1051-733-20092 Solution Set #1

Read Notes §1 *Introduction,* Skim §2 *Wave Optics and Imaging*, start to read §3 *Optical Diffraction and Imaging*

Warmup Problems

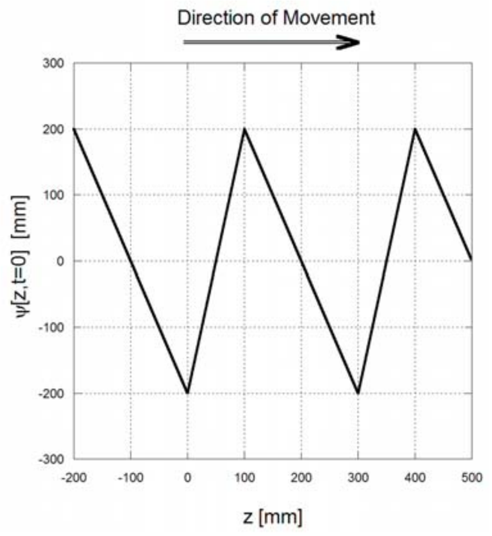
1. The speed of light in vacuum is approximately $3 \times 10^8 \frac{\mathrm{m}}{\mathrm{sec}}$. Find the wavelength of light having a frequency of 1 PHz. Compare this to the wavelength of a 60 Hz electromagnetic wave.

   Solution: (zzzz)

$$\begin{aligned} 1 \textit{ petahertz} &= 1\,\mathrm{PHz} = 10^{15}\,\mathrm{Hz} \\ \lambda &= \frac{c}{\nu} = \frac{3 \times 10^8 \frac{\mathrm{m}}{\mathrm{sec}}}{10^{15}\,\mathrm{Hz}} = \frac{3 \times 10^8 \frac{\mathrm{m}}{\mathrm{sec}}}{10^{15} \frac{\mathrm{cycle}}{\mathrm{sec}}} = 3 \times 10^{-7} \frac{\mathrm{m}}{\mathrm{cycle}} \\ &= 0.3 \times 10^{-6}\,\mathrm{m} = 0.3\,\mu\mathrm{m} = 300\,\mathrm{nm} = 3000\,\text{Å} \end{aligned}$$

$$\textit{for } \nu = 60\,\mathrm{Hz},\ \lambda = \frac{3 \times 10^8 \frac{\mathrm{m}}{\mathrm{sec}}}{60\,\mathrm{Hz}} = 0.5 \times 10^7\,\mathrm{m} = 5 \times 10^6\,\mathrm{m} = 5000\,\mathrm{km}$$

2. The figure shows the profile of a transverse wave on a string traveling in the positive z-direction at a speed of $1\,\frac{\text{m}}{\text{s}}$.



(a) Determine its spatial period.

$$\textit{From drawing}: \boxed{\lambda = 300\,\text{mm}}$$

(note that the wave is NOT harmonic, but is periodic)

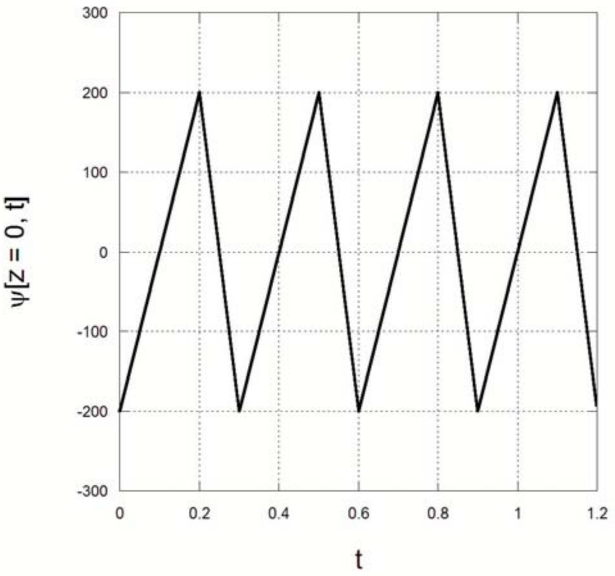
(b) Notice that as the wave passes any fixed point on the z-axis, the string at that location oscillates in time. Draw a graph of $\psi\left[t\right]$ showing how a point on the rope at $z = 0$ oscillates.

$$\text{v} = 1\frac{\text{m}}{\text{s}} \Longrightarrow \frac{1000}{300}\ \textit{cycles in}\ 1\,\text{s} = \frac{10}{3}\frac{\text{cycles}}{\text{s}} \Longrightarrow \nu = \frac{10}{3}\,\text{Hz} \Longrightarrow 1\ \textit{cycle in}\ 0.3\,\text{s}$$

Assume that the graph shows $\psi\left[z, t = 0\right]$ *.Since the wave is moving from left to right, the amplitude at the one location* $z = 0$ *"rises" more slowly from* $\psi = -200\,\text{mm}$ *to* $\psi = +200\,\text{mm}$ *in* $\frac{2}{3}$ *of a cycle and then "falls" more rapidly to* $\psi = -200\,\text{mm}$ in $\frac{1}{3}$ *of a cycle before repeating.*

$$\begin{aligned}
\psi\left[z = 0, t = 0\right] &= -200\,\text{mm} \\
\textit{rises linearly for}\ \frac{2}{3}\ \textit{of a cycle}\ (0.2\,\text{s}),\ \textit{until}\ \psi\left[z = 0, t = 0.2\,\text{s}\right] &= +200\,\text{mm} \\
\textit{decreases linearly for}\ \frac{1}{3}\ \textit{of a cycle}\ (0.1\,\text{s}),\ \textit{until}\ \psi\left[z = 0, t = 0.3\,\text{s}\right] &= -200\,\text{mm} \\
\textit{and then repeats} &:
\end{aligned}$$

Since the temporal period is $0.3\,\text{s}$*, the amplitude rises linearly for* $\frac{2}{3}$*cycle, or* $0.2\,\text{s}$*, and then decreases linearly for* $\frac{1}{3}$*cycle, or* $0.1\,\text{s}$*: the graph of* $\psi\left[z = 0, t\right]$ *has the form:*



(c) Determine the temporal frequency of the wave.

Shown in part (b)

$$\mathrm{v} \;=\; 1\frac{\mathrm{m}}{\mathrm{s}} \Longrightarrow \frac{1000}{300}\ \textit{cycles in}\ 1\,\mathrm{s} = \frac{10}{3}\frac{\textit{cycles}}{\mathrm{s}} \Longrightarrow \boxed{\nu = \frac{10}{3}\,\mathrm{Hz}}$$

3. A harmonic traveling wave is moving in the negative z-direction with an amplitude of 2 arbitrary units, a wavelength of $5\,\mathrm{m}$, and a period of $3\,\mathrm{s}$. Its displacement at $z = 0$ and $t = 0$ is zero. Write a wave equation for this wave under the following conditions:

   (a) explicitly in terms of both wavelength and period

   From the problem statement, we can write down:

$$f\,[z,t] = 2\cdot\cos\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)+\phi_0\right]$$

To evaluate ϕ_0, use the boundary condition that

$$\begin{aligned}
f\,[z=0,t=0] &= 0 = 2\cdot\cos\left[2\pi\left(\frac{0}{5\,\mathrm{m}}+\frac{0}{3\,\mathrm{s}}\right)+\phi_0\right]\\
&\Longrightarrow 2\cdot\cos\left[\phi_0\right]=0 \Longrightarrow \phi_0 = \pm\frac{\pi}{2},\pm\frac{3\pi}{2},\cdots\\
&\Longrightarrow \phi_0 = (2\ell+1)\cdot\frac{\pi}{2}\ \textit{for}\ \ell = 0,\pm1,\pm2,\cdots
\end{aligned}$$

$$\begin{aligned}
f\,[z,t] &= 2\cdot\cos\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)+(2\ell+1)\cdot\frac{\pi}{2}\right]\ \textit{for any integer value of}\ \ell\\
\ell &= -1 \Longrightarrow f\,[z,t] = 2\cdot\cos\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)-\frac{\pi}{2}\right] = +2\sin\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)\right]\\
\ell &= 0 \Longrightarrow f\,[z,t] = 2\cdot\cos\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)+\frac{\pi}{2}\right] = -2\sin\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)\right]\\
\ell &= +1 \Longrightarrow f\,[z,t] = 2\cdot\cos\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)+\frac{3\pi}{2}\right] = +2\sin\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)\right]\\
\ell &= +2 \Longrightarrow f\,[z,t] = 2\cdot\cos\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)+\frac{5\pi}{2}\right] = -2\sin\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)\right]\\
&\textit{so the signal has the form}\ \boxed{f\,[z,t] = \pm2\sin\left[2\pi\left(\frac{z}{5\,\mathrm{m}}+\frac{t}{3\,\mathrm{s}}\right)\right]}
\end{aligned}$$

(b) exhibits both propagation constant and velocity

$$\begin{aligned}
\textit{propagation constant } k_0 &\equiv \frac{2\pi}{\lambda_0} = \frac{2\pi}{5}\frac{\text{radians}}{\text{m}} \\
\textit{angular temporal frequency } \omega_0 &= 2\pi\nu_0 \\
\textit{velocity } \mathrm{v}_0 &= \lambda_0 \cdot \nu_0 = \frac{\omega_0}{k_0} = 5\,\text{m} \cdot \frac{1}{3\,\text{s}} = \frac{5}{3}\frac{\text{m}}{\text{s}}
\end{aligned}$$

$$\begin{aligned}
f[z,t] &= 2\cdot\cos\left[2\pi\left(\frac{z}{\lambda_0}+\nu_0 t\right)+\frac{(2\ell+1)}{2}\cdot\pi\right] \\
&= 2\cdot\cos\left[k_0 z-\omega_0 t+\phi_0\right] \\
&= 2\cdot\cos\left[k_0\left(z-\frac{\omega_0}{k_0}t\right)+\phi_0\right] \\
&= 2\cdot\cos\left[k_0\left(z-\mathrm{v}_0 t\right)+\phi_0\right]
\end{aligned}$$

$$\boxed{f[z,t] = \pm 2\cdot\sin\left[\frac{2\pi}{5}\frac{\text{radians}}{\text{m}}\left(z-\frac{5}{3}\frac{\text{m}}{\text{s}}\cdot t\right)\right]}$$

(c) in complex form:

$$\begin{aligned}
f[z,t] &= +2\cdot\mathrm{Re}\left\{\exp\left[+2\pi i\left(\frac{z}{\lambda_0}+\nu_0 t\right)+\frac{(2\ell+1)}{2}\cdot\pi\right]\right\} \\
&= \pm 2\cdot\mathrm{Im}\left\{\exp\left[+2\pi i\left(\frac{z}{\lambda_0}+\nu_0 t\right)\right]\right\}
\end{aligned}$$

4. A hole with diameter $d_0 = 1\,\text{mm}$ is illuminated by light with $\lambda_0 = 546\,\text{nm}$. Determine which approximation (Fresnel or Fraunhofer) is valid for observation planes located at the distances: $z_1 = 500\,\text{mm}$, $z_2 = 1\,\text{m}$, and $z_3 = 5\,\text{m}$.

The question may be expressed as determining whether the correction term in the series expansion for the phase is sufficiently small. If Fraunhofer diffraction, then all points on the surface of the aperture are in phase; in Fresnel, they need not be in phase. In the notes (§3.3), we showed that the criterion for validity of Fraunhofer diffraction is that:

$$\exp\left[+i\pi\frac{(x_0^2+y_0^2)}{\lambda_0 z}\right] \cong 1 \implies z_{\text{limit}} > \frac{r_0^2}{\lambda_0} = \frac{d_0^2}{4\lambda_0}$$

for the parameters

$$d_0 = 1\,\text{mm}, \lambda_0 = 546\,\text{nm}$$

we see that:

$$\begin{aligned}
z_{\text{limit}} &= \frac{d_0^2}{4\lambda_0} = \frac{(1\,\text{mm})^2}{4\cdot 546\,\text{nm}} \cong 0.46\,\text{m} \textit{ but } z = \ell \cdot z_{\text{limit}} \textit{ where } \ell \textit{ is "significant" (say } \ell \gtrapprox 5 \textit{ or so)} \\
z_1 &= 500\,\text{mm} \cong 1\cdot (z_1)_{\text{limit}} \implies \boxed{\textit{Fresnel diffraction for } z_1} \\
z_2 &= 1\,\text{m} \cong 2\cdot (z_1)_{\text{limit}} \implies \boxed{\textit{Fresnel diffraction for } z_2} \\
z_3 &= 5\,\text{m} \cong 10\cdot (z_1)_{\text{limit}} \implies \boxed{\textit{Fraunhofer diffraction for } z_3}
\end{aligned}$$

5. For the quadratic-phase impulse response for Fresnel diffraction without the constant phase:

$$h\left[x, y; z_1, \lambda_0\right] = \frac{1}{i\lambda_0 z_1} \exp\left[+i\frac{\pi}{\lambda_0 z_1}\left(x^2+y^2\right)\right]$$

(a) Show that the volume is unity.

$$\iint\limits_{-\infty}^{+\infty} h\left[x, y; z_1, \lambda_0\right]\, dx\, dy = \frac{1}{i\lambda_0 z_1}\left(\int_{-\infty}^{+\infty} \exp\left[+i\frac{\pi x^2}{\lambda_0 z_1}\right]\, dx\right)\cdot\left(\int_{-\infty}^{+\infty} \exp\left[+i\frac{\pi y^2}{\lambda_0 z_1}\right]\, dy\right)$$

$$= \frac{1}{i\alpha_0^2}\left(\int_{-\infty}^{+\infty} \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\, dx\right)\cdot\left(\int_{-\infty}^{+\infty} \exp\left[+i\pi\left(\frac{y}{\alpha_0}\right)^2\right]\, dy\right)$$

$$\textit{where } \alpha_0 = \sqrt{\lambda_0 z_1}$$

$$\begin{aligned}\int_{-\infty}^{+\infty} \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\, dx &= \mathcal{F}_1\left\{\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right\}\Bigg|_{\xi=0} \\ &= \left(|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\left(\alpha_0\xi\right)^2\right]\right)\Big|_{\xi=0} \\ &= |\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right] = \sqrt{\lambda_0 z_1}\cdot\exp\left[+i\frac{\pi}{4}\right]\end{aligned}$$

$$\Longrightarrow \int_{-\infty}^{+\infty} \exp\left[+i\pi\left(\frac{y}{\alpha_0}\right)^2\right]\, dy = \sqrt{\lambda_0 z_1}\cdot\exp\left[+i\frac{\pi}{4}\right]$$

$$\begin{aligned}&\frac{1}{i\lambda_0 z_1}\cdot\left(\int_{-\infty}^{+\infty} \exp\left[+i\frac{\pi x^2}{\lambda_0 z_1}\right]\, dx\right)\cdot\left(\int_{-\infty}^{+\infty} \exp\left[+i\frac{\pi y^2}{\lambda_0 z_1}\right]\, dy\right) \\ &\quad= \frac{1}{i\lambda_0 z_1}\cdot\left(\sqrt{\lambda_0 z_1}\cdot\exp\left[+i\frac{\pi}{4}\right]\right)^2 \\ &\quad= \frac{1}{i\lambda_0 z_1}\cdot\left(\sqrt{\lambda_0 z_1}\right)^2\cdot\exp\left[+i\frac{\pi}{2}\right] = \frac{i\lambda_0 z_1}{i\lambda_0 z_1} = 1\end{aligned}$$

(b) Show that the two-dimensional sinusoidal part of this function contributions ALL of the volume and that the cosine part contributes none.

(zzzz) The factor of i in the denominator of h means that we can rewrite h in the form:

$$\begin{aligned} h[x,y;z_1,\lambda_0] &= -i\cdot\frac{1}{\lambda_0 z_1}\exp\left[+i\frac{\pi}{\lambda_0 z_1}\left(x^2+y^2\right)\right] \\ &= \frac{1}{\lambda_0 z_1}\cdot -i\cdot\left(\cos\left[\frac{\pi}{\lambda_0 z_1}\left(x^2+y^2\right)\right]+i\cdot\sin\left[\frac{\pi}{\lambda_0 z_1}\left(x^2+y^2\right)\right]\right) \\ &= \frac{1}{\lambda_0 z_1}\cdot\left(\sin\left[\frac{\pi}{\lambda_0 z_1}\left(x^2+y^2\right)\right]-i\cdot\cos\left[\frac{\pi}{\lambda_0 z_1}\left(x^2+y^2\right)\right]\right) \end{aligned}$$

$$\Longrightarrow \quad 1+0\cdot i = \iint\limits_{-\infty}^{+\infty} h[x,y;z_1,\lambda_0]\ dx\ dy$$

$$= \quad \frac{1}{\lambda_0 z_1}\cdot\left(\iint\limits_{-\infty}^{+\infty}\sin\left[\frac{\pi}{\lambda_0 z_1}\left(x^2+y^2\right)\right]\ dx\ dy - i\cdot\iint\limits_{-\infty}^{+\infty}\cos\left[\frac{\pi}{\lambda_0 z_1}\left(x^2+y^2\right)\right]\ dx\ dy\right)$$

$$\Longrightarrow \quad \frac{1}{\lambda_0 z_1}\cdot\iint\limits_{-\infty}^{+\infty}\sin\left[\frac{\pi}{\lambda_0 z_1}\left(x^2+y^2\right)\right]\ dx\ dy = 1$$

$$\Longrightarrow \quad \frac{1}{\lambda_0 z_1}\cdot\iint\limits_{-\infty}^{+\infty}\cos\left[\frac{\pi}{\lambda_0 z_1}\left(x^2+y^2\right)\right]\ dx\ dy = 0$$

6. Consider propagation over the distance z_1 and then over the distance z_2, where both distances satisfy the conditions for Fresnel diffraction. Show that a single propagation over the distance $z_1 + z_2$ gives the same result as the propagations over z_1 and then over z_2.

   Done in the notes for 716. The output field at the plane specified by $z = z_1 + z_2$ in the Fresnel region due to a point source at the origin may be derived in one step by inserting this distance into the impulse response:

$$\begin{aligned}\delta[x,y] * h[x,y;z_1+z_2] &= h[x,y;z_1+z_2] \\ &= \frac{1}{i\lambda_0(z_1+z_2)} \exp\left[+2\pi i\frac{z_1+z_2}{\lambda_0}\right] \exp\left[+i\pi\frac{x^2+y^2}{\lambda_0(z_1+z_2)}\right]\end{aligned}$$

   The question is whether is expression is equivalent to the result of two Fresnel propagations in sequence:

$$\begin{aligned}\delta[x,y] * h[x,y;z_1+z_2] &= (\delta[x,y] * h[x,y;z_1]) * h[x,y;z_2] \\ &= h[x,y;z_1] * h[x,y;z_2]\end{aligned}$$

$$\begin{aligned}\Longrightarrow \left(\frac{1}{i\lambda_0 z_1} e^{+2\pi i\frac{z_1}{\lambda_0}} e^{+i\pi\left(\frac{x^2+y^2}{\lambda_0 z_1}\right)}\right) &* \left(\frac{1}{i\lambda_0 z_2} e^{+2\pi i\frac{z_2}{\lambda_0}} e^{+i\pi\left(\frac{x^2+y^2}{\lambda_0 z_2}\right)}\right) \\ &= \left(\frac{1}{i\lambda_0}\right)^2 \left(\frac{1}{z_1 z_2}\right) e^{+2\pi i\frac{z_1+z_2}{\lambda_0}} \left(e^{+i\pi\frac{x^2}{\lambda_0 z_1}} * e^{+i\pi\frac{x^2}{\lambda_0 z_2}}\right) \cdot \left(e^{+i\pi\frac{y^2}{\lambda_0 z_1}} * e^{+i\pi\frac{y^2}{\lambda_0 z_2}}\right)\end{aligned}$$

$$\mathcal{F}_1\left\{\exp\left[+i\pi\frac{x^2}{\left(\sqrt{\lambda_0 z_1}\right)^2}\right]\right\} = \left(\sqrt{\lambda_0 z_1}\right) e^{+i\frac{\pi}{4}} \exp\left[-i\pi\left(\sqrt{\lambda_0 z_1}\right)^2 \xi^2\right]$$

$$\mathcal{F}_1\left\{\exp\left[+i\pi\frac{x^2}{\left(\sqrt{\lambda_0 z_2}\right)^2}\right]\right\} = \left(\sqrt{\lambda_0 z_2}\right) e^{+i\frac{\pi}{4}} \exp\left[-i\pi\left(\sqrt{\lambda_0 z_2}\right)^2 \xi^2\right]$$

   The spectrum of the 1-D convolution over x is easy to derive:

$$\begin{aligned}&\left(\sqrt{\lambda_0 z_1}\right)\left(\sqrt{\lambda_0 z_2}\right)\left(e^{+i\frac{\pi}{4}}\right)^2 \left(\exp\left[-i\pi\lambda_0 z_1\xi^2\right] \cdot \exp\left[-i\pi\lambda_0 z_2\xi^2\right]\right) \\ &\qquad = \left(\lambda_0\sqrt{z_1 z_2}\right) e^{+i\frac{\pi}{2}} \exp\left[-i\pi\left[\lambda_0(z_1+z_2)\right]\xi^2\right]\end{aligned}$$

   which is a scaled "downchirp" function. The 1-D inverse Fourier transform yields the 1-D convolution over x:

$$\begin{aligned}\exp\left[+i\pi\frac{x^2}{\lambda_0 z_1}\right] * \exp\left[+i\pi\frac{x^2}{\lambda_0 z_2}\right] &= \left(\frac{\lambda_0\sqrt{z_1 z_2}}{\sqrt{\lambda_0(z_1+z_2)}}\right) e^{+i\frac{\pi}{4}} \exp\left[+i\pi\frac{x^2}{\lambda_0(z_1+z_2)}\right] \\ &= \left(\sqrt{\lambda_0}\sqrt{\frac{z_1 z_2}{z_1+z_2}}\right) e^{+i\frac{\pi}{4}} \exp\left[+i\pi\left(\frac{x^2}{\lambda_0(z_1+z_2)}\right)\right]\end{aligned}$$

The product of this result with the leading multiplicative factors and the analogous convolution over y yields the desired result:

$$\begin{aligned}
&(\delta[x,y] * h[x,y;z_1]) * h[x,y;z_2] \\
&\quad = \left(\frac{1}{i\lambda_0}\right)^2 \left(\frac{1}{z_1 z_2}\right)\; e^{+\frac{2\pi i}{z_1+z_2}}\; \left(\sqrt{\lambda_0}\sqrt{\frac{z_1 z_2}{z_1+z_2}} e^{+i\frac{\pi}{4}}\right)^2\; \exp\left[+i\pi\frac{x^2+y^2}{\lambda_0(z_1+z_2)}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \frac{1}{i\lambda_0(z_1+z_2)}\; e^{+\frac{2\pi i}{z_1+z_2}}\; \exp\left[+i\pi\frac{x^2+y^2}{\lambda_0(z_1+z_2)}\right]
\end{aligned}$$

7. Consider a spherical wave expanding about the point $[0, 0, -z_1]$ in a Cartesian coordinate system. The wavelength of the light is λ_0 and $z_1 > 0$.

   (a) Express the phase distribution of the spherical wave across the $[x, y]$ plane located normal to the z-axis at coordinate $z = 0$.

   The exact phase distribution is:

$$\Phi_{exact}\left[x, y; z = 0\right] = \frac{2\pi z_0}{\lambda}\sqrt{1 + \frac{x^2 + y^2}{z_0}}$$

   (b) Use the paraxial approximation to find an expression for the phase distribution of the parabolic wavefront (quadratic-phase factor) that approximates this spherical wavefront.

   The approximate phase distribution is obtained from the binomial theorem:

$$\begin{aligned}
\Phi_{exact}\left[x, y; z = 0\right] &= \frac{2\pi z_0}{\lambda}\left(1 + \frac{x^2 + y^2}{z_0}\right)^{\frac{1}{2}} \\
\Phi_{exact}\left[x, y; z = 0, \lambda_0\right] &= \frac{2\pi z_0}{\lambda_0}\left(1 + \frac{1}{2}\frac{x^2 + y^2}{z_0} - \frac{1}{8}\left(\frac{x^2 + y^2}{z_0}\right)^2 + \cdots\right) \\
\Phi_{approximate}\left[x, y; z = 0, \lambda_0\right] &= \frac{2\pi z_0}{\lambda_0} + \pi\frac{(x^2 + y^2)}{\lambda z_0}
\end{aligned}$$