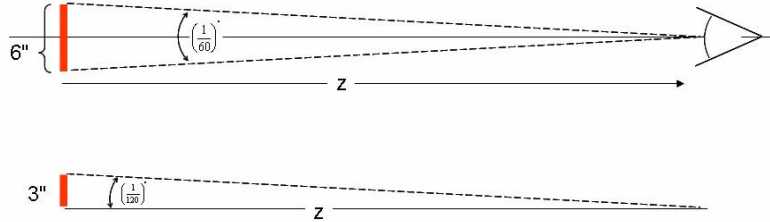


1. Assume that the resolution limit of the eye is 1 arcminute. At what distance can the eye see a black circle of diameter 6" on a white background?



One arcminute is  $(\frac{1}{60})^\circ$ , so consider a triangle with short side of 3 in and angle of  $(\frac{1}{120})^\circ$ . By (VERY BASIC) trigonometry, we have:

$$\tan \left[ \left( \frac{1}{120} \right)^\circ \right] = \frac{3 \text{ in}}{z} \implies z \cong \frac{75 \text{ mm}}{\tan \left[ \left( \frac{1}{120} \right)^\circ \right]}$$

$$\tan \left[ \left( \frac{1}{120} \right)^\circ \right] = \tan \left[ \frac{1}{120} \times \frac{\pi}{180} \text{ radians} \right] = \tan \left[ \frac{\pi}{120 \cdot 180} \right] \cong 1.454 \times 10^{-4} \text{ radians}$$

so half of the circle subtends about 0.145 milliradian = 145  $\mu$ radian. This is the radius of the resolution spot, so the angular diameter would be:

$$D_0 \cong 2.9 \times 10^{-4} \text{ radians}$$

$$z \cong \frac{150 \text{ mm}}{2.908 \times 10^{-4}} \cong 523.9 \text{ m} \cong 20,626 \text{ in} \cong 1719 \text{ ft} \cong \frac{1}{2} \text{ km}$$

2. Calculate the linear separation of two objects on the surface of the moon that may be barely resolved by the 200" telescope on Palomar Mountain; neglect atmospheric effects.

Assume that the observation is in visible light so that  $\lambda_0 \cong 0.5 \mu\text{m}$ . According to the Rayleigh limit, two Airy disk patterns with equal irradiances are "just barely" resolved if the central peak of one coincides with the first minimum of the other.

$$(\Delta\theta)_{\min} \gtrsim 1.22 \frac{\lambda_0}{d_0}$$

The distance to the moon is approximately 240,000 miles (the distance I drove my last car before its demise):

$$\begin{aligned} L &\cong 240,000 \text{ mi} \\ &\cong 2.4 \times 10^5 \text{ mi} \times 5280 \frac{\text{ft}}{\text{mi}} \times 12 \frac{\text{in}}{\text{ft}} \times 25.4 \frac{\text{mm}}{\text{in}} \cong 3.8624 \times 10^5 \text{ km} \\ d_0 &\cong 200 \text{ in} \times 25.4 \frac{\text{mm}}{\text{in}} = 5.08 \text{ m} \end{aligned}$$

$$\begin{aligned} (\Delta\theta)_{\min} &\gtrsim 1.22 \times \frac{0.5 \mu\text{m}}{5.08 \text{ m}} \cong 1.201 \times 10^{-7} \text{ radian} \\ 1 \text{ radian} &= 60 \frac{\text{arcseconds}}{\text{arcminute}} \times 60 \frac{\text{arcminutes}}{\text{degree}} \times \frac{360 \text{ degree}}{2\pi \text{ radian}} \\ &\cong 206265 \frac{\text{arcseconds}}{\text{radian}} \cong 2.0 \times 10^5 \frac{\text{arcseconds}}{\text{radian}} \\ \implies 1 \text{ arcsecond} &\cong 5 \times 10^{-6} \text{ radians} = 5 \mu\text{radians} \end{aligned}$$

useful number to remember:  $\boxed{1 \text{ arcsecond} \cong 5 \mu\text{radian}}$

$$(\Delta\theta)_{\min} \gtrsim 1.201 \times 10^{-7} \text{ radian} \cong 0.12 \mu\text{radian} \cong \left(\frac{0.12}{5}\right)'' \cong 0.024''$$

At the distance to the moon, the angular subtense of the resolvable spot has linear dimension:

$$\begin{aligned} \ell &\cong 1.201 \times 10^{-7} \cdot 3.8624 \times 10^5 \text{ km} \\ &\boxed{\ell \cong 46.4 \text{ m}} \end{aligned}$$

3. Determine the diameter of a telescope that would be required to resolve the two equally bright components of a double star whose linear separation is  $10^8$  km at a distance of 10 light years.

*The distance  $z$  in the figure in #1 is now 10 light years. Light travels at velocity*

$$v \cong 3 \cdot 10^8 \frac{\text{m}}{\text{s}}$$

*The number of seconds in a year:*

$$60 \frac{\text{s}}{\text{min}} \times 60 \frac{\text{min}}{\text{h}} \times 24 \frac{\text{h}}{\text{d}} \times 365.25 \frac{\text{d}}{\text{y}} \cong 3.156 \times 10^7 \frac{\text{s}}{\text{y}} \cong 31,000,000 \frac{\text{s}}{\text{y}}$$

*(another useful number to remember)*

*distance traveled by light in a year:*

$$z \cong 3 \cdot 10^8 \frac{\text{m}}{\text{s}} \cdot 3.156 \times 10^7 \frac{\text{s}}{\text{y}} \cong 9.468 \times 10^{12} \frac{\text{km}}{\text{y}}$$

$$\implies 10 \text{ light years} \cong 9.468 \times 10^{13} \text{ km} \lesssim 10^{14} \text{ km}$$

*The tangent of the angle is*

$$\tan \theta = \frac{\ell}{z} = \frac{10^8 \text{ km}}{9.468 \times 10^{13} \text{ km}} \cong 1.056 \times 10^{-6} \text{ radians} \cong 1.06 \mu\text{radian} \cong 0.2 \text{ arcsecond}$$

*The required diameter  $d_0$  of the telescope would satisfy:*

$$1.22 \frac{\lambda_0}{d_0} \leq 1.056 \times 10^{-6} \text{ radians}$$

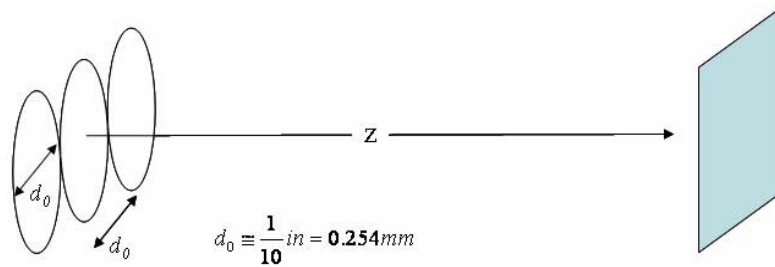
$$\implies d_0 \geq 1.22 \frac{\lambda_0}{1.056 \times 10^{-6} \text{ radians}}$$

*In visible light with  $\lambda_0 = 0.5 \mu\text{m}$ , the diameter is:*

$d_0 \geq 1.22 \frac{0.5 \mu\text{m}}{1.056 \times 10^{-6} \text{ radians}} \cong 0.578 \text{ m} = 578 \text{ mm} \cong 22.76 \text{ in} \cong 2 \text{ ft}$
---

*many advanced amateur astronomers have telescopes of this size.*

4. The neoimpressionist painter George Seurat produced paintings composed of a large number of closely spaced “dots” of unmixed pigments. The colors are mixed in the eye of the observer to blend into the desired colors. Assume that the diameters and center-to-center spacings of the dots are both  $\frac{1}{10}$  in. How far must the observer stand from the painting to obtain the necessary mixing of color?



*This is very similar to #1: we want the distance  $z_1$  to be sufficiently large so that we cannot resolve adjacent spots. One way would be to again assume that the eye can resolve about  $1'$  of arc and that we want the linear dimension of the blur to subtend at least two spots, we can use the same discussion in #1 to find:*

$$z \cong \frac{2 \cdot 2.54 \text{ mm}}{2.908 \times 10^{-4}} \cong 17.5 \text{ m} \cong 690 \text{ in} \cong 60 \text{ ft}$$

*which is huge.*

*Another way would be to assume that a diffraction-limited eye with a pupil diameter of  $d_0 \cong 2 \text{ mm}$ , which is an appropriate number for bright light (a larger diameter might be appropriate since you don't exhibit paintings under bright lights; this would make the diffraction spot smaller and require you to move back farther). The angular diameter of the full Airy disk for this estimate is:*

$$\begin{aligned} 2.44 \frac{\lambda_0}{d_0} &= 2.44 \frac{0.5 \mu\text{m}}{2 \text{ mm}} \cong 6.1 \cdot 10^{-4} \text{ radians} \cong 610 \mu\text{radians} \\ &\cong 206265 \cdot 6.1 \cdot 10^{-4} \text{ arcsec} \cong 126 \text{ arcsec} \cong 2 \text{ arcmin} \end{aligned}$$

*So we're in the same ballpark. Now assume that the Airy diffraction disk subtends at least two circles:*

$$6.1 \cdot 10^{-4} \cdot z \geq 2 \cdot 0.1 \text{ in} \cong 5 \text{ mm} \implies z \cong \frac{5 \text{ mm}}{6.1 \cdot 10^{-4}} \cong 8 \text{ m} \cong 26 \text{ ft} \cong 315 \text{ in}$$

*which is smaller because the diameter of the diffraction spot is larger.*

*As long as you explain your logic, I accept values in the range*

$$17.5 \text{ m} \geq z \geq 8 \text{ m}$$

5. A glass plate is sprayed with circular “dots” of pigment of the same size that are randomly distributed over the plate. If a distant point source of light is observed by eye through the plate, a diffuse halo is seen whose angular width is  $2^\circ$ . Estimate the diameter of the particles.

*The first point to make is that the diffraction pattern from an opaque spot on a clear background and from a clear aperture on an opaque background are very similar except at the origin:*

*Clear aperture:*

$$\begin{aligned}
 f_1[x, y] &= \text{CYL} \left[ \frac{r}{d_0} \right] \implies F_1[\xi, \eta] = \frac{\pi d_0^2}{4} \cdot \text{SOMB}[d_0 \rho] \\
 &\implies g_1[x, y] \propto F_1 \left[ \frac{x}{\lambda_0 z_1}, \frac{y}{\lambda_0 z_1} \right] \propto \text{SOMB} \left[ \frac{r}{\left( \frac{\lambda_0 z_1}{d_0} \right)} \right] \\
 &\implies I_1[x, y] \propto |g_1[x, y]|^2 \propto \text{SOMB}^2 \left[ \frac{r}{\left( \frac{\lambda_0 z_1}{d_0} \right)} \right]
 \end{aligned}$$

*Opaque spot:*

$$\begin{aligned}
 f_2[x, y] &= 1 - \text{CYL} \left[ \frac{r}{d_0} \right] \implies F_2[\xi, \eta] = \delta[\xi, \eta] - \frac{\pi d_0^2}{4} \cdot \text{SOMB}[d_0 \rho] \\
 &\implies g_2[x, y] \propto \delta \left[ \frac{x}{\lambda_0 z_1}, \frac{y}{\lambda_0 z_1} \right] - F_1 \left[ \frac{x}{\lambda_0 z_1}, \frac{y}{\lambda_0 z_1} \right] = (\lambda_0 z_1)^2 \delta[x, y] - \frac{\pi d_0^2}{4} \text{SOMB} \left[ \frac{r}{\left( \frac{\lambda_0 z_1}{d_0} \right)} \right] \\
 &\implies I_2[x, y] \propto \left| (\lambda_0 z_1)^2 \cdot \delta[x, y] - \frac{\pi d_0^2}{4} \cdot \text{SOMB} \left[ \frac{r}{\left( \frac{\lambda_0 z_1}{d_0} \right)} \right] \right|^2 \\
 &\propto (\lambda_0 z_1)^4 \cdot \delta[x, y] - 2 \cdot \frac{\pi d_0^2}{4} \cdot (\lambda_0 z_1)^2 \cdot \delta[x, y] \cdot \text{SOMB} \left[ \frac{r}{\left( \frac{\lambda_0 z_1}{d_0} \right)} \right] + \frac{\pi d_0^2}{4} \text{SOMB}^2 \left[ \frac{r}{\left( \frac{\lambda_0 z_1}{d_0} \right)} \right] \\
 &= \delta[x, y] - \frac{\pi d_0^2}{2 (\lambda_0 z_1)^2} \cdot \delta[x, y] \cdot \text{SOMB}[0, 0] + \left( \frac{\pi d_0^2}{4 (\lambda_0 z_1)^2} \right)^2 \text{SOMB}^2 \left[ \frac{r}{\left( \frac{\lambda_0 z_1}{d_0} \right)} \right] \\
 &= \left( 1 - \frac{\pi d_0^2}{2 (\lambda_0 z_1)^2} \right) \cdot \delta[x, y] + \left( \frac{\pi d_0^2}{4 (\lambda_0 z_1)^2} \right)^2 \text{SOMB}^2 \left[ \frac{r}{\left( \frac{\lambda_0 z_1}{d_0} \right)} \right]
 \end{aligned}$$

*So the irradiance “away from” the origin is proportional to the square of the besinc (sombbrero) function in both cases. The fact that the diffraction patterns are similar away from the origin for complementary objects arises from Babinet’s principle.*

If the object consists of a single paint dot, the image is as just shown, with spatial variation away from the origin in the form of:

$$I[x \neq 0, y \neq 0] \propto \text{SOMB}^2 \left[ \frac{r}{\left(\frac{\lambda_0 z_1}{d_0}\right)} \right]$$

so the angular diameter of the Airy disk is:

$$\begin{aligned} D_0 &\cong 2 \cdot 1.22 \cdot \frac{\lambda_0 z_1}{d_0} \\ \Delta\theta &\cong \frac{D_0}{z_1} \cong 2.44 \frac{\lambda_0}{d_0} \end{aligned}$$

If we have a lot of paint dots on a regularly spaced grid (described by a COMB function, the object function is:

$$f[x, y] = \text{CYL} \left[ \frac{r}{d_0} \right] * \text{COMB} \left[ \frac{x}{\Delta x}, \frac{y}{\Delta y} \right]$$

where  $\Delta x, \Delta y$  are usually larger than  $d_0$  (and often significantly larger).

$$\begin{aligned} I[x, y] &\propto \left( \pi \frac{d_0^2}{4} \text{SOMB} \left[ \frac{r}{\left(\frac{\lambda_0 z_1}{d_0}\right)} \right] \cdot (\Delta x \cdot \Delta y) \cdot \text{COMB} \left[ \frac{x}{\left(\frac{\lambda_0 z_1}{\Delta x}\right)}, \frac{y}{\left(\frac{\lambda_0 z_1}{\Delta y}\right)} \right] \right)^2 \\ &\propto \text{SOMB}^2 \left[ \frac{r}{\left(\frac{\lambda_0 z_1}{d_0}\right)} \right] \cdot \text{COMB} \left[ \frac{x}{\left(\frac{\lambda_0 z_1}{\Delta x}\right)}, \frac{y}{\left(\frac{\lambda_0 z_1}{\Delta y}\right)} \right] \end{aligned}$$

If  $(\Delta x)^2 + (\Delta y)^2 > d_0^2$ , as is usual, then many elements of the COMB function will lie within the central disk of the Sombrero, so again the central disk is determined by the diameter of the paint spot via

$$D_0 \cong 2.44 \left( \frac{\lambda_0 z_1}{d_0} \right)$$

If the spots are randomly placed, we still get an irradiance of the same form as:

$$I[x, y] \propto \text{SOMB}^2 \left[ \frac{r}{\left(\frac{\lambda_0 z_1}{d_0}\right)} \right] \cdot R \left[ \frac{x}{\left(\frac{\lambda_0 z_1}{\Delta x}\right)}, \frac{y}{\left(\frac{\lambda_0 z_1}{\Delta y}\right)} \right]$$

where  $R$  is the spectrum of the ensemble of spot locations; the central disk again is determined by the diameter of the paint spot via

$$D_0 \cong 2.44 \left( \frac{\lambda_0 z_1}{d_0} \right)$$

The “halo” may be interpreted as the diameter of the Airy pattern. Since the first zero is at  $1.22\frac{\lambda_0}{d_0}$ , we can estimate that the angular diameter of the “halo” is halfway between:

$$D_{halo} \cong 2.44 \frac{\lambda_0}{d_0}$$

If the angular **diameter** of the halo is  $2^\circ = 2 \cdot \frac{\pi}{180}$  radian  $= 3.491 \times 10^{-2}$  radian  $\cong \frac{1}{30}$  radian, then the diameter of the paint spot may be estimated:

$$\Delta\theta \cong 3.491 \times 10^{-2} \text{ radian} \cong 2.44 \left( \frac{\lambda_0 z_1}{d_0} \right) \cdot \frac{1}{z_0} = 2.44 \left( \frac{\lambda_0}{d_0} \right)$$

$$d_0 \cong \frac{2.44 \cdot \lambda_0}{3.491 \times 10^{-2}} \cong 69.9 \cdot \lambda_0$$

If  $\lambda_0 = 0.5 \mu\text{m}$  for visible light, then:

$$d_0 \cong \frac{2.44 \cdot 0.5 \mu\text{m}}{3.491 \times 10^{-2}} \cong 34.95 \mu\text{m} \cong 0.035 \text{ mm}$$

Since small scales in the object transform to large scales in the diffraction pattern, we can use the Fraunhofer diffraction to measure small objects accurately!