1051-733-20092 Solution Set #2

1. Find AND GRAPH the irradiance distribution (squared magnitude of the amplitude) of the Fresnel diffraction pattern *on the optical axis* (i.e., a function of z instead of a function of $[x, y]$) if $f[x, y] = CYL(r)$ (HINT: write down the Fresnel diffraction formula as integral and change variable to more convenient form).

$$g[x_1, y_1; z_1] = \iint_{-\infty}^{+\infty} f[x_0, y_0] \;\; h[x_1 - x_0, y_1 - y_0; z_1, \lambda_0] \;\; dx_0 \; dy_0$$

$$= \frac{1}{i\lambda_0 z_1} \exp\left[2\pi i \frac{z_1}{\lambda_0}\right] \cdot \iint_{-\infty}^{+\infty} f[x_0, y_0] \;\; \exp\left[+i\pi\left(\left(\frac{x_1 - x_0}{\sqrt{\lambda_0 z_1}}\right)^2 + \left(\frac{y_1 - y_0}{\sqrt{\lambda_0 z_1}}\right)^2\right)\right] \; dx_0 \; dy_0$$

For clarity, define:

$$k_0 \equiv \frac{2\pi}{\lambda_0}$$

$$\Longrightarrow \; g[x_1, y_1; z_1] = \frac{\exp[+ik_0 z_1]}{+i\lambda_0 z_1} \cdot \iint_{-\infty}^{+\infty} f[x_0, y_0] \;\; \exp\left[+i\frac{\pi}{\lambda_0 z_1}\left((x_1 - x_0)^2 + (y_1 - y_0)^2\right)\right] \; dx_0 \; dy_0$$

On axis $\Longrightarrow$ $[x_1, y_1] = [0, 0]$ *and the Fresnel diffraction equation in Cartesian coordinates becomes:*

$$g[0, 0; z_1] = \frac{\exp[+ik_0 z_1]}{+i\lambda_0 z_1} \iint_{-\infty}^{+\infty} f[x_0, y_0] \;\; \exp\left[+i\frac{\pi}{\lambda_0 z_1}\left(x_0^2 + y_0^2\right)\right] \; dx_0 \; dy_0$$

From this point, we can drop the subscripts on the integration coordinates.

In this problem:

$$f[x, y] = CYL\left(\sqrt{x^2 + y^2}\right)$$

Change the 2-D Fresnel integral to polar coordinates:

$$\begin{aligned}
g[0, 0; z_1] &= \frac{\exp[+ik_0 z_1]}{+i\lambda_0 z_1} \iint_{-\infty}^{+\infty} t_A[x, y] \;\; \exp\left[+i\frac{k_0}{2z_1}\left(x^2 + y^2\right)\right] \; dx \; dy \\
&= \frac{\exp[+ik_0 z_1]}{+i\lambda_0 z_1} \int_{\theta=-\pi}^{\theta=+\pi} \int_{r=0}^{r=\infty} t_A(r) \;\; \exp\left[+i\frac{k_0}{2z_1} r^2\right] \; r \; dr \; d\theta \\
&= \frac{\exp[+ik_0 z_1]}{+i\lambda_0 z_1} \int_{\theta=-\pi}^{\theta=+\pi} \int_{r=0}^{r=\infty} CYL(r) \;\; \exp\left[+i\frac{k_0}{2z_1} r^2\right] \; r \; dr \; d\theta \\
&= 2\pi \frac{\exp[+ik_0 z_1]}{+i\lambda_0 z_1} \int_0^{\frac{1}{2}} r \;\; \exp\left[+i\frac{k_0 r^2}{2z_1}\right] \; dr
\end{aligned}$$

Change variables:

$$\begin{aligned}
u &\equiv +i\frac{\pi r^2}{\lambda_0 z_1} \\
&\Longrightarrow r = 0 \Longrightarrow u = 0 \\
&\Longrightarrow r = \frac{1}{2} \Longrightarrow u = +i\frac{\pi}{4\lambda_o z_1} \\
du &= \frac{2\pi}{\lambda_0 z_1} r \; dr \Longrightarrow r \; dr = \frac{\lambda_0 z_1}{2\pi} \; du
\end{aligned}$$

$$
\begin{aligned}
g[0,0;z_1] &= 2\pi\frac{\exp[+ik_0z_1]}{+i\lambda_0 z_1}\int_{u=0}^{u=+i\frac{\pi}{4\lambda_o z_1}}\exp[u]\;\frac{\lambda_0 z_1}{2\pi}\;du\\
&= \lambda_0 z_1\frac{\exp[+ik_0z_1]}{+i\lambda_0 z_1}\int_{u=0}^{u=+i\frac{\pi}{4\lambda_o z_1}}\exp[u]\;du\\
&= -i\cdot\exp[+ik_0z_1]\left(\exp\left[+i\frac{\pi}{4\lambda_o z_1}\right]-1\right)\\
&= -i\exp[+ik_0z_1]\left(\exp\left[+i\frac{\pi}{4\lambda_o z_1}\right]-1\right)\\
&= -i\exp[+ik_0z_1]\exp\left[+i\frac{\pi}{8\lambda_o z_1}\right]\left(\exp\left[+i\frac{\pi}{8\lambda_o z_1}\right]-\exp\left[-i\frac{\pi}{8\lambda_o z_1}\right]\right)\\
&= -i\exp[+ik_0z_1]\exp\left[+i\frac{\pi}{8\lambda_o z_1}\right]\left(2i\sin\left[\frac{\pi}{8\lambda_o z_1}\right]\right)\\
&= 2\cdot\exp[+ik_0z_1]\exp\left[+i\frac{\pi}{8\lambda_o z_1}\right]\cdot\sin\left[\frac{\pi}{8\lambda_o z_1}\right]
\end{aligned}
$$

The on-axis amplitude distribution is

$$g[0,0;z_1] = 2\cdot\exp[+ik_0z_1]\exp\left[+i\frac{\pi}{8\lambda_o z_1}\right]\cdot\sin\left[\frac{\pi}{8\lambda_o z_1}\right]$$

and the irradiance is proportional to the squared magnitude:

$$I[0,0;z_1,d_0=1]\propto|g[0,0;z_1]|^2 = 4\sin^2\left[\frac{\pi}{8\lambda_0 z_1}\right]$$

Note that the argument of the sine is dimensionally correct since we took the diameter of the circular aperture to be unity. The more general argument for an aperture of diameter d_0 *would be*

$$I[0,0;z_1,d_0]\propto 4\sin^2\left[\frac{\pi d_0^2}{8\lambda_0 z_1}\right]$$

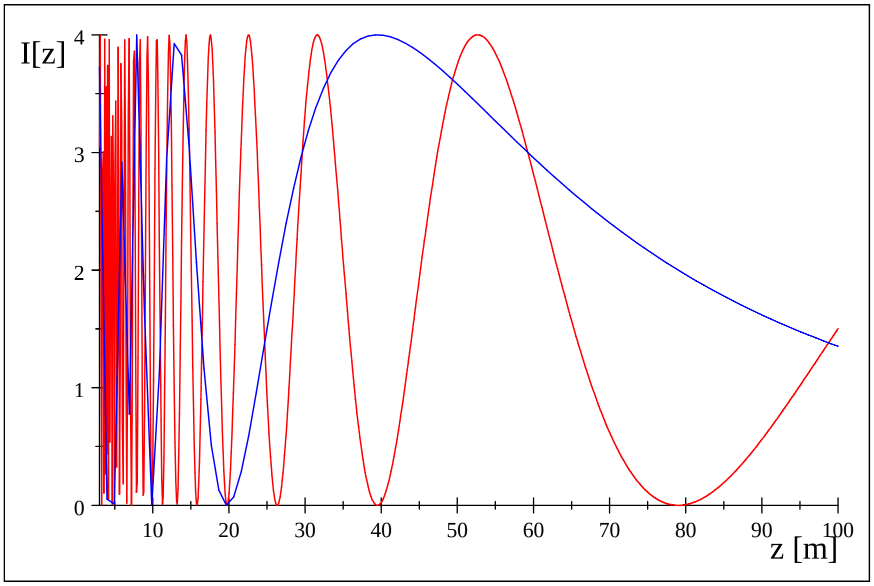
. Also note that the irradiance is proportional to the square of a sinusoid whose argument includes the reciprocal of the variable z_1*. To check the limiting behavior, evaluate the expression in the limit* $z_1\to\infty$*. The argument of the sine wave gets smaller so that we can approximate the sine by its argument:*

$$
\begin{aligned}
\lim_{z_1\to\infty}\{I[0,0;z_1]\} &= 4\cdot\lim_{z_1\to\infty}\left\{\sin^2\left[\frac{\pi}{8\lambda_0 z_1}\right]\right\}\\
&= 4\cdot\left(\frac{\pi}{8\lambda_0 z_1}\right)^2 = \left(\left(\frac{\pi}{4}\right)\cdot\frac{1}{4\lambda_0^2}\right)\cdot\frac{1}{z_1^2}\propto\frac{1}{z_1^2}
\end{aligned}
$$

This is an expression of the inverse square law.

If we assume $\lambda_0 = 632.8\,\text{nm}$ *(He:Ne laser) and recognize that we have to observe in the Fresnel diffraction region, we can graph the function. Consider the limits* $3\,\text{m}\leq z_1\leq 100\,\text{m}$*, which we showed to be in the Fresnel diffraction region if* $d_0\cong 20\,\text{mm}$ *or so.*

$$|I[0,0;z_1]|^2 = 4\sin^2\left[\frac{\pi\cdot(20\,\text{mm})^2}{8\cdot(632.8\cdot 10^{-9}\,\text{m})\cdot z_1}\right]$$



Comparison of irradiance values on axis for two values of the diameter: red $\Longrightarrow d_0 = 20\,\text{mm}$*; blue* $\Longrightarrow d_0 = 10\,\text{mm}$*. Note that the scale is compressed for the smaller diameter because the Fresnel region is closer to the aperture.*

Note that the on-axis irradiance has minima and maxima! And that they are close together near the aperture and farther apart as the distance increases.

2. The Fresnel diffraction pattern from a rectangular aperture of width b_0 propagated over the distance z_1 is $g[x,y]$. Find the width d_0 of the aperture in the same plane as the original aperture that produces a scaled replica of $g[x,y]$ if the propagation distance is $2 \cdot z_1$.

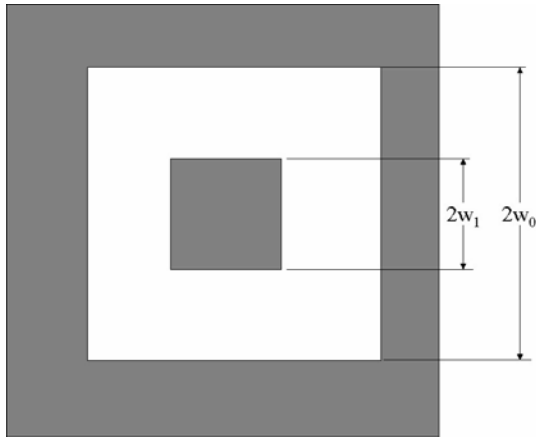
$$\begin{aligned}
g_1[x,y;z_1,b_0] &= RECT\left[\frac{x}{b_0},\frac{y}{b_0}\right] * \frac{1}{i\lambda_0 z_1}\exp\left[+2\pi i\frac{z_1}{\lambda_0}\right]\exp\left[+i\pi\frac{x^2+y^2}{\lambda_0 z_1}\right] \\
G_1[\xi,\eta;z_1,b_0] &= b_0^2 \cdot SINC[b_0\xi, b_0\eta]\cdot\exp\left[+2\pi i\frac{z_1}{\lambda_0}\right]\exp\left[-i\pi\lambda_0 z_1\left(\xi^2+\eta^2\right)\right] \\
&= b_0^2\cdot\exp\left[+2\pi i\frac{z_1}{\lambda_0}\right]\cdot\left(SINC[b_0\xi,b_0\eta]\cdot\exp\left[-i\pi\lambda_0 z_1\left(\xi^2+\eta^2\right)\right]\right) \\
&\propto SINC\left[\frac{\xi}{b_0^{-1}},\frac{\eta}{b_0^{-1}}\right]\cdot\exp\left[-i\pi\frac{\left(\xi^2+\eta^2\right)}{\left(\lambda_0 z_1\right)^{-1}}\right] \\
&= SINC\left[\frac{\xi}{b_0^{-1}},\frac{\eta}{b_0^{-1}}\right]\cdot\exp\left[-i\pi\frac{\left(\xi^2+\eta^2\right)}{\left(\sqrt{\lambda_0 z_1}\right)^{-2}}\right]
\end{aligned}$$

$$\begin{aligned}
g_2[x,y;z_2,d_0] &= RECT\left[\frac{x}{d_0},\frac{y}{d_0}\right] * \frac{1}{i\lambda_0 z_2}\exp\left[+2\pi i\frac{z_2}{\lambda_0}\right]\exp\left[+i\pi\frac{x^2+y^2}{\lambda_0 z_2}\right] \\
&= RECT\left[\frac{x}{d_0},\frac{y}{d_0}\right] * \frac{1}{i\lambda_0 z_2}\exp\left[+2\pi i\frac{z_2}{\lambda_0}\right]\exp\left[+i\pi\frac{x^2+y^2}{\lambda_0\cdot 2\cdot z_1}\right] \\
G_2[\xi,\eta;z_1,b_0] &= d_0^2\cdot\exp\left[+2\pi i\frac{z_2}{\lambda_0}\right]\cdot\left(SINC[d_0\xi,d_0\eta]\cdot\exp\left[-i\pi\lambda_0 z_2\left(\xi^2+\eta^2\right)\right]\right) \\
&= d_0^2\cdot\exp\left[+2\pi i\frac{z_2}{\lambda_0}\right]\cdot\left(SINC[d_0\xi,d_0\eta]\cdot\exp\left[-i\pi\lambda_0\cdot 2\cdot z_1\left(\xi^2+\eta^2\right)\right]\right) \\
&\propto SINC\left[\frac{\xi}{d_0^{-1}},\frac{\eta}{d_0^{-1}}\right]\cdot\exp\left[-i\pi\frac{\left(\xi^2+\eta^2\right)}{\left(\sqrt{\lambda_0\cdot 2\cdot z_1}\right)^{-2}}\right]
\end{aligned}$$

So the ratios of the scale factors of the SINC and chirp functions must be identical:

$$\begin{aligned}
\frac{b_0}{\sqrt{\lambda_0 z_1}} &= \frac{d_0}{\sqrt{\lambda_0 z_2}} = \frac{d_0}{\sqrt{2\lambda_0 z_1}} \\
&\implies d_0 = \sqrt{2}b_0
\end{aligned}$$

3. For the aperture distribution shown, where the aperture is square and has a square central obscuration, and "white" represents transparent regions and "gray" represents opaque, and assuming normally incident monochromatic plane-wave illumination with λ_0:



(a) Find an expression for the "intensity" (i.e., irradiance) distribution for light that has propagated a distance z_1 into the Fraunhofer diffraction region.

The amplitude transmittance is clearly:

$$t[x,y] = RECT\left[\frac{x}{2w_0}, \frac{y}{2w_0}\right] - RECT\left[\frac{x}{2w_1}, \frac{y}{2w_1}\right]$$

Its Fourier transform is:

$$T[\xi,\eta] = (2w_0)^2\, SINC[2w_0\xi, 2w_0\eta] - (2w_1)^2\, SINC[2w_1\xi, 2w_1\eta]$$

At the distance z_1 in the Fraunhofer diffraction region, the amplitude is:

$$\begin{aligned}
g[x,y;z_1] &= \frac{1}{i\lambda_0 z_1}\exp\left[+2\pi i\frac{z_1}{\lambda_0}\right]\cdot T\left[\frac{x}{\lambda_0 z_1}, \frac{y}{\lambda_0 z_1}\right] \equiv K_0 \cdot T\left[\frac{x}{\lambda_0 z_1}, \frac{y}{\lambda_0 z_1}\right] \\
\textit{where } K_0 &\equiv \frac{1}{i\lambda_0 z_1}\exp\left[+2\pi i\frac{z_1}{\lambda_0}\right] \\
g[x,y;z_1] &= K_0\cdot\left((2w_0)^2\, SINC\left[2w_0\frac{x}{\lambda_0 z_1}, 2w_0\frac{y}{\lambda_0 z_1}\right] - (2w_1)^2\, SINC\left[2w_1\frac{x}{\lambda_0 z_1}, 2w_1\frac{y}{\lambda_0 z_1}\right]\right) \\
&= 4K_0\cdot\left(w_0^2 SINC\left[\frac{x}{\left(\frac{\lambda_0 z_1}{2w_0}\right)}, \frac{y}{\left(\frac{\lambda_0 z_1}{2w_0}\right)}\right] - w_1^2 SINC\left[\frac{x}{\left(\frac{\lambda_0 z_1}{2w_1}\right)}, \frac{y}{\left(\frac{\lambda_0 z_1}{2w_1}\right)}\right]\right)
\end{aligned}$$

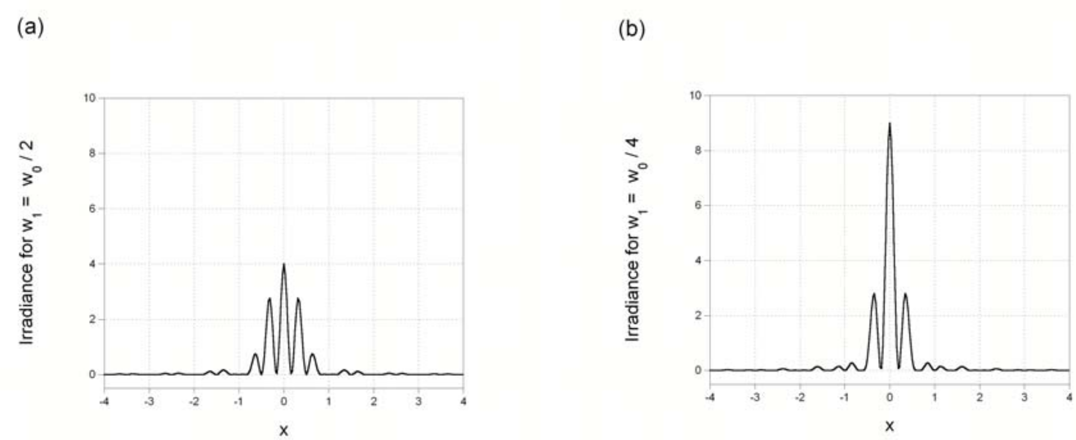
The irradiance distribution is proportional to the squared magnitude:

$$\begin{aligned} I[x,y;z_1] \propto\ & 16\cdot\left(\frac{1}{\lambda_0^2 z_1^2}\right)\cdot w_0^4 SINC^2\left[\frac{x}{\left(\frac{\lambda_0 z_1}{2w_0}\right)},\frac{y}{\left(\frac{\lambda_0 z_1}{2w_0}\right)}\right] \\ & +16\cdot\left(\frac{1}{\lambda_0^2 z_1^2}\right)\cdot w_1^4 SINC^2\left[\frac{x}{\left(\frac{\lambda_0 z_1}{2w_1}\right)},\frac{y}{\left(\frac{\lambda_0 z_1}{2w_1}\right)}\right] \\ & -16\cdot\left(\frac{1}{\lambda_0^2 z_1^2}\right)\cdot 2w_0^2 w_1^2\cdot SINC\left[\frac{x}{\left(\frac{\lambda_0 z_1}{2w_0}\right)},\frac{y}{\left(\frac{\lambda_0 z_1}{2w_0}\right)}\right] SINC\left[\frac{x}{\left(\frac{\lambda_0 z_1}{2w_1}\right)},\frac{y}{\left(\frac{\lambda_0 z_1}{2w_0}\right)}\right] \end{aligned}$$

(b) Sketch the x-profile of this distribution for the propagation distance.

$$\begin{aligned} I[x,0;z_1] \propto\ & 16\cdot\left(\frac{1}{\lambda_0^2 z_1^2}\right)\cdot w_0^4 SINC^2\left[\frac{x}{\left(\frac{\lambda_0 z_1}{2w_0}\right)},\frac{0}{\left(\frac{\lambda_0 z_1}{2w_0}\right)}\right] \\ & +16\cdot\left(\frac{1}{\lambda_0^2 z_1^2}\right)\cdot w_1^4 SINC^2\left[\frac{x}{\left(\frac{\lambda_0 z_1}{2w_1}\right)},\frac{0}{\left(\frac{\lambda_0 z_1}{2w_1}\right)}\right] \\ & -16\cdot\left(\frac{1}{\lambda_0^2 z_1^2}\right)\cdot 2w_0^2 w_1^2\cdot SINC\left[\frac{x}{\left(\frac{\lambda_0 z_1}{2w_0}\right)},\frac{0}{\left(\frac{\lambda_0 z_1}{2w_0}\right)}\right] SINC\left[\frac{x}{\left(\frac{\lambda_0 z_1}{2w_1}\right)},\frac{0}{\left(\frac{\lambda_0 z_1}{2w_0}\right)}\right] \end{aligned}$$

Graph by computer along the x axis:



Irradiance along x-axis for (a) $\frac{w_1}{w_0}=\frac{1}{2}$ and (b) $\frac{w_1}{w_0}=\frac{1}{4}$, so that less light is blocked by the central region. In these cases, the aperture may be modeled as pairs of rectangle functions with different widths and separations, so that the width parameter of the SINC-function envelope is different in the two cases.

4. A diffracting screen has a circularly symmetric amplitude transmittance function given by:

$$t_1(r) = \frac{1}{2}\left(1 + \cos\left[\beta\, r^2\right]\right) \cdot CYL\left(\frac{r}{d_0}\right)$$

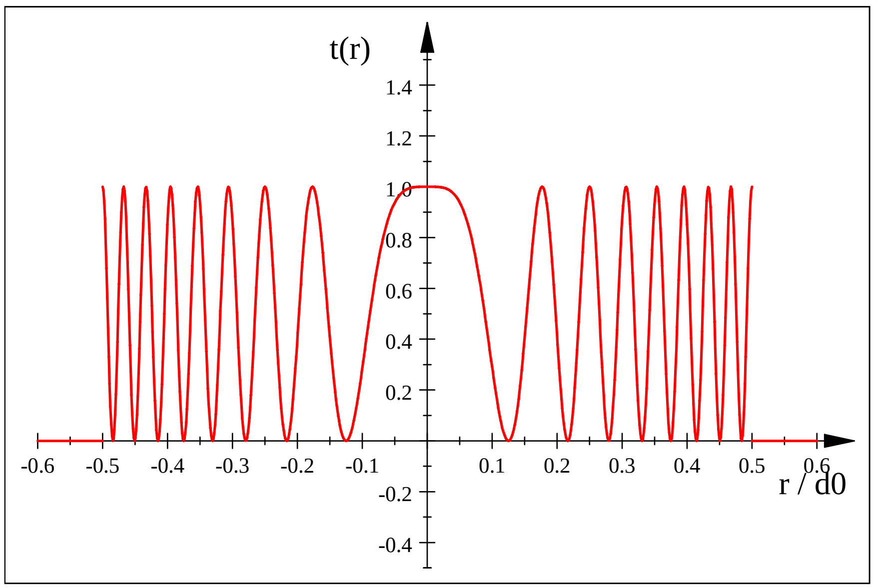
whereβ is a constant and $CYL(r)$ is defined as before:

$$CYL(r) \equiv \begin{cases} 1 & \text{for} \quad r < \frac{1}{2} \\ \frac{1}{2} & \text{for} \quad r = \frac{1}{2} \\ 0 & \text{for} \quad r > \frac{1}{2} \end{cases}$$

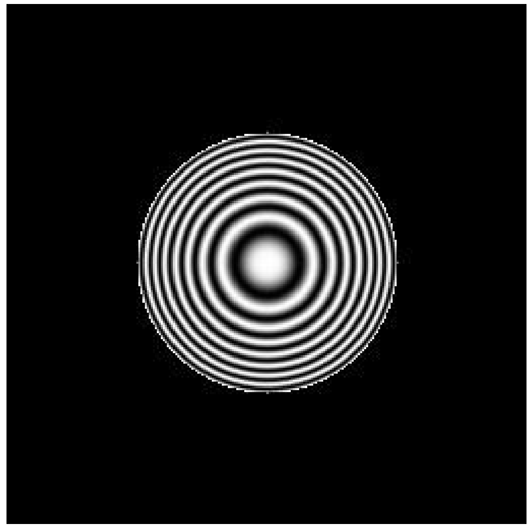
(a) (zzzz) What are the dimensions of β?

Since the argument of the cosine must be dimensionless, β must be a quantity with dimensions of length^{-2}.

(b) Sketch $t_1(r)$ for the case where d_0 is chosen so that the largest phase angle in the cosine is 16π radians.



Transmittance of the diffracting screen as function of normalized radial coordinate $\frac{r}{d_0}$; note that the transmittance goes through 8 maxima by the time the coordinate reaches the edge of the pupil, which corresponds to a phase change of 16π.



2-D image representation: white corresponds to $t = 1$ and black to $t = 0$. Note the "smooth" aspect of the variation.

(c) In what ways does this screen act like a lens? (HINT: Euler relation).

$$\begin{aligned} t_1(r) &= \frac{1}{2}\left(1+\cos\left[\beta\, r^2\right]\right)\cdot CYL\left(\frac{r}{d_0}\right) \\ &= \frac{1}{2}\left(1+\frac{1}{2}\exp\left[+i\pi\beta\, r^2\right]+\frac{1}{2}\exp\left[-i\pi\beta\, r^2\right]\right)\cdot CYL\left(\frac{r}{d_0}\right) \end{aligned}$$

This is the sum of three terms. Now recall the transmittance of a spherical lens with focal length $\mathbf{f} > 0$ using the sag-formula approximation on p.93 of the notes:

$$t(r) = p(r)\cdot\exp\left[+2\pi i\frac{n\tau}{\lambda_0}\right]\cdot\exp\left[-i\pi\frac{r^2}{\lambda_0\mathbf{f}}\right]$$

so we can equate the expressions to determine the focal length in terms of β. In the two terms:

$$\begin{aligned} \exp\left[+i\beta\, r^2\right] &= \exp\left[-i\pi\frac{r^2}{\lambda_0\mathbf{f}}\right] \implies \mathbf{f} = -\pi\frac{\lambda_0}{\beta} \\ \exp\left[-i\beta\, r^2\right] &= \exp\left[-i\pi\frac{r^2}{\lambda_0\mathbf{f}}\right] \implies \mathbf{f} = +\pi\frac{\lambda_0}{\beta} \end{aligned}$$

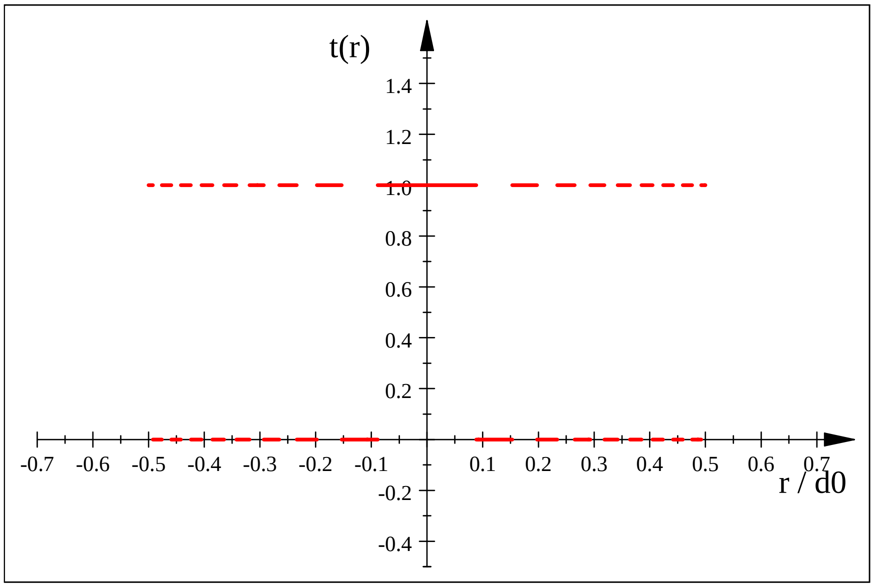
Note that the unit constant term in the sum of three terms may be viewed as:

$$1 = \lim_{\beta\to 0}\exp\left[+i\pi\beta\, r^2\right] \implies \mathbf{f} = \infty$$

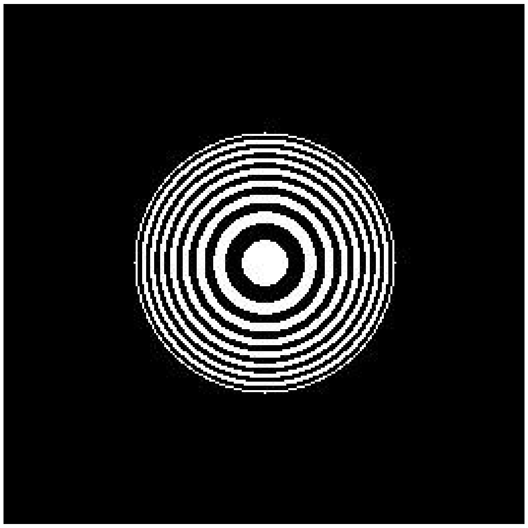
In words, the chirp transmittance function may be viewed as a sum of three lenses: one with infinite focal length (no focusing power), a negative lens and a positive lens with focal lengths $\pm\pi\frac{\lambda_0}{\beta}$

(d) Sketch the transmittance after thresholding at $t_0 = \frac{1}{2}$:

$$t_2(r) = \frac{1}{2}\left(1 + SGN\left[\cos\left[\beta\, r^2\right]\right]\right) \cdot CYL\left(\frac{r}{d_0}\right)$$



Profile of transmittance of diffracting screen as function of normalized radial coordinate $\frac{r}{d_0}$*; note that the transmittance goes through 8 maxima by the time the coordinate reaches the edge of the pupil, which corresponds to a phase change of 16*π*.*



2-D image representation: white corresponds to $t = 1$ *and black to* $t = 0$*. Note the thresholded nature of the variation.*

The thresholding creates sharp transitions in the transmittance at the local changes. We already know that sinusoidal components with large spatial frequencies are necessary to create these transitions. This device is known as a "Fresnel Zone Plate."

(e) Describe the *qualitative* difference in performance in the systems using $t_1(r)$ and $t_2(r)$.

The sharp transitions add additional sinusoidal components to the transmittance, which has the effect of adding quadratic-phase terms with different values of β *and hence of the focal length.*

5. Consider a grating whose transmittance is a sinusoidal function of x:

$$f[x,y] = \left(\frac{1}{2} + \frac{\alpha_0}{2}\cos\left[2\pi\frac{x}{L_0}\right]\right)\cdot 1[y]$$

where $\alpha_0 \leq 1$.

(a) Use the transfer function of light propagation in the Fresnel diffraction region to find the spectrum of the Fresnel diffraction pattern at some propagation distance z_1.

First, find the spectrum of the signal; by inspection:

$$F[\xi,\eta] = \frac{1}{2}\delta[\xi,\eta] + \frac{\alpha_0}{4}\delta\left[\xi + \frac{1}{L_0},\eta\right] + \frac{\alpha_0}{4}\delta\left[\xi - \frac{1}{L_0},\eta\right]$$

The transfer function of propagation (including the time dependence) is:

$$H[\xi,\eta;z_1,t] = \exp\left[2\pi i\left(\frac{z_1}{\lambda_0} - \nu_0 t\right)\right]\cdot\exp\left[-i\pi\lambda_0 z_1\left(\xi^2+\eta^2\right)\right]$$

but we can ignore the time:

$$\Longrightarrow H[\xi,\eta;z_1] = \exp\left[2\pi i\frac{z_1}{\lambda_0}\right]\cdot\exp\left[-i\pi\lambda_0 z_1\left(\xi^2+\eta^2\right)\right]$$

The spectrum after propagating is:

$$\begin{aligned}
G[\xi,\eta;z_1] &= F[\xi,\eta]\cdot H[\xi,\eta;z_1] \\
&= \exp\left[2\pi i\frac{z_1}{\lambda_0}\right]\cdot\left(\frac{1}{2}\delta[\xi,\eta] + \frac{\alpha_0}{4}\delta\left[\xi+\frac{1}{L_0},\eta\right] + \frac{\alpha_0}{4}\delta\left[\xi-\frac{1}{L_0},\eta\right]\right) \\
&\quad\cdot\exp\left[-i\pi\lambda_0 z_1\left(\xi^2+\eta^2\right)\right] \\
&= \exp\left[2\pi i\frac{z_1}{\lambda_0}\right]\cdot\frac{1}{2}\delta[\xi,\eta]\cdot\exp\left[-i\pi\lambda_0 z_1\left(0^2+0^2\right)\right] \\
&\quad+\exp\left[2\pi i\frac{z_1}{\lambda_0}\right]\cdot\frac{\alpha_0}{4}\delta\left[\xi+\frac{1}{L_0},\eta\right]\cdot\exp\left[-i\pi\lambda_0 z_1\left(\left(-\frac{1}{L_0}\right)^2+0^2\right)\right] \\
&\quad+\exp\left[2\pi i\frac{z_1}{\lambda_0}\right]\cdot\frac{\alpha_0}{4}\delta\left[\xi-\frac{1}{L_0},\eta\right]\cdot\exp\left[-i\pi\lambda_0 z_1\left(\left(+\frac{1}{L_0}\right)^2+0^2\right)\right]
\end{aligned}$$

$$G[\xi,\eta;z_1] = \exp\left[2\pi i\frac{z_1}{\lambda_0}\right]\cdot\left(\frac{1}{2}\delta[\xi,\eta] + \frac{\alpha_0}{4}\exp\left[-i\pi\frac{\lambda_0 z_1}{L_0^2}\right]\left(\delta\left[\xi+\frac{1}{L_0},\eta\right]+\delta\left[\xi-\frac{1}{L_0},\eta\right]\right)\right)$$

so the "image amplitude" is:

$$g[x,y;z_1] = \exp\left[2\pi i\frac{z_1}{\lambda_0}\right]\cdot\left(\frac{1}{2}\cdot 1 + \frac{\alpha_0}{2}\exp\left[-i\pi\frac{\lambda_0 z_1}{L_0^2}\right]\cdot\cos\left[2\pi\frac{x}{L_0}\right]\right)$$

and the irradiance is:

$$
\begin{aligned}
I[x,y;z_1] &\propto \left|\frac{1}{2}\cdot 1+\frac{\alpha_0}{2}\exp\left[-i\pi\frac{\lambda_0 z_1}{L_0^2}\right]\cdot\cos\left[2\pi\frac{x}{L_0}\right]\right|^2 \\
&= \left(\frac{1}{2}+\frac{\alpha_0}{2}\exp\left[-i\pi\frac{\lambda_0 z_1}{L_0^2}\right]\cdot\cos\left[2\pi\frac{x}{L_0}\right]\right)\left(\frac{1}{2}+\frac{\alpha_0}{2}\exp\left[+i\pi\frac{\lambda_0 z_1}{L_0^2}\right]\cdot\cos\left[2\pi\frac{x}{L_0}\right]\right) \\
&= \frac{1}{4}+\frac{\alpha_0^2}{4}\cos^2\left[2\pi\frac{x}{L_0}\right]+\frac{2\alpha_0}{4}\cos\left[2\pi\frac{x}{L_0}\right]\cdot\left(\exp\left[-i\pi\frac{\lambda_0 z_1}{L_0^2}\right]+\exp\left[+i\pi\frac{\lambda_0 z_1}{L_0^2}\right]\right) \\
&= \frac{1}{4}+\frac{\alpha_0^2}{4}\cos^2\left[2\pi\frac{x}{L_0}\right]+\frac{2\alpha_0}{4}\cos\left[2\pi\frac{x}{L_0}\right]\cdot\left(2\cos\left[\pi\frac{\lambda_0 z_1}{L_0^2}\right]\right) \\
&= \frac{1}{4}\left(1+\alpha_0^2\cos^2\left[2\pi\frac{x}{L_0}\right]+2\alpha_0\cos\left[2\pi\frac{x}{L_0}\right]\cdot\left(2\cos\left[\pi\frac{\lambda_0 z_1}{L_0^2}\right]\right)\right)
\end{aligned}
$$

(b) Find a relationship for the propagation distances z_1 that produce replicas of the amplitude of the original grating function, even though the system includes no lenses. Your relationship should demonstrate that the same result is obtained for many propagation distances z_1, which means that the periodic grating forms images of itself at various distances in the Fresnel diffraction region. These are called *Talbot images*, after William Henry Fox Talbot, who first explained them. Note that Talbot images are produced for any periodic object, not just for gratings.

The irradiance pattern is a function of z_1:

$$
I[x,y;z_1]\propto 1+\alpha_0^2\cos^2\left[2\pi\frac{x}{L_0}\right]+2\alpha_0\cos\left[2\pi\frac{x}{L_0}\right]\cdot\left(2\cos\left[\pi\frac{\lambda_0 z_1}{L_0^2}\right]\right)
$$

and will give the same result for all distances z_1 *such that:*

$$
\cos\left[\pi\frac{\lambda_0 z_1}{L_0^2}\right] = 1 \Longrightarrow \pi\frac{\lambda_0 z_1}{L_0^2}=2\pi\cdot\ell \text{ where } \ell=1,2,3,\cdots
$$

distance increment between replicas $\Longrightarrow \boxed{\Delta z_1=\frac{2L_0^2}{\lambda_0}}$

so if the period L_0 *is lengthened, then the distance increment between replicas of the pattern are lengthened as* L_0^2. *This happens for any periodic function, not just sinusoids.*