Lecture 12: Image Processing and 2D Transforms

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October 18, 2005

Abstract

The Fourier transform provides information about the global frequency-domain characteristics of an image. The Fourier description can be computed using discrete techniques, which are natural for digital images. Here we focus on the relationship between the spatial and frequency domains.

2D Fourier Transform

Let $f(x, y)$ be a 2D function that may have infinite support. The 2D Fourier transform pair is defined

$$\mathcal{F}(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathsf{f}(x, y) e^{-i2\pi(ux+vy)} dx dy$$

$$\mathsf{f}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{F}(u, v) e^{i2\pi(ux+vy)} du dv$$

We are interested in transforms related to images, which are defined on a finite support. If an image has width A and height B with the origin at the center, then

$$\mathcal{F}(u, v) = \int_{-A/2}^{A/2} \int_{-B/2}^{B/2} \mathsf{f}(x, y) e^{-i2\pi(ux+vy)} dx dy$$

where $\mathsf{f}(x, y)$ represents the image brightness at point (x, y).

This expression assumes that $\mathsf{f}(x, y)$ is extended with $\mathsf{f}(x, y) = 0$ outside the image frame.

Periodic Extension

If it is assumed that $\mathsf{f}(x, y)$ is extended *periodically* outside the image frame then we can use a Fourier series.

$$\mathsf{f}(x, y) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} F(u, v) e^{i2\pi(ux/A + vy/B)}$$

Note the following:

1. (u, v) take on integer values.
2. (x, y) take on a continuum of values.
3. $\mathsf{f}(x, y)$ is periodic with

$$\mathsf{f}(x + nA, y + mB) = \mathsf{f}(x, y)$$

   for any integers (n, m).

Fourier Series Coefficient

Multiply both sides of the Fourier series expression by $e^{-i2\pi(ax/A+by/B)}$and integrate over the image frame.

$$\int_{-A/2}^{A/2}\int_{-B/2}^{B/2} \mathsf{f}(x,y)e^{-i2\pi(\frac{ax}{A}+\frac{by}{B})}dxdy$$

$$= \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty} F(u,v)\int_{-A/2}^{A/2}\int_{-B/2}^{B/2} e^{i2\pi\left(\frac{x(u-a)}{A}+\frac{y(v-b)}{B}\right)}dxdy$$

$$= ABF(a,b)$$

Fourier Series Coefficient (cont)

Hence, the Fourier series coefficient is

$$F(a,b) = \frac{1}{AB}\int_{-A/2}^{A/2}\int_{-B/2}^{B/2} \mathsf{f}(x,y)e^{-i2\pi(\frac{ax}{A}+\frac{by}{B})}dxdy$$

$$= \frac{1}{AB}\mathcal{F}\left(\frac{a}{A},\frac{b}{B}\right) \quad \text{for integer } (a,b)$$

Bandwidth Limited Approximation

Suppose that $\mathcal{F}(u,v) \approx 0$ outside a region

$$\mathcal{S}_{uv} = \{u,v| -W_u \le u \le W_u, -W_v \le v \le W_v\}$$

Then it can be extended periodically outside $\mathcal{S}_{uv}$ and expressed as a Fourier series over the spatial domain.

$$\mathcal{F}(u,v) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y) e^{-i2\pi\left(\frac{ux}{2W_u}+\frac{vy}{2W_v}\right)}$$

Note that:

1. (x,y) take on integer values.
2. (u,v) take on a continuum of values.
3. $\mathcal{F}(u,v)$ is periodic with

$$\mathcal{F}(u+2nW_u, v+2mW_v) = \mathcal{F}(u,v) \quad \text{for any integers } (n,m)$$

Fourier Series Coefficient

We can use an analysis similar to the above to find the value of $f(s,t)$ at integer values of (s,t). We find

$$f(s,t) = \frac{1}{4W_uW_v}\mathsf{f}\left(\frac{s}{2W_u}, \frac{t}{2W_v}\right)$$

The image plane is sampled on a grid with spacing $\left(\frac{1}{2W_u}, \frac{1}{2W_v}\right)$. The number of samples that are needed in each dimension are

$$N = 2AW_u$$

$$M = 2BW_v$$

Discrete Fourier Transform

We have already seen that

$$F(a,b) = \frac{1}{AB}\mathcal{F}\left(\frac{a}{A}, \frac{b}{B}\right)$$

All of the information we need is provided by samples of $\mathcal{F}(u,v)$ on a grid with spacing $\left(\frac{1}{A}, \frac{1}{B}\right)$. The number of points need are

$$N = 2AW_u$$

$$M = 2BW_v$$

We need the same number of values to describe the image, whether the values are from the spatial domain or the frequency domain.

The samples of $\mathcal{F}$ are complex numbers, but the information is the same because of symmetries.

Discrete Fourier Transform

We can now construct the DFT pair by substituting into

$$\mathcal{F}(u,v) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y) e^{-i2\pi(\frac{ux}{2W_u} + \frac{vy}{2W_v})}$$

$$ABF(a,b) = \mathcal{F}\left(\frac{a}{A}, \frac{b}{B}\right) = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} \frac{1}{4W_u W_v} \mathsf{f}(\frac{x}{2W_u}, \frac{y}{2W_v}) e^{-i2\pi\left(\frac{ax}{2AW_u} + \frac{by}{2BW_v}\right)}$$

$$F(a,b) = \frac{1}{MN} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} \mathsf{f}(\frac{x}{2W_u}, \frac{y}{2W_v}) e^{-i2\pi\left(\frac{ax}{N} + \frac{by}{M}\right)}$$

Discrete Fourier Transform

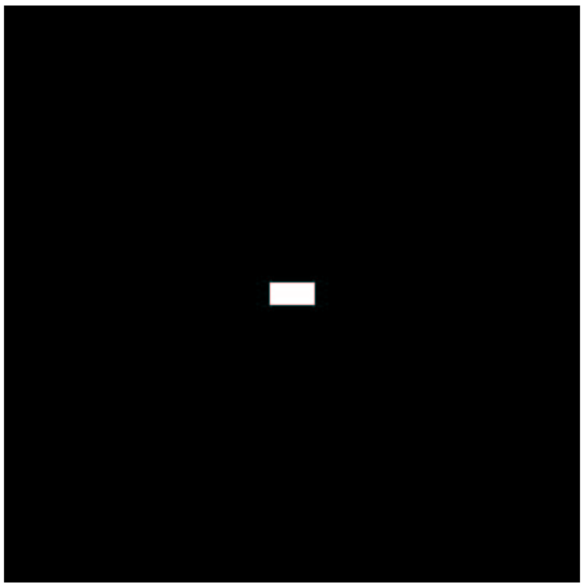
It is conventional to replace f with f without keeping track of the extra $4W_uW_v$ factor. We'll follow convention and worry about the scaling when we do numerical computations. We'll also replace (a, b) by (u, v). This produces the transform pair

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x,y) e^{-i2\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)}$$

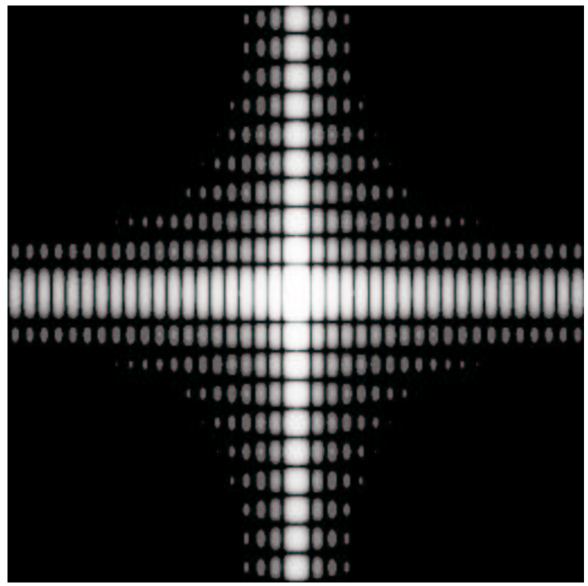
$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u,v) e^{i2\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)}$$

It is often the case that the origin of an array that represents $F(u, v)$ is at a corner. This does no harm because $F(u, v)$ is periodic with period (N, M), but one must pay attention.

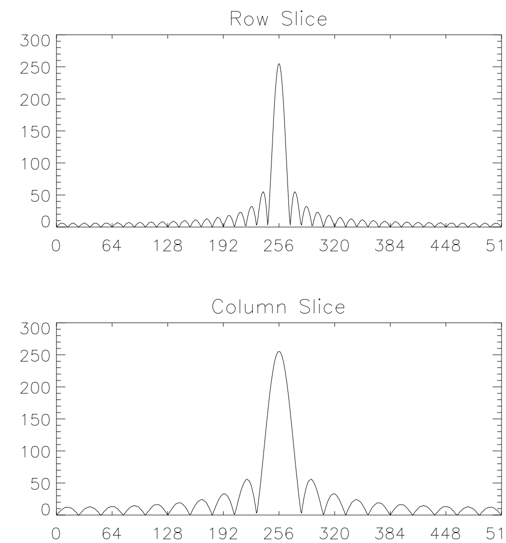
Example: G&W Figure 4.3



40×20 Rectangle

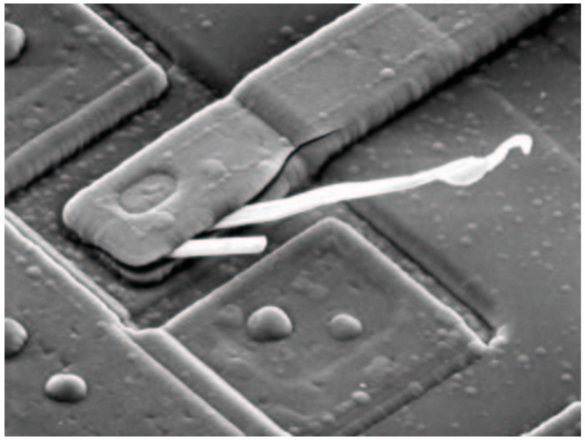


$|F(u,v)|$



Row and Column Profiles

Example: G&W Figure 4.6



Original

Spectrum

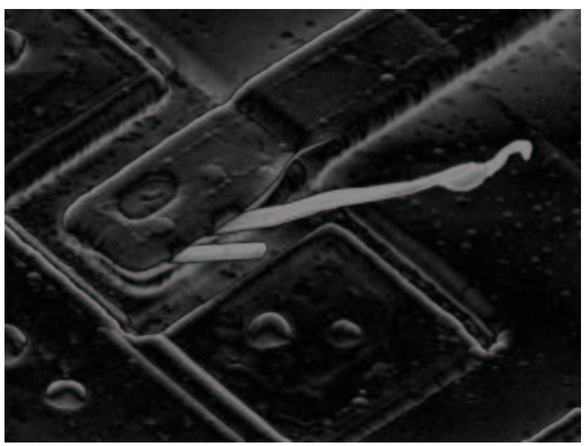
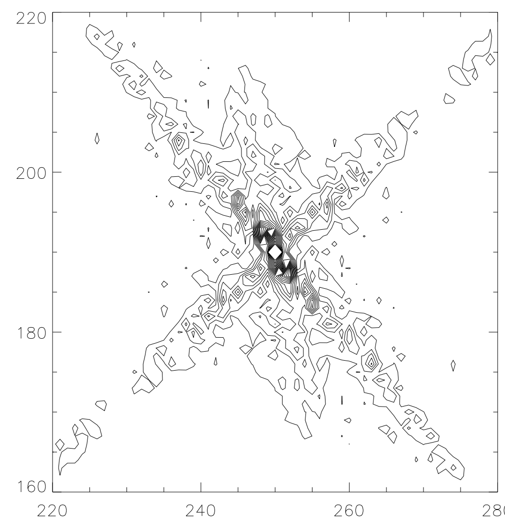


Image with $F(0, 0) = 0$



Spectrum contours

Fourier Image Components

An image, represented by $f(x,y)$ is the sum of a set of component images

$$f(x,y;u,v) = F(u,v)e^{i2\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)}$$

We can write the image as

$$f(x,y) = \sum_{u=0}^{N-1}\sum_{v=0}^{M-1} f(x,y;u,v)$$

It is important to understand the nature of the image components.

What is $f(x, y; u, v)$?

1. $F(u, v)$ is just a complex number. There are MN of them–one for each (u, v) combination.

2. $F(u, v) = A(u, v)e^{i\theta(u,v)}$ where $A(u, v)$ is the size and $\theta(u, v)$ is the angle of the complex number.

3. It is a compact way to write

$$
\begin{aligned}
f(x, y; u, v) &= F(u, v)e^{i2\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)} \\
&= A(u, v)\cos\left[2\pi\left(\frac{ux}{N} + \frac{vy}{M}\right) + \theta(u, v)\right] \\
&\quad + iA(u, v)\sin\left[2\pi\left(\frac{ux}{N} + \frac{vy}{M}\right) + \theta(u, v)\right]
\end{aligned}
$$

What is $F(u,v)e^{i2\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)}$?

4. If $f(x,y)$ is real then $F(u,v){=}F^*(N-u,M-v)$. This means that $A(N-u,M-v) = A(u,v)$ and $\theta(N-u,M-v) = -\theta(u,v)$.

5. We can combine the (u,v) and $(N-u,M-v)$ terms as

$$F(u,v)e^{i2\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)} + F(N-u,M-v)e^{i2\pi\left(\frac{(N-u)x}{N}+\frac{(M-v)y}{M}\right)}$$
$$= 2A(u,v)\cos\left[2\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)+\theta(u,v)\right]$$

6. Similarly, for real $f(x,y)$, the $(u,M-v)$ and $(N-u,v)$ terms can be combined since $F(u,M-v) = F^*(N-u,v)$

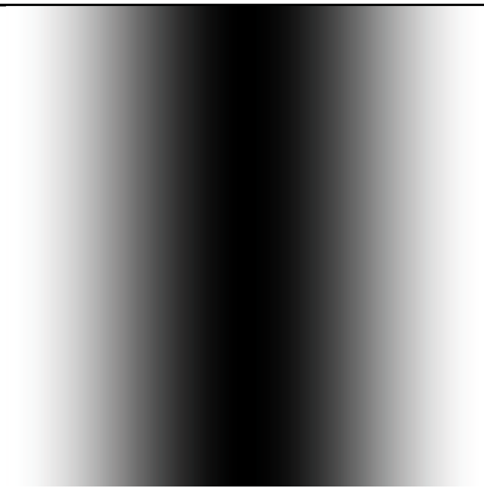
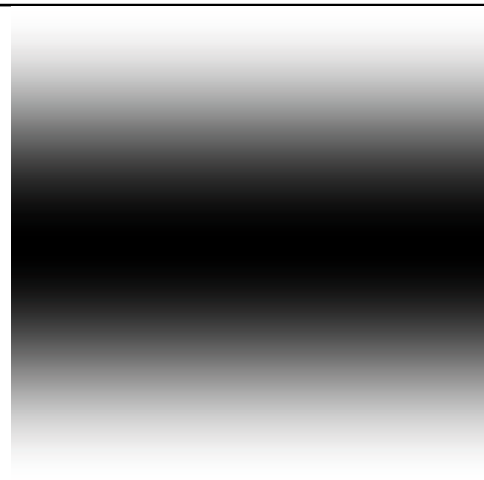
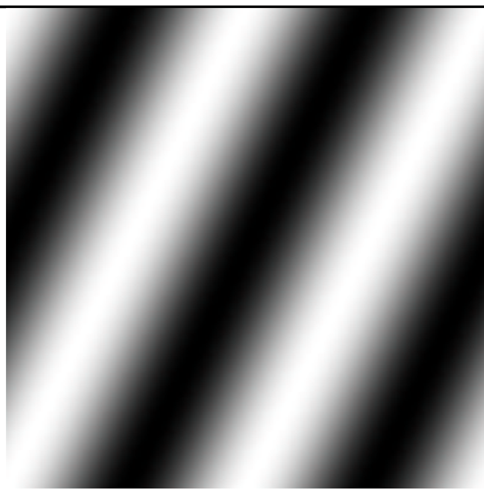
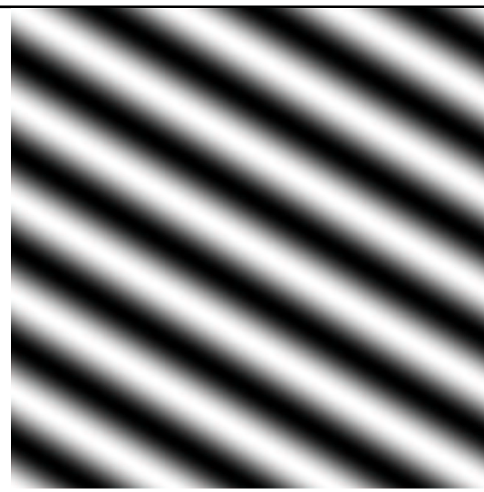
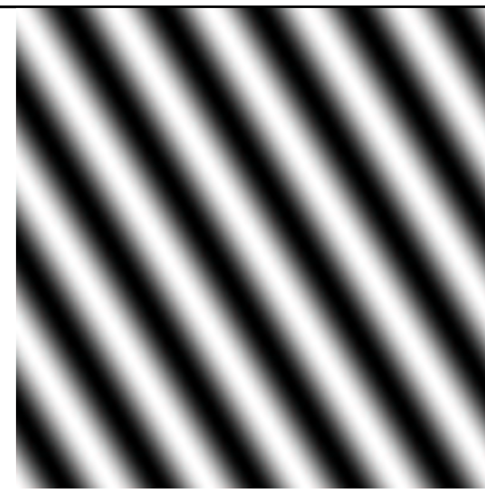
$$F(u,M-v)e^{i2\pi\left(\frac{ux}{N}+\frac{(M-v)y}{M}\right)} + F(N-u,v)e^{i2\pi\left(\frac{(N-u)x}{N}+\frac{vy}{M}\right)}$$
$$= 2A(u,M-v)\cos\left[2\pi\left(\frac{ux}{N}-\frac{vy}{M}\right)+\theta(u,M-v)\right]$$

The long version

$$f(x,y) = A[0,0] + \sum_{u=1}^{N/2-1} 2A(u,0)\cos\left[2\pi\frac{ux}{N} + \theta(u,0)\right]$$
$$+ \sum_{v=1}^{M/2-1} 2A(0,v)\cos\left[2\pi\frac{vy}{M} + \theta(0,v)\right]$$
$$+ \sum_{u=1}^{N/2-1}\sum_{v=1}^{M/2-1} 2A(u,v)\cos\left[2\pi\left(\frac{ux}{N} + \frac{vy}{M}\right) + \theta(u,v)\right]$$
$$+ \sum_{u=1}^{N/2-1}\sum_{v=1}^{M/2-1} 2A(u,M-v)\cos\left[2\pi\left(\frac{ux}{N} - \frac{vy}{M}\right) + \theta(u,M-v)\right]$$

Examples of some of the components are shown on the next slide.

Some Components

		
$u = 1 \; v = 0$	$u = 0 \; v = 1$	$u = 1 \; v = 2$
		
$u = 2 \; v = -1$	$u = 3 \; v = 5$	$u = 5 \; v = 3$

Filtering

Let $H(u, v)$ be the system function of a linear shift-invariant filter.

Let $f(x, y)$ be the input image and $g(x, y)$ be the output image.

The LSI filter operates independently on each of the exponential function image components.

$$f(x, y; u, v) = F(u, v)e^{i2\pi\left(\frac{xu}{N}+\frac{yv}{M}\right)}$$

The corresponding component of the output image is

$$\begin{aligned} g(x, y; u, v) &= H(u, v)f(x, y; u, v) \\ &= H(u, v)F(u, v)e^{i2\pi\left(\frac{xu}{N}+\frac{yv}{M}\right)} \end{aligned}$$

The complete output image is

$$g(x, y) = \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} g(x, y; u, v)$$

Filtering

The output image $g(x,y)$ has a DFT

$$G(u,v) = \frac{1}{MN}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} g(x,y)e^{-i2\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)}$$

$$g(x,y) = \sum_{u=0}^{N-1}\sum_{v=0}^{M-1} G(u,v)e^{i2\pi\left(\frac{ux}{N}+\frac{vy}{M}\right)}$$

where

$$G(u,v) = H(u,v)F(u,v)$$

The filter operates by changing the magnitude and phase of each (u,v) component by multiplying $F(u,v)$ by $H(u,v)$.

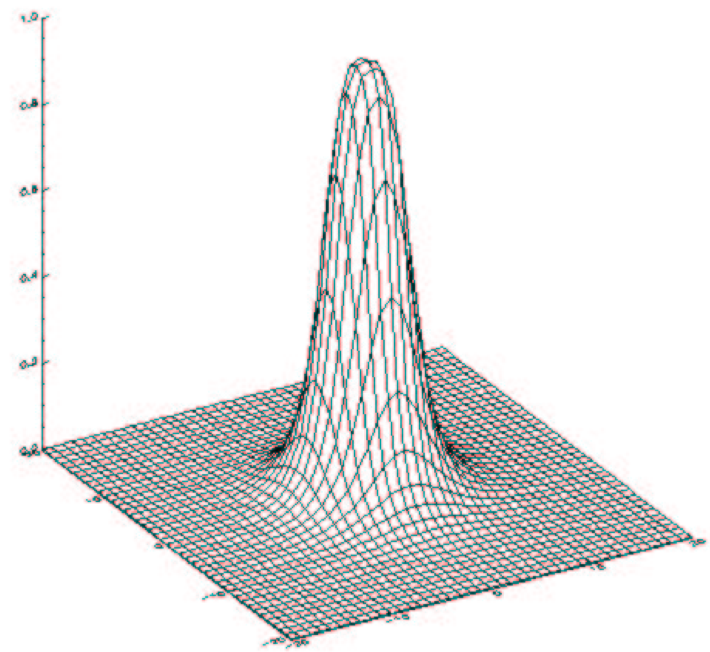
Filter Construction

An image can be processed by multiplication with a filter function

$$G(u, v) = F(u, v)H(u, v)$$

Butterworth lowpass filter:

$$H(u, v) = \frac{1}{1 + \left(\frac{u^2+v^2}{D_0^2}\right)^n}$$



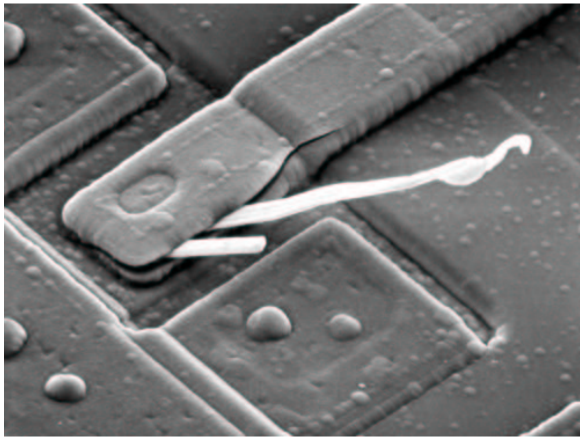
Butterworth filter of order $n = 2$

Constructing a Filter Array

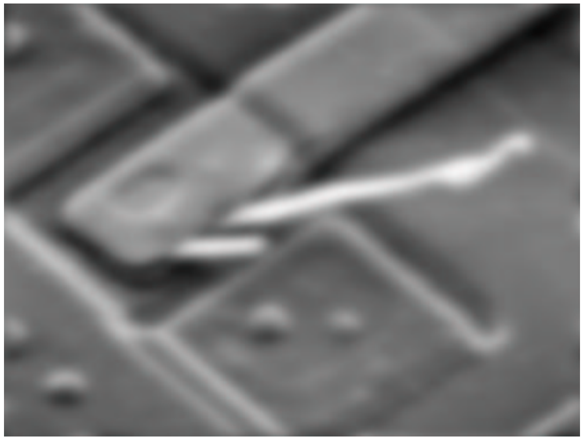
Given an image of size $N \times M$, we need to construct a filter of the same size. For a Butterworth LP of order p do the following:

```
x=Indgen(N)-N/2 ;x-axis (centered)
y=Indgen(M)-M/2 ;y-axis (centered
u=x#replicate(1,M) ;The u-coordinate plane
v=y##replicate(1,N) ;The v-coordinate plane
D=(u^2+v^2)/D0^2
H=1/(1+D^p)
surface,H,u,v
```

Filtering with LPF

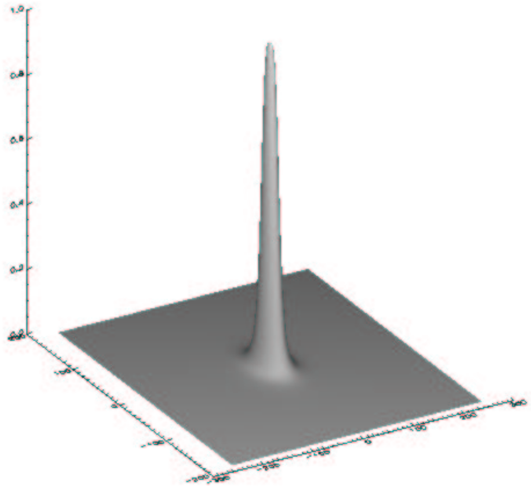


Original



After LPF

Spectrum



$|H_{LP}(u,v)|$

HPF Characteristic

A highpass filter can be constructed from a lowpass filter by subtraction.

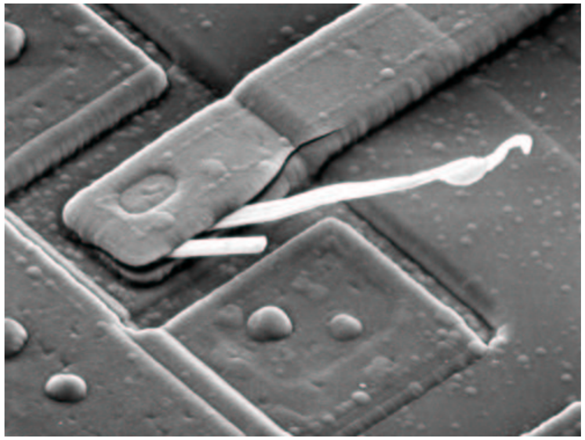
$$H_{HP}(u,v) = H_{LP}(0,0) - H_{LP}(u,v)$$

For the Butterworth filter

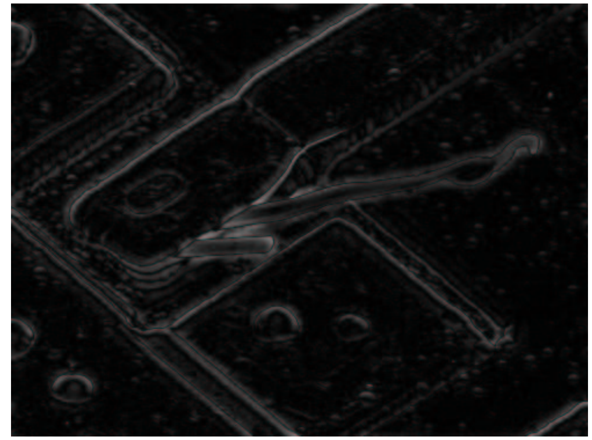
$$H_{HP}(u,v) = 1 - \frac{1}{1 + \left(\frac{u^2+v^2}{D_0^2}\right)^n}$$

$$= \frac{\left(\frac{u^2+v^2}{D_0^2}\right)^n}{1 + \left(\frac{u^2+v^2}{D_0^2}\right)^n}$$

Filtering with HPF

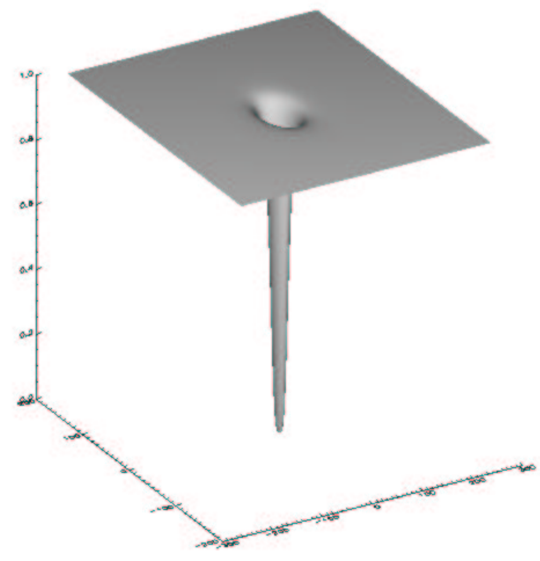


Original



After HPF

Spectrum



$|H_{HP}(u, v)|$

Filter Construction

```
;Construct coordinate arrays of size NxM
u=(Findgen(N)-N/2)#Replicate(1,M)
v=(Findgen(M)-M/2)##Replicate(1,N)
;Construct a lowpass filter
D0=30
LPF=(u^2+v^2) LT D0^2
;Show the filter response function
Window,/free,xsize=300,ysize=300
Shade_Surf,LPF,u,v
```

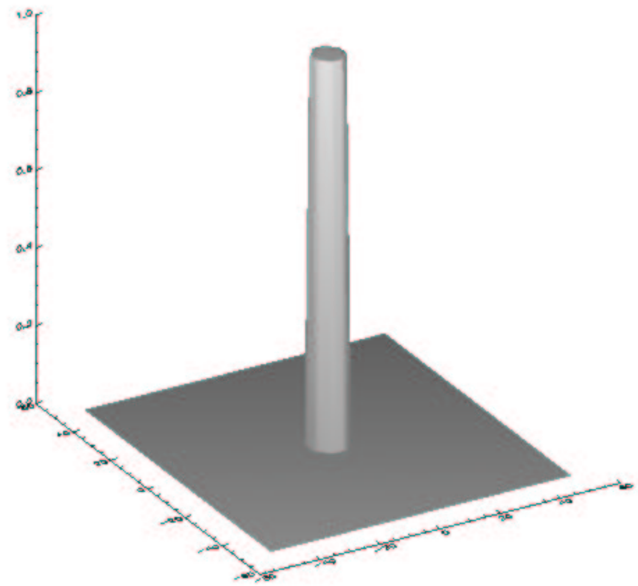


Image Filtering

```
A=Read_Image('restest.jpg')
Sa=Size(A,/dimensions)
N=Sa[0] & M=Sa[1]
Window,/free,xsize=N,ysize=M
TV,A
AF=FFT(A)
AFS=Shift(AF,N/2,M/2)
AFSH=LPF*AFS
A2=Abs(FFT(AFSH,/Inverse))
Window,/free,xsize=N,ysize=M
TVSCL,A2
```