a free electron density of $N = 8.48 \times 10^{28}$ electrons/ m^3 (this number is based on the fact that each copper atom contributes a single conduction electron to the solid). From Eq. 1.71, the plasma frequency of copper comes out to be $\nu_p = \omega_p/2\pi = 2.61 \times 10^{15}$ Hz, which is in the near UV. Since the plasma frequency is much lower than the frequency

of gamma rays, x-rays, and even radiation in the far UV, these radiations are easily transmitted through copper, as well as other metallic conductors. Another example concerns transmission of radiation through the ionosphere, which contains ionized gas. Here, N , the density of these ions, is roughly seventeen orders of magnitude lower than the value for a typical metal. This gives a plasma frequency of only a few megahertz, which explains why the atmosphere is transparent in the microwave region.

Figure 19 is a plot of n and k vs. ω/ω_p for the particularly simple case of $\tau^{-1} \rightarrow 0$, i.e., no damping. With this idealization, the plasma frequency plays the role of a critical frequency. At frequencies below this value, the index is purely imaginary—this means that the wave is completely reflected at the boundary of the medium. Above ω_p , the index is purely real, and the conductor is transparent. Of course, any real conducting medium is characterized by a finite value of τ , and the actual curves will not be quite as simple as the ones shown here. However, except for at very low frequencies ($\omega \ll \tau^{-1}$) where the effects of damping are extremely important, there is a remarkable resemblance between the curves of Fig. 19 and those for real conductors [27].

1.7 Polarization

Up to this point, it has been assumed that the electric-field vector of an EM wave is linearly polarized along a particular coordinate direction. In reality, a state of linear polarization corresponds to only one of a number of possible types of polarization, as discussed in this section.

1.7.1 Types of Polarization

Consider a monochromatic electromagnetic plane wave (frequency ω , wavenumber k) propagating in the $+z$ -direction. The most general form for the electric-field vector is a superposition (i.e., sum) of two mutually orthogonal, linearly-polarized waves, one polarized along the x -direction and the other polarized along the y -direction:

$$\mathbf{E}(z, t) = \hat{i}E_x(z, t) + \hat{j}E_y(z, t). \quad (1.98)$$

Each component is just a harmonic wave with its own amplitude (E_{0x} , E_{0y}) and its own initial phase:

$$\begin{aligned} E_x(z, t) &= E_{0x} \cos(kz - \omega t) \\ E_y(z, t) &= E_{0y} \cos(kz - \omega t + \Delta). \end{aligned} \quad (1.99)$$

Without any loss of generality, the initial phase of the x -component of the wave has been set to zero, and Δ represents the phase difference between the y -component and the x -component. Depending on the value of Δ and the relative values of the two amplitudes, E_{0x} and E_{0y} , one obtains different types of polarization for the propagating wave. The different cases are presented below: