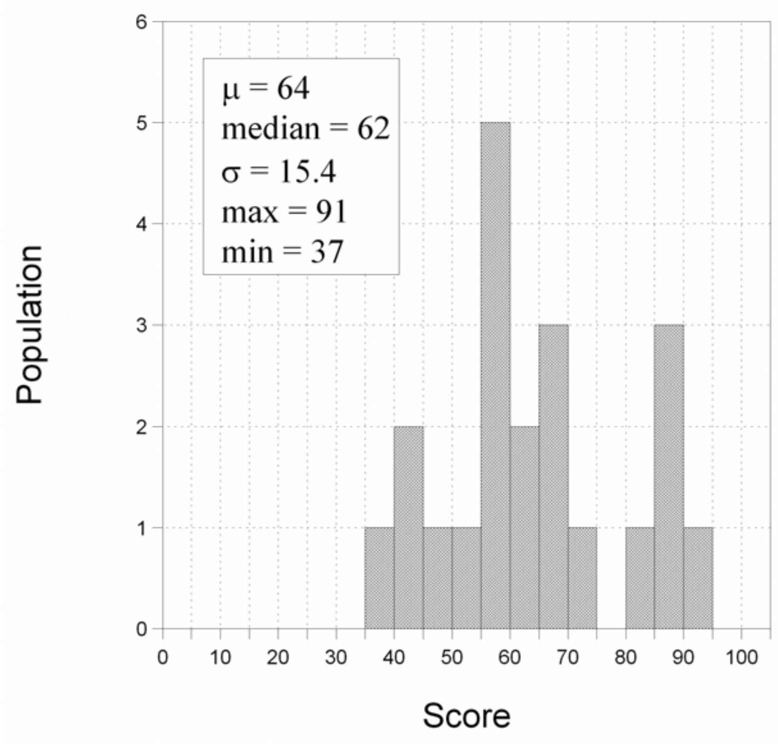
# SIMG-733 Optics-20092 Solutions to Midterm Exam

## Statistics and Comments:

Many people are still not sketching before writing down equations. Some wrote down equations with no explanation, e.g., the depth of field equation, which is the "what" rather than the (more important) "why." I advise that it typically does little good to "memorize" the equations – you should get a feeling about the graphical situation, which often gives you the equation. The graphical solution likely will stick with you much longer than the memorized equation(s).

## Amended Histogram (after updated grades)



## Average per problem (each out of possible 20):

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\mu$ | 15.11 | 9.79 | 12.42 | 12.33 | 12.44 | 14.19 | 13.00 |
| Max | 20 | 15 | 20 | 18 | 20 | 20 | 20 |
| Min | 8 | 6 | 6 | 5 | 5 | 5 | 8 |

This indicates that the greatest difficulty was with problem #2 (on depth of field), where, again, few people drew a good sketch of the situation. Problems 3 (complementary aperture),4 (aperture function to generate derivative), and 5 (diffraction grating) gave the next most difficulty. Note that the maximum score of 20 was attained by someone on all problems except #2 and #4.

1. In the homework, you demonstrated that the cascade of propagation of light from a 2-D function $f[x,y]$ over the distance $z_1$ in the Fresnel region, followed by second propagation over the distance $z_2$ in the Fresnel region yields the same result as a propagation of light from $f[x,y]$over the distance $z_1+z_2$. Consider the 1-D case, i.e., consider propagation of light from the 1-D function $f[x]$ over the distances $z_1$ and $z_2$. Is this propagation equivalent to that over the distance $z_1+z_2$? Explain.

   *The "doubly diffracted" amplitude pattern in 2-D may be evaluated via either a convolution over the distance $z_1+z_2$ or over the same distance in two steps. If the first, the impulse response is:*

$$h[x,y;z_1+z_2] = \frac{1}{i\lambda_0(z_1+z_2)} \exp\left[+2\pi i \frac{z_1+z_2}{\lambda_0}\right] \exp\left[+i\pi \frac{x^2+y^2}{\lambda_0(z_1+z_2)}\right]$$

   *If the second, the impulse response is the convolution:*

$$\left(\frac{1}{i\lambda_0 z_1} e^{+2\pi i \frac{z_1}{\lambda_0}} e^{+i\pi\left(\frac{x^2+y^2}{\lambda_0 z_1}\right)}\right) * \left(\frac{1}{i\lambda_0 z_2} e^{+2\pi i \frac{z_2}{\lambda_0}} e^{+i\pi\left(\frac{x^2+y^2}{\lambda_0 z_2}\right)}\right)$$

$$= \left(\frac{1}{i\lambda_0}\right)^2 \left(\frac{1}{z_1 z_2}\right) e^{+2\pi i \frac{z_1+z_2}{\lambda_0}} \left(e^{+i\pi \frac{x^2}{\lambda_0 z_1}} * e^{+i\pi \frac{x^2}{\lambda_0 z_2}}\right) \cdot \left(e^{+i\pi \frac{y^2}{\lambda_0 z_1}} * e^{+i\pi \frac{y^2}{\lambda_0 z_2}}\right)$$

   *You showed that these were equal. But if the problem is recast in 1-D where the impulse response is the 1-D analogue:*

$$\begin{aligned} h[x;z_1] &= \frac{1}{i\lambda_0 z_1} \exp\left[+2\pi i \frac{z_1}{\lambda_0}\right] \exp\left[+i\pi\left(\frac{x^2}{\lambda_0 z_1}\right)\right] \\ h[x;z_2] &= \frac{1}{i\lambda_0 z_2} \exp\left[+2\pi i \frac{z_2}{\lambda_0}\right] \exp\left[+i\pi\left(\frac{x^2}{\lambda_0 z_2}\right)\right] \end{aligned}$$

   *so that we must check to see if:*

$$\begin{aligned} h[x;z_1] * h[x;z_2] &= h[x,y;z_1+z_2] \\ &= \frac{1}{i\lambda_0(z_1+z_2)} \exp\left[+2\pi i \frac{(z_1+z_2)}{\lambda_0}\right] \exp\left[+i\pi\left(\frac{x^2}{\lambda_0(z_1+z_2)}\right)\right] \end{aligned}$$

$$\frac{1}{i\lambda_0 z_1} \exp\left[+2\pi i \frac{z_1}{\lambda_0}\right] \exp\left[+i\pi\left(\frac{x^2}{\lambda_0 z_1}\right)\right] * \frac{1}{i\lambda_0 z_2} \exp\left[+2\pi i \frac{z_2}{\lambda_0}\right] \exp\left[+i\pi\left(\frac{x^2}{\lambda_0 z_2}\right)\right]$$

$$= \frac{1}{i\lambda_0 z_1} \cdot \frac{1}{i\lambda_0 z_2} \exp\left[+2\pi i \frac{(z_1+z_1)}{\lambda_0}\right] \left(\exp\left[+i\pi\left(\frac{x^2}{\lambda_0 z_1}\right)\right] * \exp\left[+i\pi\left(\frac{x^2}{\lambda_0 z_2}\right)\right]\right)$$

$$\begin{aligned} \mathcal{F}_1\left\{\exp\left[+i\pi\left(\frac{x^2}{\lambda_0 z_1}\right)\right]\right\} &= \sqrt{\lambda_0 z_1} \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi\lambda_0 z_1 \xi^2\right] \\ \mathcal{F}_1\left\{\exp\left[+i\pi\left(\frac{x^2}{\lambda_0 z_2}\right)\right]\right\} &= \sqrt{\lambda_0 z_2} \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi\lambda_0 z_2 \xi^2\right] \end{aligned}$$

$$\exp\left[+i\pi\left(\frac{x^2}{\lambda_0 z_1}\right)\right] * \exp\left[+i\pi\left(\frac{x^2}{\lambda_0 z_2}\right)\right]$$
$$= \mathcal{F}_1^{-1}\left\{\sqrt{\lambda_0 z_1}\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\lambda_0 z_1\xi^2\right]\cdot\sqrt{\lambda_0 z_2}\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\lambda_0 z_2\xi^2\right]\right\}$$

$$\begin{aligned}
&= \mathcal{F}_1^{-1}\left\{\lambda_0\sqrt{z_1 z_2}\exp\left[+i\frac{\pi}{2}\right]\exp\left[-i\pi\lambda_0\left(z_1+z_2\right)\xi^2\right]\right\} \\
&= \left(i\lambda_0\sqrt{z_1 z_2}\right)\mathcal{F}_1^{-1}\left\{\exp\left[-i\pi\lambda_0\left(z_1+z_2\right)\xi^2\right]\right\} \\
&= \left(i\lambda_0\sqrt{z_1 z_2}\right)\frac{1}{\sqrt{\lambda_0\left(z_1+z_2\right)}}\exp\left[-i\frac{\pi}{4}\right]\exp\left[+i\pi\frac{x^2}{\lambda_0\left(z_1+z_2\right)}\right] \\
&= \sqrt{\frac{i\lambda_0 z_1 z_2}{z_1+z_2}}\exp\left[+i\pi\frac{x^2}{\lambda_0\left(z_1+z_2\right)}\right] \neq \frac{1}{i\lambda_0\left(z_1+z_2\right)}\exp\left[+i\pi\left(\frac{x^2}{\lambda_0\left(z_1+z_2\right)}\right)\right]
\end{aligned}$$

*The Fourier transform in the second dimension provides the necessary scale factors to obtain the correct result.*

2. Images of objects at different distances from the optic are created by a lens with a certain focal length $\mathbf{f}_1 = 50\,\text{mm}$ and aperture diameter $d_1 = 35\,\text{mm}$. The lens is used in monochromatic light with wavelength $\lambda_0 = 0.5\,\mu\text{m}$. The lens is used to image an object with "depth", i.e., its distance from the lens ranges from $(z_1)_{\text{far}}$ to $(z_1)_{\text{near}}$. Alternatively, you could consider several objects in the field of view at distances in this interval.

   (a) Explain the concept of "depth of field" for this lens, use diagrams.

   *The f/number of this lens is:*

$$\text{f}/\# = \frac{\mathbf{f}_1}{d_1} = \frac{50\,\text{mm}}{35\,\text{mm}} = \frac{10}{7} \cong 1.4$$

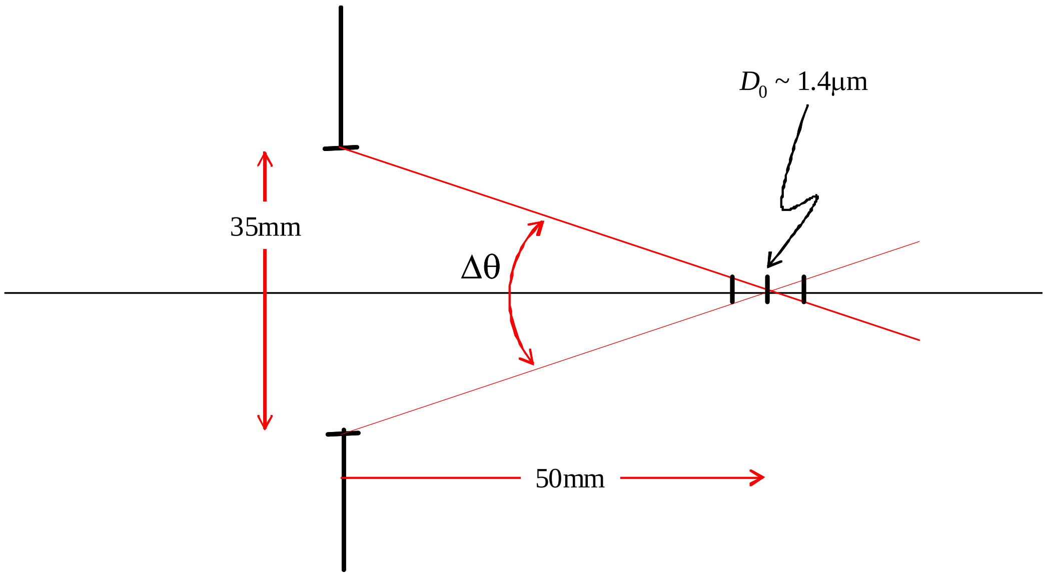
   *(the prescription was modeled on the Nikon 50mm f/1.4 "normal" lens for 35mm cameras*

   *If the lens is used "wide open", then the cone of light exiting the lens towards the image plane spans a full angle of:*

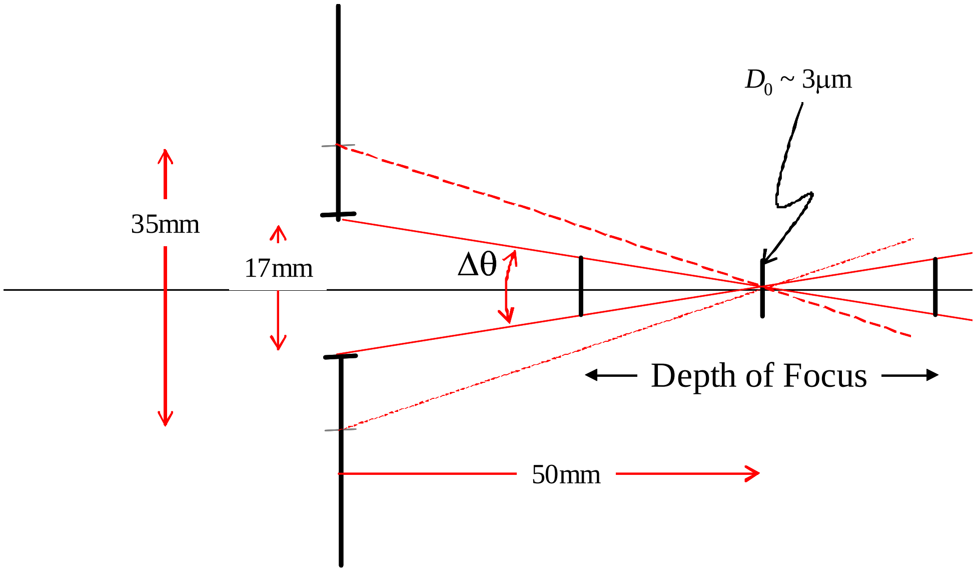
$$\theta = \tan^{-1}\left[\frac{35\,\text{mm}}{50\,\text{mm}}\right] = \tan^{-1}\left[\frac{5}{7}\right]$$

   *By using our "rule of thumb," we estimate that the diameter of the Airy disk pattern in visible light is approximately the f/# measured in micrometers:*

$$D_1 \cong 1.4\,\mu\text{m}$$



(b) Explain the effect of "stopping down" the lens (reducing the diameter of the aperture while maintaining the image distance).



*Stopping down the lens both increases the diameter of the Airy disk and decreases the cone angle of the light from the lens to the image. In this way, the depth of focus is increased by two factors of the f/#.*

(c) Describe the effect of stopping down the lens on the "quality" of images obtained of an object at the distance $z_1$ that exactly satisfies the imaging equation for a fixed focal length **f** and fixed image distance $z_2$; in other words, what are the relative "qualities" of the images before and after stopping down the lens.

*Since the lens is stopped down to create a smaller aperture diameter, the diameter of the Airy disk is increased, so the entire image is "blurrier", but the range of distances that appear equally "blurrier" is now larger*

3. Determine the forms of the Fraunhofer diffraction patterns for an aperture function consisting of transparent circular holes is similar to that for the "complementary aperture," i.e., replace the transparent holes with opaque circular spots and opaque background with a transparent background.

*I'll just do the general case; we need to evaluate the patterns resulting from two two aperture functions* $f_1[x,y]$ *where* $0 \leq f_1 \leq 1$ *and* $f_2[x,y] = 1 - f_1[x,y]$. *The Fraunhofer diffraction patterns are proportional to the Fourier transforms of the aperture functions:*

$$\begin{aligned}
\mathcal{F}_2\{f_1[x,y]\}|_{\xi\to\frac{x}{\lambda_0 z_2},\eta\to\frac{y}{\lambda_0 z_2}} &= F\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] \\
\mathcal{F}_2\{f_2[x,y]\}|_{\xi\to\frac{x}{\lambda_0 z_2},\eta\to\frac{y}{\lambda_0 z_2}} &= \mathcal{F}_2\{1[x,y] - f[x,y]\}|_{\xi\to\frac{x}{\lambda_0 z_2},\eta\to\frac{y}{\lambda_0 z_2}} \\
&= \delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] - F\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]
\end{aligned}$$

*The observed irradiance in each case is proportional to the squared magnitude:*

$$\begin{aligned}
I_1[x,y] &\propto \left|F_1\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right|^2 \\
I_2[x,y] &\propto \left|F_2\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right|^2 = \left|\delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] - F\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right|^2 \\
&= \delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] - \delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] \\
&\quad \cdot\left(F\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] + F^*\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right) + \left|F\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right|^2 \\
&= \delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] - \delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] \\
&\quad \cdot\left(F\left[\frac{0}{\lambda_0 z_2},\frac{0}{\lambda_0 z_2}\right] + F^*\left[\frac{0}{\lambda_0 z_2},\frac{0}{\lambda_0 z_2}\right]\right) + \left|F\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right|^2 \\
&= \left(\delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\cdot(1 - 2\cdot\mathrm{Re}\{F[0,0]\})\right) + \left|F\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right|^2
\end{aligned}$$

*So the two irradiance patterns are identical except at the origin of coordinates!*

*In the specific problem, you can write the aperture function as, say:*

$$\begin{aligned}
f_1[x,y] &= \left(\frac{1}{\Delta x}\cdot\frac{1}{\Delta y}\cdot COMB\left[\frac{x}{\Delta x},\frac{y}{\Delta y}\right]\right) * CYL\left(\frac{r}{d_0}\right) \\
F_1\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] &= COMB\left[\frac{x}{\left(\frac{\lambda_0 z_2}{\Delta x}\right)},\frac{y}{\left(\frac{\lambda_0 z_2}{\Delta y}\right)}\right]\cdot\frac{\pi d_0^2}{4} SOMB\left(\frac{\sqrt{x^2+y^2}}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right) \\
\left|F_1\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right|^2 &= \left|COMB\left[\frac{x}{\left(\frac{\lambda_0 z_2}{\Delta x}\right)},\frac{y}{\left(\frac{\lambda_0 z_2}{\Delta y}\right)}\right]\cdot\frac{\pi d_0^2}{4} SOMB\left(\frac{\sqrt{x^2+y^2}}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right)\right|^2
\end{aligned}$$

$$\begin{aligned}
f_2\left[x,y\right] &= 1\left[x,y\right]-\left(\frac{1}{\Delta x}\cdot\frac{1}{\Delta y}\cdot COMB\left[\frac{x}{\Delta x},\frac{y}{\Delta y}\right]\right)*CYL\left(\frac{r}{d_0}\right)\\
F_2\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] &= \delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]-COMB\left[\frac{x}{\left(\frac{\lambda_0 z_2}{\Delta x}\right)},\frac{y}{\left(\frac{\lambda_0 z_2}{\Delta y}\right)}\right]\cdot\frac{\pi d_0^2}{4}SOMB\left(\frac{\sqrt{x^2+y^2}}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right)\\
\left|F_2\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right|^2 &= \left|\delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]-COMB\left[\frac{x}{\left(\frac{\lambda_0 z_2}{\Delta x}\right)},\frac{y}{\left(\frac{\lambda_0 z_2}{\Delta y}\right)}\right]\cdot\frac{\pi d_0^2}{4}SOMB\left(\frac{\sqrt{x^2+y^2}}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right)\right|^2\\
&= \delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]+\left|COMB\left[\frac{x}{\left(\frac{\lambda_0 z_2}{\Delta x}\right)},\frac{y}{\left(\frac{\lambda_0 z_2}{\Delta y}\right)}\right]\cdot\frac{\pi d_0^2}{4}SOMB\left(\frac{\sqrt{x^2+y^2}}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right)\right|^2\\
&\quad -2\cdot\left(\frac{\pi d_0^2}{4}\right)^2 SOMB^2\left(\frac{0}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right)\cdot\delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\\
&= \left(1-2\cdot\left(\frac{\pi d_0^2}{4}\right)^2\right)\cdot\delta\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\\
&\quad +\left|COMB\left[\frac{x}{\left(\frac{\lambda_0 z_2}{\Delta x}\right)},\frac{y}{\left(\frac{\lambda_0 z_2}{\Delta y}\right)}\right]\cdot\frac{\pi d_0^2}{4}SOMB\left(\frac{\sqrt{x^2+y^2}}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right)\right|^2
\end{aligned}$$

4. Construct an aperture function $p[x,y]$ that, if acting on coherent (monochromatic) light will produce an output amplitude $g[x,y]$ of the form:

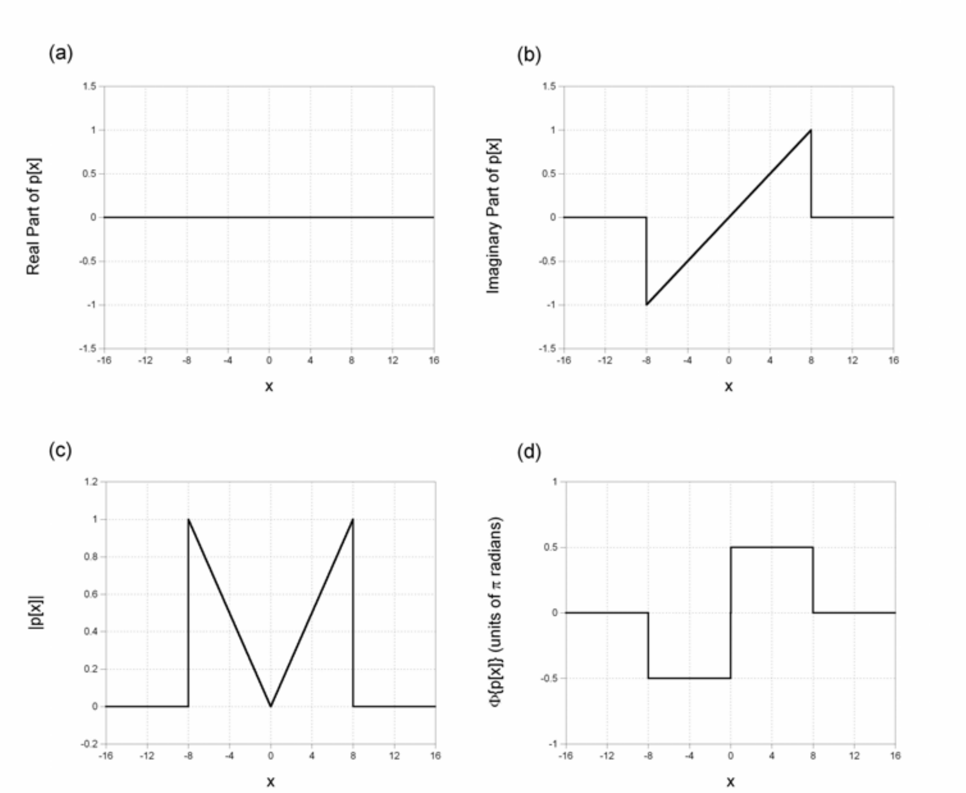
$$g[x,y] \cong \frac{\partial}{\partial x} f[x,y]$$

Describe how a practical such aperture may be constructed from glass, etc., and also describe any limitations in the success of the result.

*We know that the aperture function must be "proportional to" the transfer function of the desired impulse response in coherent light. Since the impulse response is a derivative, the transfer function must be:*

$$\begin{aligned} H[\xi,\eta] &= \mathcal{F}_2[\delta'[x]\cdot\delta[y]] = +2\pi i\xi\cdot 1[\eta] \\ &\Longrightarrow p[x,y] = +2\pi i\frac{x}{\lambda_0\mathbf{f}}\cdot 1\left[\frac{y}{\lambda_0\mathbf{f}}\right] + 2\pi i\frac{x}{\lambda_0\mathbf{f}}\cdot 1[y] \end{aligned}$$

*So we need the pupil function to induce an amplitude function that is proportional to* $+2\pi i x = 2\pi x\cdot\exp\left[+i\frac{\pi}{2}\right]$ *on the incoming light. It is helpful to plot the function to be generated, which I'll do as a 1-D function of x:*



*x-axis profile of aperture function to evaluate the 1-D derivative; the function induces an increase in the magnitude proportional to the distance from the origin and a change in phase of π radians in the domain from negative to positive coordinates. Note that the magnitude cannot exceed unity (maximum transmittance) and the aperture width must be finite.*

*So we must vary the magnitude from 0 at the origin to unity at the edge of the aperture* *and have a phse change of $\pi$ radians as you pass from negative to positive coordinates. We create the variation in magnitude by "apodizing" the aperture (varying its transmittance) by using a photographic transparency.*

*Note that the maximum magnitude is limited to unity (since this is a passive system that cannot inject power) and that the aperture size must be constrained to satisfy the Fraunhofer condition.*

*The phase change of $\pi$ radians requires an additional phase delay on one side of the origin, which may be implemented via a thin piece of glass (which has practical issues). Perhaps contrary to your intuition, this* ***is*** *a practical device.*

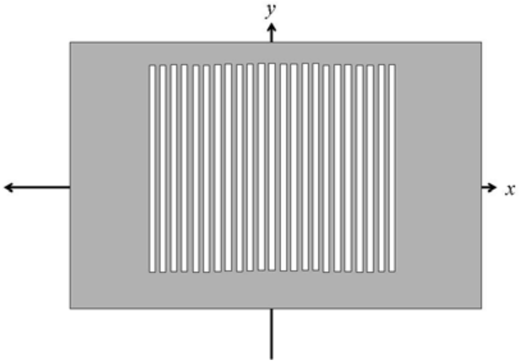
5. Light with wavelength $\lambda_0$ propagates from a point source located on axis at a very large distance "to the left" of the plane $z = 0$, i.e., at some coordinate $z \ll 0$. The wavefronts from the source observed at the origin are appropriatedly modeled as plane waves; assume that the amplitude of the sinusoidal oscillations is 1. The "aperture" at the plane $z = 0$ is a square with side dimension of 25 mm. Within the square, the aperture function consists of alternating transmitting and opaque regions with equal widths of $10\,\mu$m. The light transmitted through the aperture then traverses a large distance $z_2$ to the observation plane in the Fraunhofer region.

   (a) Write down an equation for and sketch $t[x, y]$; label relevant quantities.

$$
\begin{aligned}
t[x,y] &= RECT[x-2n] = RECT[x] * \delta[x-2n] \\
\sum_{n=-\infty}^{+\infty} RECT[x-2n] &= \sum_{n=-\infty}^{+\infty} \left(RECT[x] * \delta[x-2n]\right) \\
&= RECT[x] * \sum_{n=-\infty}^{+\infty} \delta[x-2n] \\
&= RECT[x] * \sum_{n=-\infty}^{+\infty} \delta\left[2\left(\frac{x}{2}-n\right)\right] \\
&= RECT[x] * \frac{1}{2}\sum_{n=-\infty}^{+\infty} \delta\left[\frac{x}{2}-n\right] \\
&= RECT[x] * \frac{1}{2} COMB\left[\frac{x}{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&t[x,y;z=0] \\
&= \frac{1}{2\cdot 20\,\mu\text{m}}\left(RECT\left[\frac{x}{10\,\mu\text{m}}\right] * COMB\left[\frac{x}{20\,\mu\text{m}}\right]\right)\cdot\left(RECT\left[\frac{x}{25\,\text{mm}}\right]\cdot RECT\left[\frac{y}{25\,\text{mm}}\right]\right)
\end{aligned}
$$

*This is a "square-wave" grating with about 50 bars per mm along the x-direction. The drawing shows an approximation (though with fewer than number of bars)*



*This may look familiar – it is a mathematical model of a diffraction grating. The (hidden) goal of the rest of the problem is to show that the grating disperses white light.*

(b) Derive the expression for the amplitude pattern observed at the distance $z_2$ from the aperture. Include relevant numerical factors in the expression. You need not sketch this, but it may help to do so.

$$\begin{aligned} g[x,y;z_2] &= \frac{1}{i\lambda_0 z_2}\exp\left[+2\pi i\frac{z_2}{\lambda_0}\right]\cdot \mathcal{F}_2\{t[x,y]\}|_{\xi\to\frac{x}{\lambda_0 z_2},\eta\to\frac{y}{\lambda_0 z_2}} \\ &= \frac{1}{i\lambda_0 z_2}\exp\left[+2\pi i\frac{z_2}{\lambda_0}\right]\cdot T\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] \\ &= K_1\cdot T\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] \end{aligned}$$

$$t[x,y] \\ = \frac{1}{2\cdot 20\,\mu\text{m}}\left(RECT\left[\frac{x}{10\,\mu\text{m}}\right]*COMB\left[\frac{x}{20\,\mu\text{m}}\right]\right)\cdot\left(RECT\left[\frac{x}{25\,\text{mm}}\right]\cdot RECT\left[\frac{y}{25\,\text{mm}}\right]\right)$$

$$\begin{aligned} T[\xi,\eta] = \frac{1}{40\,\mu\text{m}}&\cdot\left(10\,\mu\text{m}\cdot SINC[10\,\mu\text{m}\cdot\xi]\cdot 20\,\mu\text{m}\cdot COMB[20\,\mu\text{m}\cdot\xi]\right) \\ &*\left((25\,\text{mm})^2\cdot SINC[25\,\text{mm}\cdot\xi]\cdot SINC[25\,\text{mm}\cdot\eta]\right) \end{aligned}$$

$$\begin{aligned} g[x,y] &\propto T\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] \\ &= 5\,\mu\text{m}\cdot\left(SINC\left[10\,\mu\text{m}\cdot\frac{x}{\lambda_0 z_2}\right]\cdot COMB\left[20\,\mu\text{m}\cdot\frac{x}{\lambda_0 z_2}\right]\right) \\ &\quad *\left((25\,\text{mm})^2\cdot SINC\left[25\,\text{mm}\cdot\frac{x}{\lambda_0 z_2}\right]\cdot SINC\left[25\,\text{mm}\cdot\frac{y}{\lambda_0 z_2}\right]\right) \end{aligned}$$

*which is a "wide" SINC function along the x-direction modulated by a COMB function whose separation is half as wide convolved with a "very narrow" 2-D SINC function; the scale factor for the narrow SINC is smaller than that of the wide SINC function by the factor of:*

$$\frac{10\,\mu\text{m}}{25\,\text{mm}} = 4\cdot 10^{-4}$$

*so there are 2500 narrow SINCs within the width parameter of the wide SINC; therefore we approximate the narrow SINC function as a Dirac delta function:*

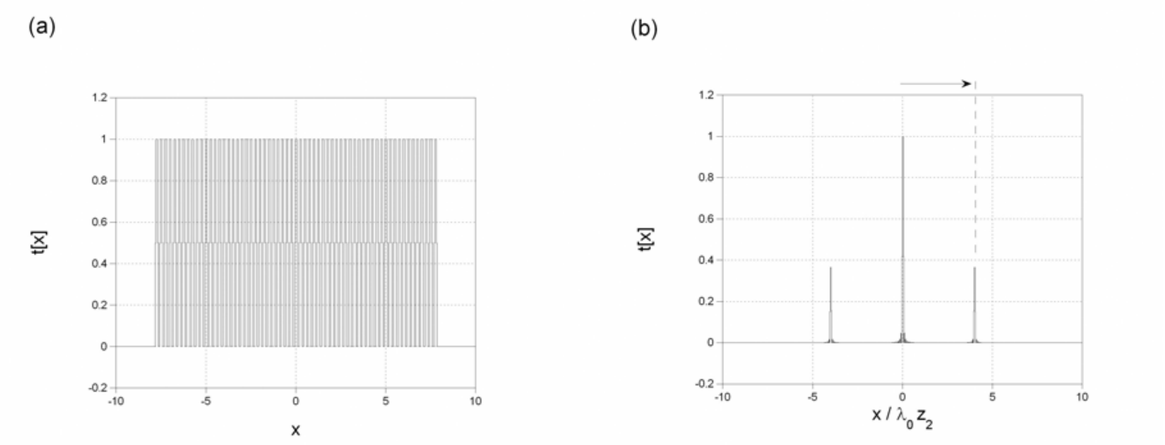
$$\begin{aligned} g[x,y] &\propto T\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] \\ &= 5\,\mu\text{m}\cdot\left(SINC\left[10\,\mu\text{m}\cdot\frac{x}{\lambda_0 z_2}\right]\cdot COMB\left[20\,\mu\text{m}\cdot\frac{x}{\lambda_0 z_2}\right]\right) \\ &\propto SINC\left[10\,\mu\text{m}\cdot\frac{x}{\lambda_0 z_2}\right]\cdot\sum_{k=-\infty}^{+\infty}\delta\left[\frac{x}{\left(\frac{\lambda_0 z_2}{20\,\mu\text{m}}\right)}-k\right] \\ &= SINC\left[\frac{x}{\left(\frac{\lambda_0 z_2}{10\,\mu\text{m}}\right)}\right]\cdot\left(\frac{\lambda_0 z_2}{20\,\mu\text{m}}\right)\sum_{k=-\infty}^{+\infty}\delta\left[x-k\cdot\frac{\lambda_0 z_2}{20\,\mu\text{m}}\right] \end{aligned}$$

*which is a SINC function multiplied by a series of Dirac delta functions. Note the wavelength dependence in the translation parameter in the Dirac delta function – the translation gets larger as $\lambda_0$ increases. In words, the output is a modulated COMB function whose amplitude decreases with increasing radial distance in the x direction. The even orders of the COMB vanish under the action of the SINC. The elements of the COMB represent the diffracted light at those positions.*

(c) Evaluate AND SKETCH the irradiance pattern at the same location.

*The irradiance is the squared magnitude:*

$$I[x,y] \propto |g[x,y]|^2 \propto \left| SINC\left[\frac{x}{\left(\frac{\lambda_0 z_2}{10\,\mu\text{m}}\right)}\right]\right|^2 \cdot \sum_{k=-\infty}^{+\infty} \delta\left[x - k \cdot \frac{\lambda_0 z_2}{20\,\mu\text{m}}\right]$$



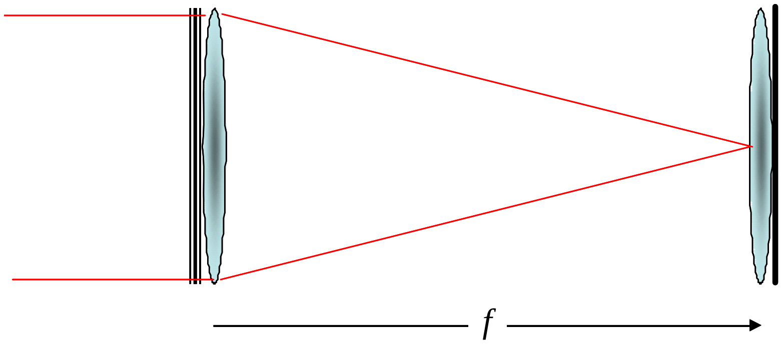
*(a) approxation of transmittance function $t[x]$; (b) approximation of irradiance pattern for a single incident wavelength. The displacement of the first order is proportional to the wavelength, which is the working mechanism for diffraction gratings.*

(d) Describe in words AND SKETCH the irradiance pattern that would be observed if the point source radiates the same amplitude at different wavelengths $\lambda$ spanning the entire visible spectrum.

*If multiple wavelengths, you get different Dirac delta function (or, more accurately, approximations thereto) for each wavelength that are displaced from the origin by distances proportional to $\lambda$; each appears in the color of its wavelength, so you get the spectrum of the white light.*

6. An object $f[x,y]$ is illuminated by a monochromatic plane wave with wavelength $\lambda_0$. *Immediately after* the object is a lens with focal length $\mathbf{f}$ whose diameter is sufficiently large so that all light transmitted through the object passes through the lens; in other words, the lens is in contact with the object. The light propagates a distance $z_2 = \mathbf{f}$, where it encounters an identical lens with focal length $\mathbf{f}$. The observation plane immediately follows (is in contact with) the lens.

   (a) Sketch the system.



   (b) Determine the amplitude and irradiance patterns at the observation plane in terms of the parameters of the system and $f[x,y]$.

   *The first lens imposes a quadratic-phase factor on the light from the object transparency $f[x,y]$*

$$g[x,y] = \left[\left(f[x,y]\cdot p[x,y]\exp\left[-i\pi\frac{x^2+y^2}{\lambda_0\mathbf{f}}\right]\right) * \left(\frac{1}{i\lambda_0\mathbf{f}}\exp\left[+2\pi i\frac{\mathbf{f}}{z_0}\right]\exp\left[+i\pi\frac{x^2+y^2}{\lambda_0\mathbf{f}}\right]\right)\right] \cdot\left(p[x,y]\exp\left[-i\pi\frac{x^2+y^2}{\lambda_0\mathbf{f}}\right]\right)$$

   *but the lens is assumed large enough so that $p[x,y]\to 1[x,y]$*

$$g[x,y] = \frac{1}{i\lambda_0\mathbf{f}}\exp\left[+2\pi i\frac{\mathbf{f}}{z_0}\right] \cdot\left(\left[\left(f[x,y]\cdot\exp\left[-i\pi\frac{x^2+y^2}{\lambda_0\mathbf{f}}\right]\right) * \left(\exp\left[+i\pi\frac{x^2+y^2}{\lambda_0\mathbf{f}}\right]\right)\right]\cdot\left(\exp\left[-i\pi\frac{x^2+y^2}{\lambda_0\mathbf{f}}\right]\right)\right)$$

*For simplicity, consider in 1-D:*

$$\begin{aligned}
g[x] &\propto \left(f[x]\cdot\exp\left[-i\pi\frac{x^2}{\lambda_0\mathbf{f}}\right]\right) * \left(\exp\left[+i\pi\frac{x^2}{\lambda_0\mathbf{f}}\right]\right)\cdot\left(\exp\left[-i\pi\frac{x^2}{\lambda_0\mathbf{f}}\right]\right) \\
&= \exp\left[-i\pi\frac{x^2}{\lambda_0\mathbf{f}}\right]\cdot\int_{-\infty}^{+\infty} f[\alpha]\exp\left[-i\pi\frac{\alpha^2}{\lambda_0\mathbf{f}}\right]\cdot\exp\left[+i\pi\frac{(x-\alpha)^2}{\lambda_0\mathbf{f}}\right]\,d\alpha \\
&= \exp\left[-i\pi\frac{x^2}{\lambda_0\mathbf{f}}\right]\cdot\int_{-\infty}^{+\infty} f[\alpha]\exp\left[-i\pi\frac{\alpha^2}{\lambda_0\mathbf{f}}\right]\cdot\exp\left[+i\pi\frac{x^2+\alpha^2-2x\alpha}{\lambda_0\mathbf{f}}\right]\,d\alpha \\
&= \left(\exp\left[-i\pi\frac{x^2}{\lambda_0\mathbf{f}}\right]\cdot\exp\left[+i\pi\frac{x^2}{\lambda_0\mathbf{f}}\right]\right)\cdot\int_{-\infty}^{+\infty} f[\alpha]\left(\exp\left[-i\pi\frac{\alpha^2}{\lambda_0\mathbf{f}}\right]\cdot\exp\left[+i\pi\frac{\alpha^2}{\lambda_0\mathbf{f}}\right]\right) \\
&\quad\cdot\exp\left[-i\pi\frac{2x\alpha}{\lambda_0\mathbf{f}}\right]\,d\alpha \\
&= 1\cdot\int_{-\infty}^{+\infty} f[\alpha]\cdot(1)\cdot\exp\left[-i\pi\frac{2x\alpha}{\lambda_0\mathbf{f}}\right]\,d\alpha \\
&= \int_{-\infty}^{+\infty} f[\alpha]\cdot\exp\left[-2\pi i\frac{x}{\lambda_0\mathbf{f}}\cdot\alpha\right]\,d\alpha = F[\xi]\big|_{\xi\to\frac{x}{\lambda_0\mathbf{f}}} = F\left[\frac{x}{\lambda_0\mathbf{f}}\right]
\end{aligned}$$

*This is an M-C-M chirp Fourier transform, so we can see that the 2-D output may be written in the form:*

$$\begin{aligned}
g[x,y] &= \left(\frac{1}{i\lambda_0\mathbf{f}}\exp\left[+2\pi i\frac{\mathbf{f}}{z_0}\right]\right)\cdot F\left[\frac{x}{\lambda_0\mathbf{f}},\frac{x}{\lambda_0\mathbf{f}}\right] \\
&\propto F\left[\frac{x}{\lambda_0\mathbf{f}},\frac{x}{\lambda_0\mathbf{f}}\right]
\end{aligned}$$

7. Consider two monochromatic point sources emit radiation with wavelength $\lambda_0$ and the same phase. The sources are separated by distance $d_0$ along the y-axis. Light emitted by from one of the sources immediately passes through a pane of glass with refractive index $n_0$, thickness $\ell_0$, and plane-parallel sides. The thickness is such that to the light emerging from the glass is exactly out of phase relative to the light from the other source. Light from both sources then propagates a distance $z_2$ to the observation plane.

   (a) Determine the thickness $\ell_0$ of the glass and comment on the practicality of using glass to induce the phase delay.

   *The "optical thickness" of the glass is $n \cdot \ell_0$. The term "out of phase" means that the phase difference is $\pi$ radians (NOT $\frac{\pi}{2}$ as several respondents did). This means that the phase of the light exiting the glass must be $\pi$ radians:*

$$\begin{aligned}\Delta\phi\left[z = \ell_0, n_1\right] &= \pi \\ &= 2\pi\frac{n \cdot \ell_0}{\lambda_0} \Longrightarrow 2\frac{n \cdot \ell_0}{\lambda_0} = 1 \Longrightarrow \ell_0 = \frac{\lambda_0}{2 \cdot n}\end{aligned}$$

   *The optical thickness of the glass must be half a wavelength:*

$$n \cdot \ell_0 = \frac{1}{2}\lambda_0 \Longrightarrow \ell_0 = \frac{\lambda_0}{2 \cdot n}$$

   *If $n = 1.5$ (as we usually assume for glass), then the thickness is:*

$$\ell_0 = \frac{\lambda_0}{2 \cdot 1.5} = \frac{\lambda_0}{3} \sim \frac{0.5\,\mu\text{m}}{3} = \frac{1}{6}\,\mu\text{m}$$

   *which is too thin to be practical, but the principle is sound.*

   (b) Determine the irradiance pattern observed if the distance $z_2$ is in the Fresnel diffraction region.

   *The source function is a pair of Dirac delta functions with a phase shift applied to one:*

$$\begin{aligned}f\left[x, y\right] &= \delta\left[x, y - \frac{d_0}{2}\right] + \delta\left[x, y + \frac{d_0}{2}\right]\exp\left[+i\pi\right] \\ &= \delta\left[x, y - \frac{d_0}{2}\right] - \delta\left[x, y + \frac{d_0}{2}\right]\end{aligned}$$

*The amplitude in the Fresnel diffraction region is obtained by propagating:*

$$\begin{aligned}
g\left[x,y;z_2\right] &= f\left[x,y;z=0,\lambda_0\right] * \frac{1}{i\lambda_0 z_2}\exp\left[+2\pi i\frac{z_2}{\lambda_0}\right]\exp\left[+i\pi\frac{x^2+y^2}{\lambda_0 z_2}\right] \\
&= K_2\cdot\left(f\left[x,y;z=0,\lambda_0\right] * \exp\left[+i\pi\frac{x^2+y^2}{\lambda_0 z_2}\right]\right) \quad \textit{where } K_2 \equiv \frac{1}{i\lambda_0 z_2}\exp\left[+2\pi i\frac{z_2}{\lambda_0}\right] \\
&= K_2\cdot\left(\delta\left[x,y-\frac{d_0}{2}\right]-\delta\left[x,y+\frac{d_0}{2}\right]\right) * \exp\left[+i\pi\frac{x^2+y^2}{\lambda_0 z_2}\right] \\
&= K_2\cdot\exp\left[+i\pi\frac{x^2}{\lambda_0 z_2}\right]\left(\exp\left[+i\pi\frac{\left(y-\frac{d_0}{2}\right)^2}{\lambda_0 z_2}\right]-\exp\left[+i\pi\frac{\left(y+\frac{d_0}{2}\right)^2}{\lambda_0 z_2}\right]\right) \\
&= K_2\cdot\exp\left[+i\pi\frac{x^2}{\lambda_0 z_2}\right]\left(\exp\left[+i\pi\frac{y^2+\left(\frac{d_0}{2}\right)^2-yd_0}{\lambda_0 z_2}\right]-\exp\left[+i\pi\frac{y^2+\left(\frac{d_0}{2}\right)^2+yd_0}{\lambda_0 z_2}\right]\right) \\
&= K_2\cdot\exp\left[+i\pi\frac{x^2+y^2}{\lambda_0 z_2}\right]\cdot\exp\left[+i\pi\frac{\left(\frac{d_0}{2}\right)^2}{\lambda_0 z_2}\right]\left(\exp\left[-i\pi\frac{yd_0}{\lambda_0 z_2}\right]-\exp\left[+i\pi\frac{yd_0}{\lambda_0 z_2}\right]\right) \\
&= -2i\cdot K_2\cdot\exp\left[+i\pi\frac{x^2+y^2}{\lambda_0 z_2}\right]\cdot\exp\left[+i\pi\frac{\left(\frac{d_0}{2}\right)^2}{\lambda_0 z_2}\right]\cdot\sin\left[\pi\frac{yd_0}{\lambda_0 z_2}\right]
\end{aligned}$$

*The irradiance is proportional to the squared magnitude:*

$$\begin{aligned}
I\left[x,y;z_2.\lambda_0\right] &\propto \left|-2i\cdot K_2\cdot\exp\left[+i\pi\frac{x^2+y^2}{\lambda_0 z_2}\right]\cdot\exp\left[+i\pi\frac{\left(\frac{d_0}{2}\right)^2}{\lambda_0 z_2}\right]\cdot\sin\left[\pi\frac{yd_0}{\lambda_0 z_2}\right]\right|^2 \\
&= \frac{4}{\lambda_0^2 z_2^2}\cdot\sin^2\left[\pi\frac{yd_0}{\lambda_0 z_2}\right] \\
\textit{Recall} &: \quad \sin^2\left[\theta\right]=\frac{1}{2}\left(1-\cos\left[2\theta\right]\right) \\
&\Longrightarrow \quad I\left[x,y;z_2.\lambda_0\right]\propto\frac{4}{\lambda_0^2 z_2^2}\cdot\frac{1}{2}\left(1-\cos\left[2\pi\frac{y}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right]\right)
\end{aligned}$$

$$\boxed{I\left[x,y;z_2.\lambda_0\right]\propto\frac{2}{\lambda_0^2 z_2^2}\cdot\left(1-\cos\left[2\pi\frac{y}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right]\right)}$$

*which is a biased cosine pattern with a minimum at the origin and a period:*

$$D_0=\frac{\lambda_0 z_2}{d_0}$$

(c) Determine the irradiance pattern observed if the distance $z_2$ is in the Fraunhofer diffraction region.

*In the Fraunhofer region, the observed pattern is proportional to the squared magnitude of the 2-D Fourier transform of the source function:*

$$I[x,y;z_2,\lambda_0] \propto \left|\frac{1}{i\lambda_0 z_2}\exp\left[+2\pi i\frac{z_2}{\lambda_0}\right]\cdot F\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right|^2$$

$$\begin{aligned}
f[x,y] &= \delta\left[x,y-\frac{d_0}{2}\right]-\delta\left[x,y+\frac{d_0}{2}\right]\\
&\Longrightarrow F[\xi,\eta]=1[\xi]\cdot\exp\left[-2\pi i\cdot\eta\cdot\frac{d_0}{2}\right]-1[\xi]\cdot\exp\left[+2\pi i\cdot\eta\cdot\frac{d_0}{2}\right]\\
&= (-1)\cdot 2i\cdot 1[\xi]\cdot\sin[\pi d_0\xi]\\
F\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right] &= -2i\cdot 1\left[\frac{x}{\lambda_0 z_2}\right]\cdot\sin\left[\pi d_0\frac{y}{\lambda_0 z_2}\right]=-2i\cdot 1\left[\frac{x}{\lambda_0 z_2}\right]\cdot\sin\left[\pi\frac{y}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right]\\
\left|F\left[\frac{x}{\lambda_0 z_2},\frac{y}{\lambda_0 z_2}\right]\right|^2 &= 4\cdot 1\left[\frac{x}{\lambda_0 z_2}\right]\cdot\sin^2\left[\pi\frac{y}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right]\\
&= 2\cdot 1\left[\frac{x}{\lambda_0 z_2}\right]\cdot\left(1-\cos\left[2\pi\frac{y}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right]\right)\\
D_0 &= \frac{\lambda_0 z_2}{d_0}
\end{aligned}$$

$$\boxed{I[x,y;z_2,\lambda_0]\propto\frac{2}{\lambda_0^2 z_2^2}\cdot\left(1-\cos\left[2\pi\frac{y}{\left(\frac{\lambda_0 z_2}{d_0}\right)}\right]\right)}$$

(d) (OPTIONAL BONUS) Describe how you can create a "practical" pane of glass that induces an appropriate phase delay in the light from one source.

*The key point to recognize here is that we get the same result in monochromatic light if the phase induced by the glass is an odd multiple of $\pi$ radians, so we can make the glass much thicker.*