1051-733-20092 Solution Set #6

1. A plano-concave lens with focal length $\mathbf{f}_1 = 120\,\text{mm}$ is placed in contact with a plano-convex lens with $\mathbf{f}_2 = 75\,\text{mm}$. Find the refractive power of the combination.

   The tricky (i.e., "mean") part of this problem is that you must recognize that the plano-concave lens has a negative focal length:

$$\frac{1}{\mathbf{f}_{\text{eff}}} = \frac{1}{\mathbf{f}_1} + \frac{1}{\mathbf{f}_2} - \frac{t}{\mathbf{f}_1 \cdot \mathbf{f}_2} \Longrightarrow \frac{1}{\mathbf{f}_{\text{eff}}} = \frac{1}{-120\,\text{mm}} + \frac{1}{75\,\text{mm}} - \frac{0\,\text{mm}}{-120\,\text{mm} \cdot 75\,\text{mm}}$$

$$\frac{1}{\mathbf{f}_{\text{eff}}} = +\frac{1}{200\,\text{mm}} \Longrightarrow \boxed{\mathbf{f}_{\text{eff}} = +200\,\text{mm}}$$

2. Determine the ratio of the focal lengths of a thin glass with surface radii of curvature R_1 and R_2 if used with both surfaces in water and with both in air. The respective refractive indices of water and air are $n = 1.33$ and $n = 1.0$.

$$\frac{n_1}{z_1} + \frac{n_2}{z_1'} = \frac{n_2 - n_1}{R_1}$$
$$\frac{n_2}{z_2} + \frac{n_3}{z_2'} = \frac{n_3 - n_2}{R_2}$$

In this situation, $n_2 = n_{\text{glass}}$ and $n_1 = n_3$. If the lens is thin, then $z_2 = -z_1'$

$$\frac{n_1}{z_1} + \frac{n_{\text{glass}}}{-z_2} = \frac{n_{\text{glass}} - n_1}{R_1}$$
$$\frac{n_{\text{glass}}}{z_2} + \frac{n_1}{z_2'} = \frac{n_1 - n_{\text{glass}}}{R_2}$$

Add them together:

$$\frac{n_1}{z_1} + \left(\frac{n_{\text{glass}}}{-z_2} + \frac{n_{\text{glass}}}{z_2}\right) + \frac{n_1}{z_2'} = \frac{n_{\text{glass}} - n_1}{R_1} + \frac{n_1 - n_{\text{glass}}}{R_2}$$
$$\frac{n_1}{z_1} + \frac{n_1}{z_2'} = (n_{\text{glass}} - n_1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Divide both sides by n_1:

$$\frac{1}{z_1} + \frac{1}{z_2'} = \left(\frac{n_{\text{glass}} - n_1}{n_1}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Now recognize that the left-hand side is the focal length:

$$\frac{1}{z_1} + \frac{1}{z_2'} = \frac{1}{\mathbf{f}_{(n_1)}} = \left(\frac{n_{\text{glass}} - n_1}{n_1}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

In air, $n_1 = 1$; in water $n_1 \cong 1.33$:

$$\frac{1}{\mathbf{f}_{\text{air}}} = \left(\frac{n_{\text{glass}} - 1}{1}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$
$$\frac{1}{\mathbf{f}_{\text{water}}} = \left(\frac{n_{\text{glass}} - 1.33}{1.33}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

So the ratios of the two focal lengths is:

$$\frac{\mathbf{f}_{\text{air}}}{\mathbf{f}_{\text{water}}} = \frac{\left(\frac{1}{n_{\text{glass}}-1}\right)}{\left(\frac{1.33}{n_{\text{glass}}-1.33}\right)} = \frac{1}{1.33} \cdot \left(\frac{n_{\text{glass}} - 1.33}{n_{\text{glass}} - 1}\right)$$

$$\frac{\mathbf{f}_{\text{water}}}{\mathbf{f}_{\text{air}}} = \frac{\left(\frac{1.33}{n_{\text{glass}}-1.33}\right)}{\left(\frac{1}{n_{\text{glass}}-1}\right)} = \boxed{\frac{\mathbf{f}_{\text{water}}}{\mathbf{f}_{\text{air}}} = 1.33 \cdot \left(\frac{n_{\text{glass}} - 1}{n_{\text{glass}} - 1.33}\right)}$$

In the common case of $n_{\text{glass}} = 1.5$, *we have:*

$$\frac{\mathbf{f}_{\text{water}}}{\mathbf{f}_{\text{air}}} = 1.33 \cdot \left(\frac{1.5 - 1}{1.5 - 1.33}\right) \cong 3.91$$

The trend fits with intuition; the refractive index of water is closer to that of glass, so we expect the focal length to increase; if water had the same index as glass, the focal length would be infinite.

3. The objective lens of a microscope has focal length $\mathbf{f} = 10\,\text{mm}$ and an ocular with $\mathbf{f} = 25\,\text{mm}$. Determine the distance between the lenses and the transverse magnfication if the object is in sharp focus when it is located at a distance $z = 10.5\,\text{mm}$ from the objective.

One point (the "trick" to the problem) to realize is that the microscope produces an image at the standard "near point" of the eye, i.e., at a distance of $250\,\text{mm}$ "behind" the eyepiece lens (which is the closest that the typical observer can see it in focus, thus the image must be virtual). The distance from the object to the object means that an image is formed by the objective at the distance:

$$\begin{aligned}
\frac{1}{(z_2)_{\text{obj}}} &= \frac{1}{\mathbf{f}_{\text{obj}}} - \frac{1}{(z_1)_{\text{obj}}} = \frac{1}{10\,\text{mm}} - \frac{1}{10.5\,\text{mm}} \\
&\Longrightarrow (z_2)_{\text{obj}} = \left(\frac{1}{10\,\text{mm}} - \frac{1}{10.5\,\text{mm}}\right)^{-1} = +210\,\text{mm} \\
(M_T)_{\text{obj}} &= -\frac{(z_2)_{\text{obj}}}{(z_1)_{\text{obj}}} = -\frac{210\,\text{mm}}{10.5\,\text{mm}} = -20
\end{aligned}$$

This image is then viewed by the eyepiece to create an image at the distance $z = -250\,\text{mm}$ (virtual image BEHIND second lens). The imaging equation for the second lens is:

$$\begin{aligned}
\frac{1}{(z_1)_{\text{eye}}} + \frac{1}{(z_2)_{\text{eye}}} &= \frac{1}{(z_1)_{\text{eye}}} + \frac{1}{-250\,\text{mm}} = \frac{1}{\mathbf{f}_{\text{eye}}} = \frac{1}{25\,\text{mm}} \\
&\Longrightarrow (z_1)_{\text{eye}} = \left(\frac{1}{25\,\text{mm}} - \frac{1}{-250\,\text{mm}}\right)^{-1} = +\frac{250}{11}\,\text{mm} \cong 22.7\,\text{mm}
\end{aligned}$$

which means that the distance from the objective lens to the eyepiece lens must be the sum of the image distance from the former and the object distance for the latter:

$$t = (z_2)_{\text{obj}} + (z_1)_{\text{eye}} = 210\,\text{mm} + \frac{250}{11}\,\text{mm} = \boxed{t = \frac{2560}{11}\,\text{mm} \cong 232.73\,\text{mm}}$$

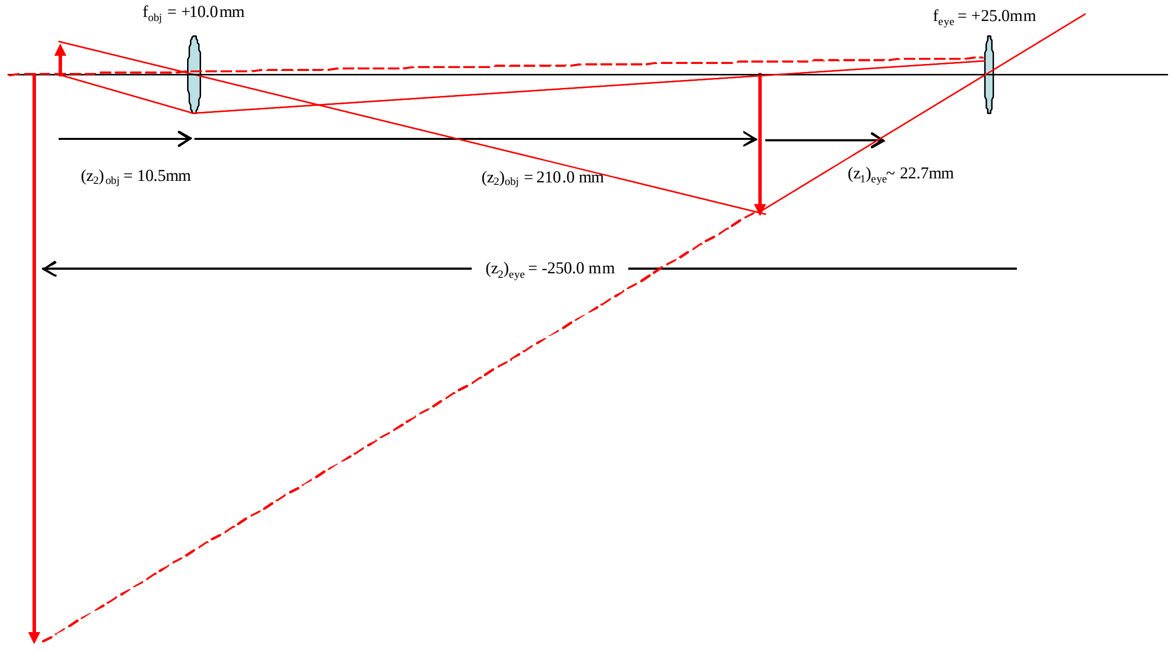
The transverse magnification of the second image is:

$$(M_T)_{\text{eye}} = -\frac{(z_2)_{\text{eye}}}{(z_1)_{\text{eye}}} = -\frac{-250\,\text{mm}}{\frac{250}{11}\,\text{mm}} = +11$$

So the transverse magnification of the entire system is:

$$M_T = (M_T)_{\text{obj}} \times (M_T)_{\text{eye}} = (-20)\cdot(+11) = \boxed{M_T = -220}$$

So the image is inverted, as suggested by the sketch.



f_obj = +10.0mm
f_eye = +25.0mm
(z2)obj = 10.5mm
(z2)obj = 210.0 mm
(z1)eye~ 22.7mm
(z2)eye = -250.0 mm

4. A distant object is observed through a telescope consisting of an objective lens with $\mathbf{f} = 300\,\text{mm}$ and a single eyepiece lens with $\mathbf{f} = 50\,\text{mm}$. The telescope is adjusted so that the final image is located at a distance of 400 mm from the eye lens.

   (a) Determine the distance between the two lenses.

   The second sentence means that the "telescope" is not being used as a telescope, i.e., as an optical system with "no power" so that the angles of the entering and exiting rays are identical. If the object distance from the first lens is infinite, then the image created by that lens is formed at its focal point, i.e., at the distance $(z_2)_{\text{obj}} = 300\,\text{mm}$. *If the image is to be viewed by eye, then it must be virtual, so the distance from the eyepiece is* $(z_2)_{\text{eye}} = -400\,\text{mm}$. *The corresponding object distance for the eyepiece (second lens) must satisfy:*

$$\frac{1}{(z_1)_{\text{eye}}} + \frac{1}{(z_2)_{\text{eye}}} = \frac{1}{\mathbf{f}_2}$$

$$\Longrightarrow (z_1)_{\text{eye}} = \left(\frac{1}{50\,\text{mm}} - \frac{1}{-400\,\text{mm}}\right)^{-1} = \frac{400}{9}\,\text{mm} \cong 44.4\,\text{mm} \textit{ for virtual image}$$

   In this case, the distance between the lenses is:

$$(z_2)_{\text{obj}} + (z_1)_{\text{eye}} = \boxed{t = 300\,\text{mm} + \tfrac{400}{9}\,\text{mm} \cong 344.44\,\text{mm} \textit{ for virtual image}}$$

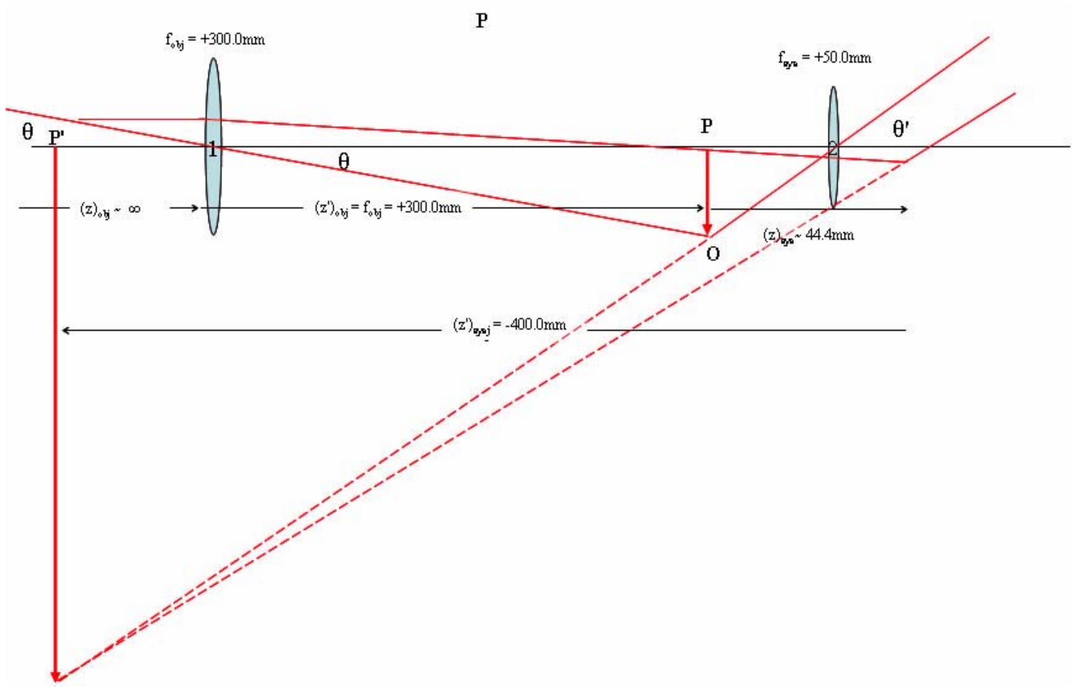
   If you interpreted the statement of the object distance to be that a real image is formed (that could be used to generate an image on a sensor), then the corresponding object distance for the second lens is:

$$(z_1)_{\text{eye}} = \left(\frac{1}{50\,\text{mm}} - \frac{1}{+400\,\text{mm}}\right)^{-1} = \frac{400}{7}\,\text{mm} \cong 57.1\,\text{mm} \textit{ for real image}$$

   and the distance between the lenses is:

$$(z_2)_{\text{obj}} + (z_1)_{\text{eye}} = \boxed{t = 300\,\text{mm} + \tfrac{400}{7}\,\text{mm} \cong 357.1\,\text{mm} \textit{ for real image}}$$

(b) Make a careful diagram of the system that traces a ray bundle from a lateral (off-axis) point on the object to the retina.



(c) Calculate the magnifying power of the system.

NOTE that the "magnifying power" is NOT the transverse magnification; the transverse magnification is the ratio of the "widths" of the two images, which cannot be calculated for an object at an infinite distance away. The "magnifying power" is the ratio of the exiting to entering ray-angle tangents, usually for the chief ray.

$$
\begin{aligned}
MP &= \frac{\tan[\theta']}{\tan[\theta]} \\
\tan[\theta] &= \frac{\overline{\mathbf{PO}}}{\overline{\mathbf{P1}}} = \frac{\overline{\mathbf{PO}}}{\mathbf{f}_1} = \frac{\overline{\mathbf{PO}}}{300\,\text{mm}} \\
\tan[\theta'] &= \frac{\overline{\mathbf{PO}}}{\overline{\mathbf{P2}}}
\end{aligned}
$$

$$
\boxed{MP = \frac{300\,\text{mm}}{44.4\,\text{mm}} \cong 6.76}
$$

5. An optical system is composed of two lenses and a stop. The first lens L_1 has diameter $d_1 = 100\,\text{mm}$ and focal length $\mathbf{f}_1 = 60\,\text{mm}$; the diameter of lens L_2 is $d_2 = 40\,\text{mm}$ and its focal length is $\mathbf{f}_2 = -75\,\text{mm}$. The diameter of the stop (iris) is $d_s = 60\,\text{mm}$. The distance from L_1 to the stop is 10 mm and from the stop to L_2 is 20 mm. Determine the positions of the image-space focal point $\mathbf{F}'$, the object-space focal point $\mathbf{F}$, the locations and sizes of the pupils, the locations of the principal points, and the angular field of view. Sketch the system.

To do this, I used the matrix formulation presented in the notes. First need to find the "vertex-to-vertex matrix" $\mathcal{M}_V$, which is a sequential cascade of refraction + transfer + refraction + transfer ... . For the first lens, the refraction matrix is:

$$\mathcal{R}_1 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\mathbf{f}_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.06\,\text{m}} & 1 \end{bmatrix}$$

The transfer from the first lens to the stop is:

$$\mathcal{T}_1 = \begin{bmatrix} 1 & \frac{t}{n} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{0.01\,\text{m}}{1} \\ 0 & 1 \end{bmatrix}$$

The stop has no power (infinite focal length), so its matrix is:

$$\mathcal{R}_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\infty} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The second transfer is:

$$\mathcal{T}_2 = \begin{bmatrix} 1 & \frac{0.02\,\text{m}}{1} \\ 0 & 1 \end{bmatrix}$$

and the matrix for the last lens is:

$$\mathcal{R}_3 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\mathbf{f}_2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{-0.075\,\text{m}} & 1 \end{bmatrix}$$

To find $\mathbf{F}'$, we need to find the intersection of the ray from ∞ with the optical axis. To find the field of view, we need to find the chief ray, which goes through the center of the stop

$$\begin{aligned}
\mathcal{M}_V &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ -\frac{1}{-0.075\,\text{m}} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{0.02\,\text{m}}{1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{0.01\,\text{m}}{1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.06\,\text{m}} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0.03\,\text{m} \\ -\frac{10}{\text{m}} & \frac{7}{5} \end{bmatrix}
\end{aligned}$$

The effective focal length of the system is the negative of the reciprocal of element C

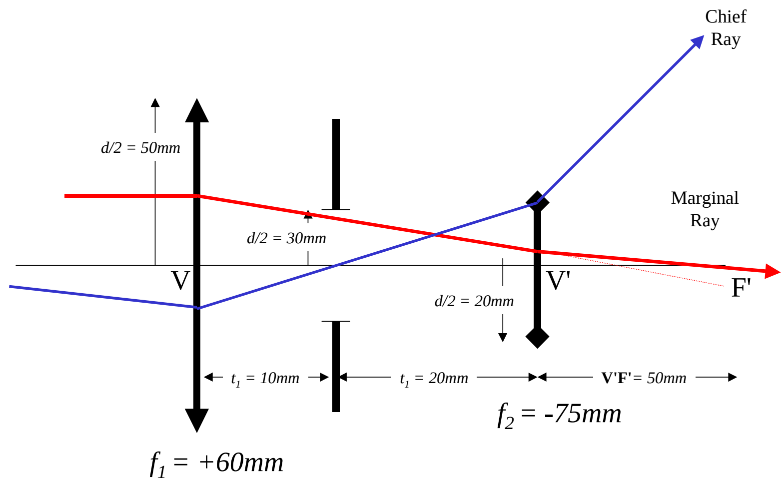
$$\mathbf{f}_{eff} = -\frac{1}{C} = \frac{1}{10}\,\text{m} = +100\,\text{mm}$$

Now cast out a "provisional marginal ray" from an object at ∞ with ray vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The output ray is:

$$\begin{bmatrix} \frac{1}{2} & 0.03\,\text{m} \\ -\frac{10}{\text{m}} & \frac{7}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{10}{\text{m}} \end{bmatrix}$$

The distance to from the rear lens to the image-space focal point $\mathbf{F}'$ *is the ratio of the components of the output vector*

$$\overline{\mathbf{V}'\mathbf{F}'} = -\frac{\left(\frac{1}{2}\right)}{\left(-\frac{10}{\mathrm{m}}\right)} = \frac{1}{20}\,\mathrm{m} = +50\,\mathrm{mm}$$



The front focal point is found by "reversing" the system:

$$. \left(\mathcal{M}_V\right)_{reversed} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.06\,\mathrm{m}} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{0.01\,\mathrm{m}}{1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{0.02\,\mathrm{m}}{1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{-0.075\,\mathrm{m}} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{7}{5} & 0.03\,\mathrm{m} \\ -\frac{10.0}{\mathrm{m}} & \frac{1}{2} \end{bmatrix} = \left(\mathcal{M}_V\right)^{-1}_{reversed}$$

Throw in the provisional marginal ray:

$$\begin{bmatrix} \frac{7}{5} & 0.03\,\mathrm{m} \\ -\frac{10.0}{\mathrm{m}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ -\frac{10.0}{\mathrm{m}} \end{bmatrix}$$

and calculate the ratio of the ray height to the ray angle to find the distance:

$$\overline{\mathbf{FV}} = -\frac{\left(\frac{7}{5}\right)}{\left(-\frac{10.0}{\mathrm{m}}\right)} = +0.14\,\mathrm{m} = +140\,\mathrm{mm}$$

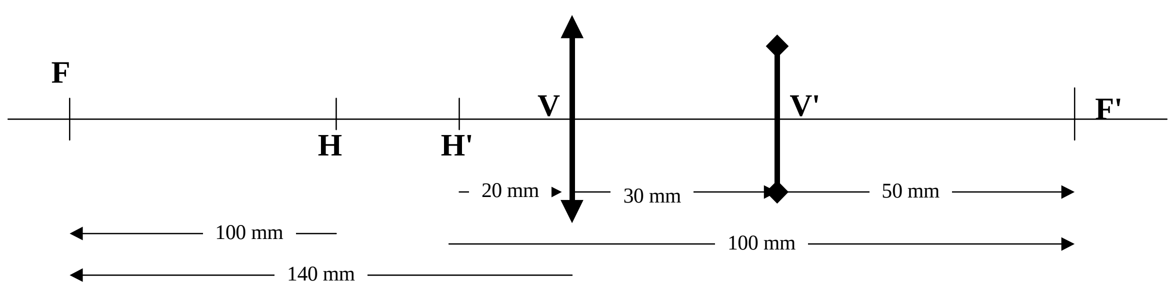


Diagram of optical system, showing the vertices, focal points, and principal points (which are one focal length from each focal point)

We now use the locations of the focal points and the focal length to find the principal points. Since the focal length is $\mathbf{f} = +\,100\,\text{mm}$*, then the object space principal point* $\mathbf{H}$ *is located such that* $\overline{\mathbf{FH}} = +100\,\text{mm}$*. Since the distance* $\overline{\mathbf{FV}} = +140\,\text{mm}$*, the distance* $\overline{\mathbf{HV}} = (140\,\text{mm} - 100\,\text{mm}) = 40\,\text{mm}$*. The distance from the image-space vertex to the image-space focal point is* $\overline{\mathbf{V'F'}} = 50\,\text{mm}$ *and* $\overline{\mathbf{H'F'}} = 100\,\text{mm} \implies \overline{\mathbf{V'H'}} = -50\,\text{mm}$.

We now have to find the chief ray, which goes through the center of the stop. The angle of the ray through the center of the stop to lens L_1 *is the ratio of the "semidiameter" to the distance:* $\frac{50\,\text{mm}}{10\,\text{mm}} = 5$*. The angle of the ray through* L_2 *is* $\frac{20\,\text{mm}}{20\,\text{mm}} = 1$*; so the chief ray is constrained by lens* L_2*.The entrance pupil is the image of the stop seen through* L_1*. The distance is* $10\,\text{mm}$ *and the focal length is* $f_1 = +60\,\text{mm}$*, so the distance to the entrance pupil is:*

$$\frac{1}{s'} = \frac{1}{60} - \frac{1}{10} = -\frac{5}{60} = -\frac{1}{12} \Longrightarrow s' = -12\,\text{mm}$$

The entrance pupil is virtual, its transverse magnification is:

$$M_T = -\frac{s'}{s} = -\frac{(-12)}{10} = +1.2$$

For the exit pupil, the focal length of the lens that the stop is imaged "through" is $\mathbf{f}_2 = -75\,\text{mm}$ *and the distance* $s = +20\,\text{mm}$*. The distance to the exit pupil is:*

$$\frac{1}{s'} = \frac{1}{-75} - \frac{1}{20} = -\frac{19}{300} \Longrightarrow s' = -\frac{300}{19} \simeq -15.8\,\text{mm}$$

6. A magnifying lens with focal length $\mathbf{f} = 60\,\text{mm}$ is used to view an object by a person whose closest focus is 250 mm. If the person holds the glass close to the eye, determine the best position of the object.

   Use the imaging equation:

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{\mathbf{f}} \implies \frac{1}{z_1} = \frac{1}{\mathbf{f}} - \frac{1}{z_2}$$

$$\implies z_1 = \left(\frac{1}{\mathbf{f}} - \frac{1}{z_2}\right)^{-1} = \left(\frac{1}{60\,\text{mm}} - \frac{1}{-250\,\text{mm}}\right)^{-1} = \frac{1500}{31}\,\text{mm} \cong +48.4\,\text{mm}$$

$$\boxed{z_1 \cong +48.4\,\text{mm}}$$

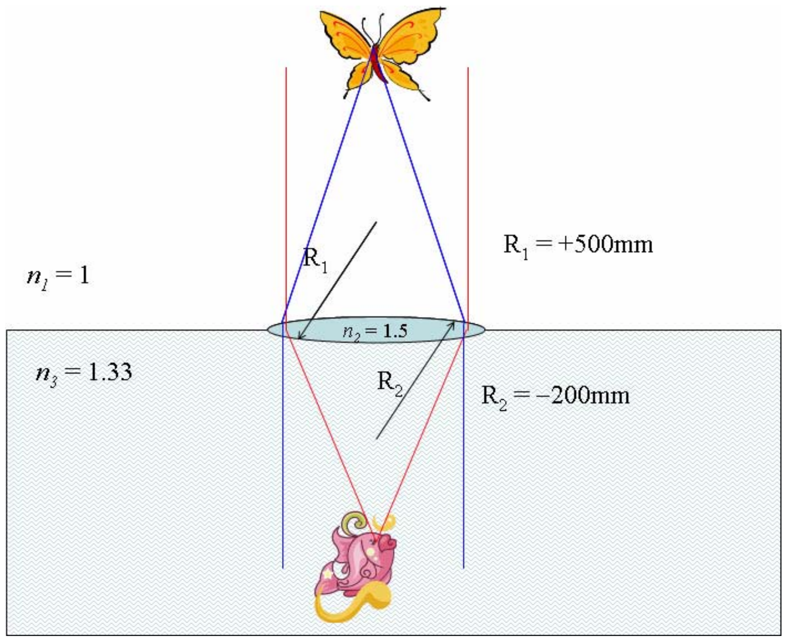
7. An object of diameter 50 mm is placed 333 mm from a double-concave lens with power 8 diopters. Determine the position and size of the image and characterize its nature as real or virtual.

$$\begin{aligned}
\frac{1}{z_1} + \frac{1}{z_2} &= \frac{1}{\mathbf{f}} = -8\,\text{m}^{-1} \\
z_1 &= 333\,\text{mm} \implies z_2 = \left(-8\,\text{m}^{-1} - \frac{1}{333\,\text{mm}}\right)^{-1} \cong \boxed{z_2 \cong -90.884\,\text{mm}} \\
M_T &= -\frac{z_2}{z_1} = -\frac{-90.884\,\text{mm}}{333\,\text{mm}} = \boxed{M_T \cong 0.273}
\end{aligned}$$

   Since the image distance is negative, the image is virtual. Since $M_T < 1$, the image is "minified."

8. A thin biconvex lens is supported with its symmetry axis vertical on the surface of a pond of water. The lower surface of the lens with radius of curvature $R = 200\,\text{mm}$ is in contact with the water, while the upper surface in air has radius $R = 500\,\text{mm}$. The refractive index of the water is $n = 1.33$ and of the air is $n = 1.0$. An insect hovering over the lens and a fish in the water each see parallel rays of light.

(a) Determine the distances of the insect and the fish from the lens.
First, make a sketch:



Since the insect and the fish both "see" parallel rays, they must both be at the respecive focal points. Recall the imaging equation for different media:

$$\frac{n_2 - n_1}{R_1} + \frac{n_3 - n_2}{R_2} = \frac{1}{\mathbf{f}} = \frac{n_1}{\mathbf{f}_1} = \frac{n_3}{\mathbf{f}_3}$$

where the subscripts 1 *and* 3 *refer to the media of the object and image spaces, respectively. In the first case for the fish,* $R_1 = +200\,\text{mm}$ *and* $R_2 = -500\,\text{mm}$

$$\frac{1.5 - 1}{+0.5\,\text{m}} + \frac{1.33 - 1.5}{-0.2\,\text{m}} = 1.85\,\text{m}^{-1} \implies \mathbf{f}_1 = \frac{1\,\text{m}}{1.85} \cong 540.5\,\text{mm}$$

$$\mathbf{f}_3 = n_3 \cdot \mathbf{f}_1 = 1.33 \cdot \frac{1\,\text{m}}{1.85} \cong 718.9\,\text{mm}$$

: *The distance from the insect to the lens is*

$$\boxed{\textit{distance from insect to lens} = \mathbf{f}_1 \cong 540.5\,\text{mm}}$$

$$\boxed{\textit{distance from fish to lens} = \mathbf{f}_3 = n_3 \cdot \mathbf{f}_1 \cong 718.9\,\text{mm}}$$

(b) Determine the effect of reversing the lens.

This just swaps the radii of curvature so that $R_1 =$

$$\frac{1.5-1}{+0.2\,\mathrm{m}} + \frac{1.33-1.5}{-0.5\,\mathrm{m}} \cong \frac{2.84}{\mathrm{m}} \implies \mathbf{f}_1 = \frac{1\,\mathrm{m}}{2.84} \cong 352.1\,\mathrm{mm}$$

$$\mathbf{f}_3 = n_3 \cdot \mathbf{f}_1 = 1.33 \cdot \frac{1\,\mathrm{m}}{2.84} \cong 468.3\,\mathrm{mm}$$

$$\boxed{\textit{distance from insect to lens} = \mathbf{f}_1 \cong 352.1\,\mathrm{mm}}$$

$$\boxed{\textit{distance from fish to lens} = \mathbf{f}_3 = n_3 \cdot \mathbf{f}_1 \cong 468.3\,\mathrm{mm}}$$