IMGS-616-20141 Solutions to Optional HW #9

OBSERVATION: *Many of you just draw conclusions from equations rather than sketching out the problem. I say AGAIN, you often (i.e., nearly ALWAYS) do better when you sketch out the problem FIRST and then introduce the equations. For example, if you sketched out the spectra in problem #7 first, the answer becomes obvious!*

1. The impulse response of an imaging system is:

$$h[x] = \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]$$

Evaluate the expression for the inverse filter $w[x]$

$$\begin{aligned}
H[\xi] &= |\alpha_0| \cdot \exp\left[+i \cdot \frac{\pi}{4}\right] \cdot \exp\left[-i\pi\left(\alpha_0 \xi\right)^2\right] \\
W[\xi] &= \frac{1}{H[\xi]} = \frac{1}{|\alpha_0|} \cdot \exp\left[-i \cdot \frac{\pi}{4}\right] \cdot \exp\left[+i\pi\left(\alpha_0 \xi\right)^2\right] \\
w[x] &= \left(\frac{1}{|\alpha_0|} \cdot \exp\left[-i \cdot \frac{\pi}{4}\right]\right) \cdot \frac{1}{|\alpha_0|} \cdot \exp\left[+i \cdot \frac{\pi}{4}\right] \cdot \exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]
\end{aligned}$$

$$\boxed{w[x] = \frac{1}{|\alpha_0|^2} \cdot \exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]}$$

so the inverse filter for an "upchirp" (quadratic-phase function with a "+" sign in the exponent) is a "downchirp" with the same chirp rate.

Lots of the answers did not include all steps and missed the constant factor.

2. The transfer function of a filter is:

$$H_1[\xi] = 1[\xi] \cdot \exp[i \cdot 2\pi \cdot (1 - 2 \cdot RECT[\xi])]$$

(a) Characterize this filter (highpass, ...)

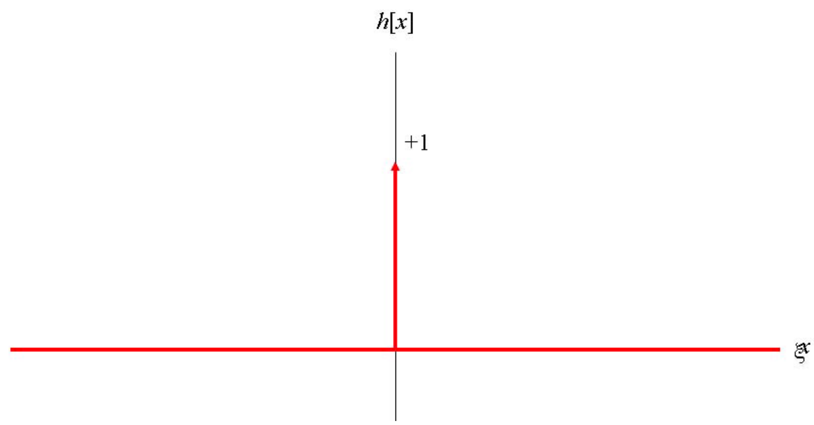
$$H_1[\xi] = 1[\xi] \cdot \begin{cases} \exp[+i \cdot 2\pi] = +1 & \text{if} \quad RECT[\xi] = 1 \implies |\xi| < \frac{1}{2} \\ \exp[+i \cdot 0] = +1 & \text{if} \quad RECT[\xi] = 1 \implies |\xi| = \frac{1}{2} \\ \exp[-i \cdot 2\pi] = +1 & \text{if} \quad RECT[\xi] = 0 \implies |\xi| < \frac{1}{2} \end{cases}$$

$$\boxed{H_1[\xi] = 1[\xi] \implies \textit{identity operator}}$$

allpass

(b) Evaluate and sketch the impulse response $h[x]$

$$\boxed{h[x] = \delta[x]}$$



Find and sketch the outputs $g[x]$ for the following inputs:

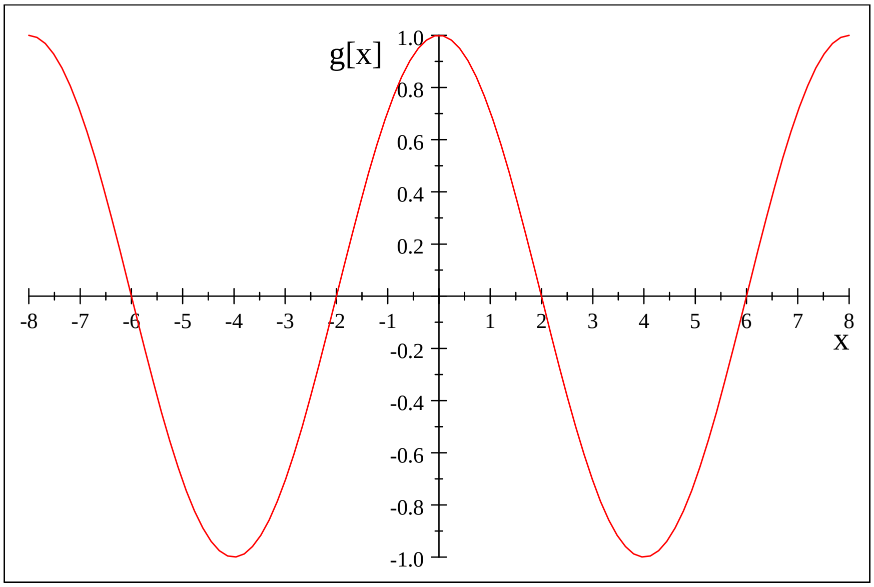
(c)

$$f_1[x] = \cos\left[2\pi \frac{x}{8}\right]$$

$$F_1[\xi] = \frac{1}{2}\left(\delta\left[\xi + \frac{1}{8}\right] + \delta\left[\xi - \frac{1}{8}\right]\right)$$

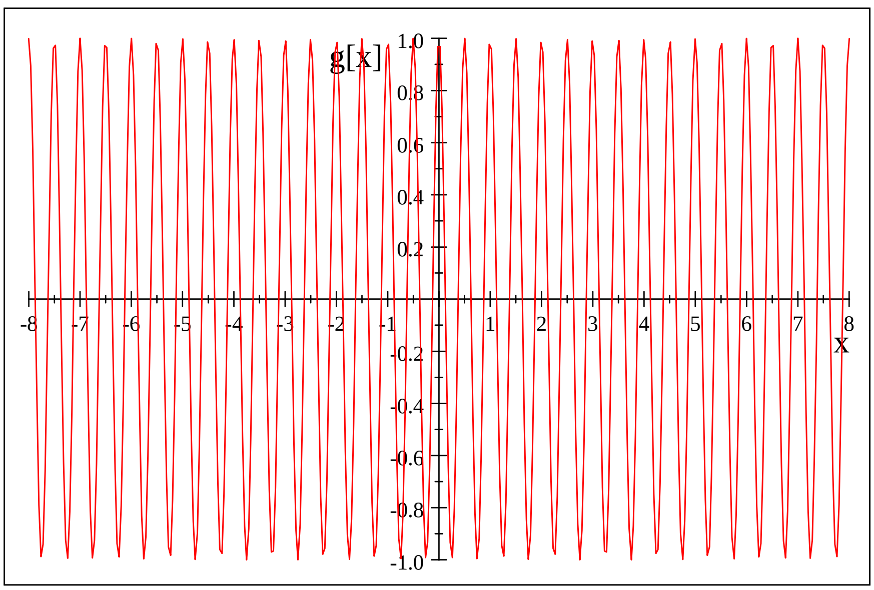
$$G_1[\xi] = F_1[\xi] \cdot H_1[\xi] = F_1[\xi]$$

$$\implies g_1[x] = f_1[x] = \cos\left[2\pi \frac{x}{8}\right]$$



(d)

$$\begin{aligned} f_2[x] &= \cos[4\pi x] = \cos[2\pi \cdot 2 \cdot x] \\ F_2[\xi] &= \frac{1}{2}\left(\delta[\xi+2] + \delta[\xi-2]\right) \\ G_2[\xi] &= F_2[\xi] \cdot H_2[\xi] = F_2[\xi] \\ g_2[x] &= f_2[x] \end{aligned}$$



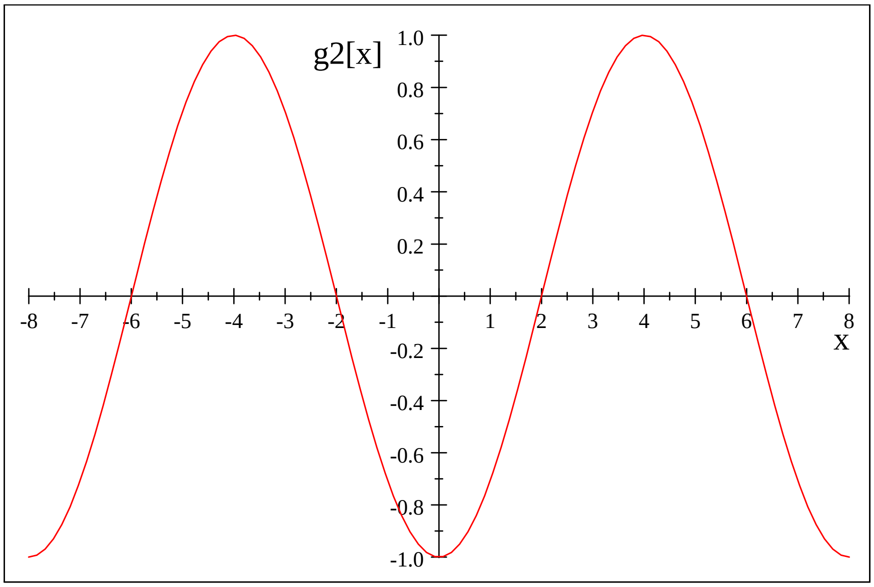
(e) Repeat (c) and (d) for the modified transfer function:

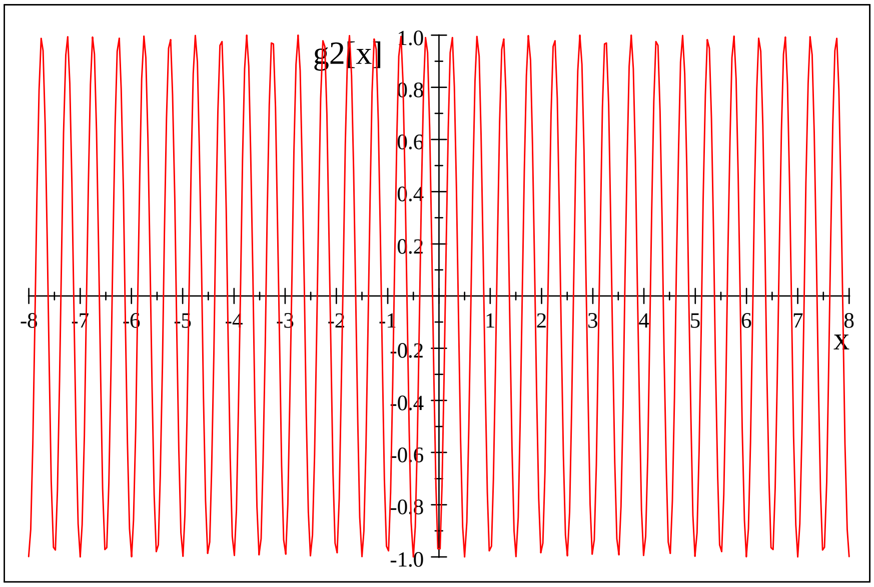
$$\begin{aligned} H_2[\xi] &= 1[\xi] \cdot \exp\left[i \cdot 2\pi \cdot \left(\frac{1}{2} - RECT[\xi]\right)\right] \\ &= 1[\xi] \cdot \exp[+i \cdot \pi] \cdot \exp[-i \cdot 2\pi \cdot RECT[\xi]] \end{aligned}$$

$$H_2[\xi] = \begin{cases} \exp[+i\pi] = -1 & \text{if} \quad RECT[\xi] = 0 \\ \exp[+i \cdot 0] = +1 & \text{if} \quad RECT[\xi] = \frac{1}{2} \\ \exp[-i\pi] = -1 & \text{if} \quad RECT[\xi] = 1 \end{cases}$$

$$= \begin{cases} -1 & \text{if} \quad \xi < -\frac{1}{2} \\ +1 & \text{if} \quad \xi = -\frac{1}{2} \\ -1 & \text{if} \quad -\frac{1}{2} < \xi < +\frac{1}{2} \\ +1 & \text{if} \quad \xi = +\frac{1}{2} \\ -1 & \text{if} \quad \xi > +\frac{1}{2} \end{cases}$$

This transfer function inverts all frequencies except $\xi = \pm\frac{1}{2}$; since $F_1\left[\xi = \pm\frac{1}{2}\right] = F_2\left[\xi = \pm\frac{1}{2}\right] = 0$, the zeros of the transfer function have no impact on the input spectra and the outputs are inverted replicas of the inputs.





3. The transfer function of a 1-D system is:

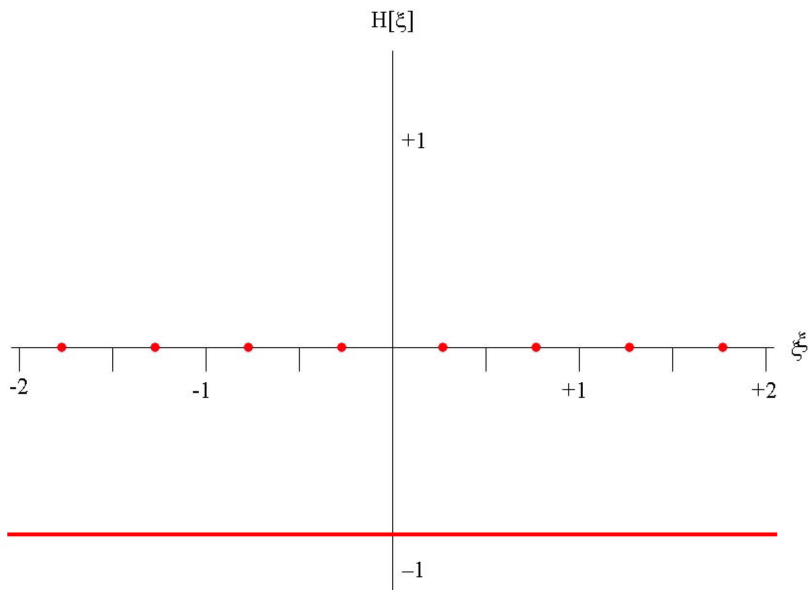
$$H[\xi] = \left(1 - 2 \cdot RECT\left[\frac{\xi}{0.5}\right] * COMB[\xi]\right) \cdot \exp\left[+i\pi\left(1 - COMB[\xi] * RECT[2\xi]\right)\right]$$

(a) Sketch this transfer function as (real, imaginary) parts and as (magnitude, phase).

$$\begin{aligned}
\Phi_H[\xi] &= \pi \cdot (1 - COMB[\xi] * RECT[2\xi]) \\
&= \begin{cases} 0 & \text{if} \quad COMB[\xi] * RECT[2\xi] = 1 \\ +\frac{\pi}{2} & \text{if} \quad COMB[\xi] * RECT[2\xi] = \frac{1}{2} \\ +\pi & \text{if} \quad COMB[\xi] * RECT[2\xi] = 0 \end{cases} \\
&\implies \exp\left[+i\pi\left(1 - COMB[\xi] * RECT[2\xi]\right)\right] = \begin{cases} 1 & \text{if} \quad COMB[\xi] * RECT[2\xi] = 1 \\ +i & \text{if} \quad COMB[\xi] * RECT[2\xi] = \frac{1}{2} \\ -1 & \text{if} \quad COMB[\xi] * RECT[2\xi] = 0 \end{cases} \\
|H[\xi]| &= \left(1 - 2 \cdot RECT\left[\frac{\xi}{0.5}\right] * COMB[\xi]\right) = \begin{cases} -1 & \text{if} \quad COMB[\xi] * RECT[2\xi] = 1 \\ 0 & \text{if} \quad COMB[\xi] * RECT[2\xi] = \frac{1}{2} \\ +1 & \text{if} \quad COMB[\xi] * RECT[2\xi] = 0 \end{cases} \\
&\implies H[\xi] = \begin{cases} +1 \cdot -1 = -1 & \text{if} \quad COMB[\xi] * RECT[2\xi] = 1 \\ +i \cdot 0 = 0 & \text{if} \quad COMB[\xi] * RECT[2\xi] = \frac{1}{2} \\ -1 \cdot +1 = -1 & \text{if} \quad COMB[\xi] * RECT[2\xi] = 0 \end{cases}
\end{aligned}$$

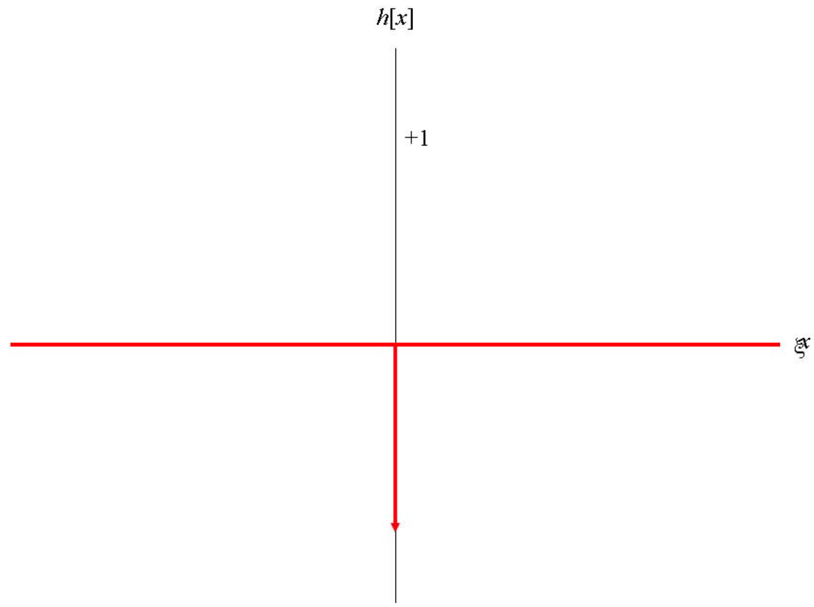
which evaluates to -1 *everywhere except at the zeros of the MTF, so it "inverts" sinusoidal components at those frequencies and blocks those at the zero crossings,* $\xi = \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4}, \cdots$ Classify this filter (e.g., highpass, ...).

allpass except for the isolated single frequencies



(b) Derive and sketch the impulse response $h[x]$.

$$h[x] = \mathcal{F}^{-1}\{H[\xi]\} = -\delta[x]$$



(c) Derive and sketch the inverse filter $w[x]$ for this impulse response.

$$W[\xi] = \frac{1}{H[\xi]} = \begin{cases} -1 & \text{if} \quad COMB[\xi] * RECT[2\xi] = 1 \\ 0 & \text{if} \quad COMB[\xi] * RECT[2\xi] = \frac{1}{2} \\ -1 & \text{if} \quad COMB[\xi] * RECT[2\xi] = 0 \end{cases}$$
$$= H[\xi]$$

$$w[x] = h[x] = -\delta[x]$$

This filter is its own inverse filter.

4. Determine the areas of:

(a) $\cos[2\pi x] \cdot \cos\left[\pi x^2\right]$

Use Rayleigh's Theorem:

$$\int_{-\infty}^{+\infty} f[x] \cdot m^*[x]\ dx = \int_{-\infty}^{+\infty} F[\xi] \cdot M^*[\xi]\ dx$$

$$\begin{aligned}
f[x] &= \cos[2\pi x] \implies F[\xi] = \frac{1}{2}\delta[\xi+1] + \frac{1}{2}\delta[\xi-1] \\
m[x] &= \cos\left[\pi x^2\right] = \operatorname{Re}\left\{\exp\left[+i\pi x^2\right]\right\} = \operatorname{Re}\left\{\cos\left[\pi x^2\right] + i \cdot \sin\left[\pi x^2\right]\right\} \\
\mathcal{F}\left\{\exp\left[+i\pi x^2\right]\right\} &= \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi\xi^2\right] \\
&= \exp\left[-i\pi\left(\xi^2 - \frac{1}{4}\right)\right] = \cos\left[\pi\left(\xi^2 - \frac{1}{4}\right)\right] - i \cdot \sin\left[\pi\left(\xi^2 - \frac{1}{4}\right)\right] \\
\mathcal{F}\left\{\cos\left[\pi x^2\right]\right\} &= \cos\left[\pi\left(\xi^2 - \frac{1}{4}\right)\right] = M[\xi] = M^*[\xi]
\end{aligned}$$

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \cos[2\pi x] \cdot \cos\left[\pi x^2\right]\ dx \\
= &\int_{-\infty}^{+\infty} \left(\frac{1}{2}\delta[\xi+1] + \frac{1}{2}\delta[\xi-1]\right) \cdot \cos\left[\pi\left(\xi^2 - \frac{1}{4}\right)\right]\ d\xi \\
= &\frac{1}{2}\int_{-\infty}^{+\infty} \delta[\xi+1] \cdot \cos\left[\pi\left(\xi^2 - \frac{1}{4}\right)\right]\ d\xi + \frac{1}{2}\int_{-\infty}^{+\infty} \delta[\xi-1] \cdot \cos\left[\pi\left(\xi^2 - \frac{1}{4}\right)\right]\ d\xi \\
= &\frac{1}{2}\int_{-\infty}^{+\infty} \delta[\xi+1] \cdot \cos\left[\pi\left((-1)^2 - \frac{1}{4}\right)\right]\ d\xi + \frac{1}{2}\int_{-\infty}^{+\infty} \delta[\xi-1] \cdot \cos\left[\pi\left((+1)^2 - \frac{1}{4}\right)\right]\ d\xi \\
= &\frac{1}{2} \cdot \cos\left[\pi\left((-1)^2 - \frac{1}{4}\right)\right]\int_{-\infty}^{+\infty} \delta[\xi+1]\ d\xi + \frac{1}{2} \cdot \cos\left[\pi\left((+1)^2 - \frac{1}{4}\right)\right]\int_{-\infty}^{+\infty} \delta[\xi-1]\ d\xi \\
= &\frac{1}{2} \cdot \cos\left[\pi\left((-1)^2 - \frac{1}{4}\right)\right] \cdot 1 + \frac{1}{2} \cdot \cos\left[\pi\left((+1)^2 - \frac{1}{4}\right)\right] \cdot 1 \\
= &2 \cdot \frac{1}{2} \cdot \cos\left[\pi\left(\frac{3}{4}\right)\right] = \cos\left[\pi \cdot \frac{3}{4}\right]
\end{aligned}$$

$$\boxed{\int_{-\infty}^{+\infty} \cos[2\pi x] \cdot \cos\left[\pi x^2\right]\ dx = -\frac{1}{\sqrt{2}}}$$

(b) $SINC^4[x]$

$$\begin{aligned}
SINC^4[x] &= SINC^2[x] \cdot SINC^2[x] \\
f[x] &= m[x] = SINC^2[x] \\
F[\xi] &= M[\xi] = TRI[\xi]
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{+\infty} SINC^4[x]\ dx &= \int_{-\infty}^{+\infty} SINC^2[x]\cdot SINC^2[x]\ dx \\
&= \int_{-\infty}^{+\infty} TRI[\xi]\cdot TRI[\xi]\ d\xi \\
&= \int_{-1}^{0} (1+\xi)^2\ d\xi + \int_{0}^{+1} (1-\xi)^2\ d\xi \\
&= \int_{-1}^{0} \left(1+\xi^2+2\xi\right)\ d\xi + \int_{0}^{+1} \left(1+\xi^2-2\xi\right)\ d\xi \\
&= \left(\xi+\frac{\xi^3}{3}+2\cdot\frac{\xi^2}{2}\right)\Bigg|_{\xi=-1}^{\xi=0} + \left(\xi+\frac{\xi^3}{3}-2\cdot\frac{\xi^2}{2}\right)\Bigg|_{\xi=0}^{\xi=1} \\
&= \left(+1+\frac{1}{3}-1\right)+\left(1+\frac{1}{3}-1\right)
\end{aligned}$$

$$\boxed{\int_{-\infty}^{+\infty} SINC^4[x]\ dx = \frac{2}{3}}$$

5. An imaging system acting on the 1-D input function $f[x]$ includes the following steps:

(a) evaluate $\mathcal{F}\{f[x]\}|_{\xi\to\frac{x}{(\alpha_0)^2}}$ where α_0 is a constant parameter with units of "length";

(b) multiply by the real-valued pupil function $p[x]$ (which has finite support) to form $g_1[x]$

(c) evaluate $\mathcal{F}\{g_1[x]\}|_{\xi\to\frac{x}{(\alpha_0)^2}}$

For an input function of your choice, evaluate and sketch the output function if the pupil $p[x] = RECT\left[\frac{x}{b_0}\right]$. Note that this is a real imaging system where the first and last steps propagate light a long distance to the Fraunhofer diffraction region.

Solution: *Part of the reason for the problem was to see if you chose a "useful" function to evaluate the result for. Some of you selected $f[x] = 1[x] \implies F[\xi] = \delta[\xi]$, i.e., only a single frequency in the function. The end result indicates almost nothing about the action of the system, so that is not a useful choice and received no credit. The most useful choice is an off-axis impulse, which has both all frequencies and some phase variation and the result will tell us plenty about the action of the system. The three steps to evaluate the output generated by the off-axis impulse is:*

$$(a)\quad : \quad \mathcal{F}\{\delta[x-x_0]\}|_{\xi\to\frac{x}{(\alpha_0)^2}} = 1[\xi]\cdot\exp[-i\cdot 2\pi\cdot\xi x_0]|_{\xi\to\frac{x}{(\alpha_0)^2}} = \exp\left[-i\cdot 2\pi\cdot\frac{x}{\alpha_0^2}\cdot x_0\right] = \exp -i\cdot 2\pi\cdot\left(\frac{x_0}{\alpha_0^2}\right)$$

$$(b):\quad g_1[x] = \exp\left[-i\cdot 2\pi\cdot\left(\frac{x_0}{\alpha_0^2}\right)\cdot x\right]\cdot p[x] = \exp\left[-i\cdot 2\pi\cdot\left(\frac{x_0}{\alpha_0^2}\right)\cdot x\right]\cdot RECT\left[\frac{x}{b_0}\right]$$

$$(c):\quad \mathcal{F}\{g_1[x]\}|_{\xi\to\frac{x}{(\alpha_0)^2}} = \mathcal{F}\left\{\exp\left[-i\cdot 2\pi\cdot\left(\frac{x_0}{\alpha_0^2}\right)\cdot x\right]\cdot p[x]\right\}\Bigg|_{\xi\to\frac{x}{(\alpha_0)^2}}$$

$$= \mathcal{F}\left\{\exp\left[-i\cdot 2\pi\cdot\left(\frac{x_0}{\alpha_0^2}\right)\cdot x\right]\cdot RECT\left[\frac{x}{b_0}\right]\right\}\Bigg|_{\xi\to\frac{x}{(\alpha_0)^2}}$$

$$= \left(\delta\left[\xi+\left(\frac{x_0}{\alpha_0^2}\right)\right] * |b_0|\cdot SINC[b_0\xi]\right)\Bigg|_{\xi\to\frac{x}{(\alpha_0)^2}}$$

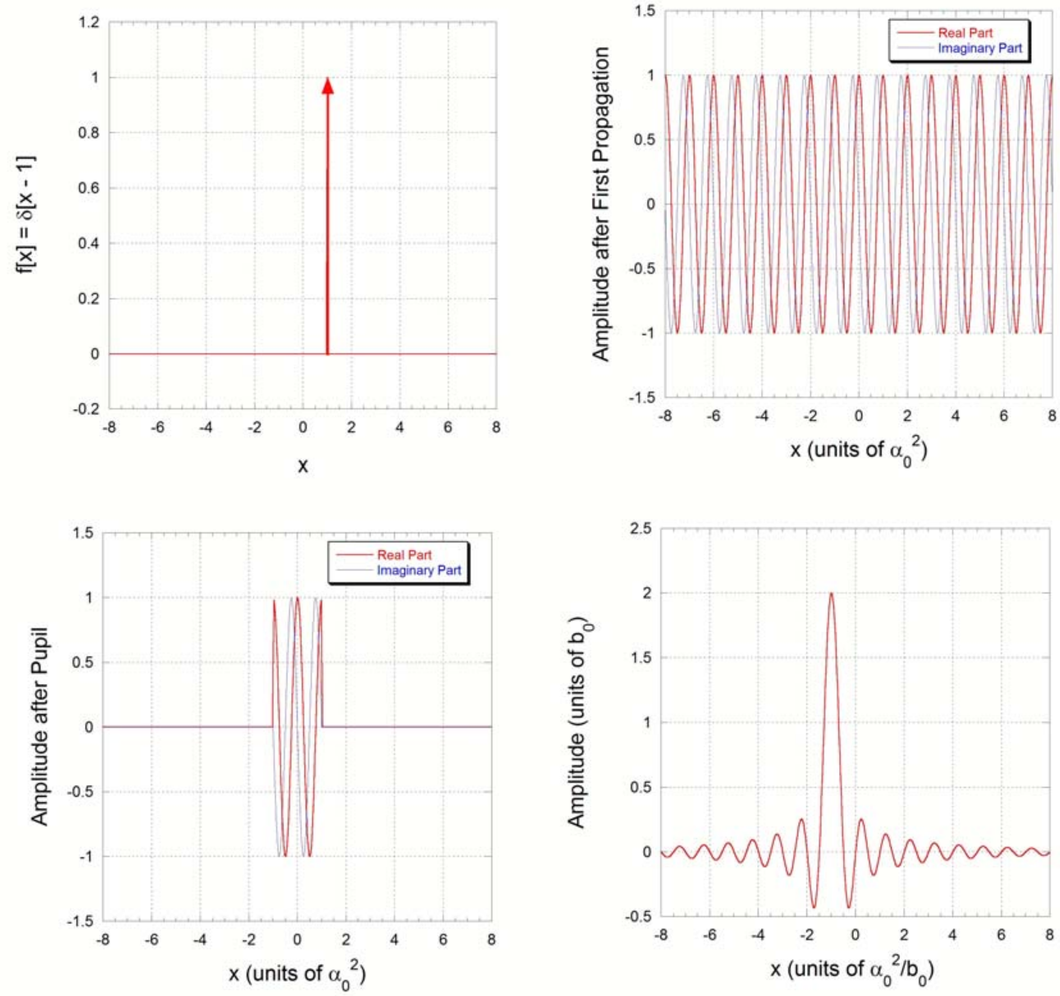
$$= |b_0|\, SINC\left[b_0\cdot\left(\xi+\frac{x_0}{\alpha_0^2}\right)\right]\Bigg|_{\xi\to\frac{x}{(\alpha_0)^2}}$$

$$= |b_0|\, P\left[b_0\left(\frac{x}{\alpha_0^2}+\frac{x_0}{\alpha_0^2}\right)\right] = P\left[\frac{x+x_0}{\alpha_0^2/b_0}\right] =$$

So the output generated by an "off-axis" impulse is an off-axis replica of the Fourier transform of the aperture function, though the translation is in the OPPOSITE direction from that of the original Dirac delta. ***See Figure 21.21 in the book****. Also not that we can substitute $x_0 = 0$ to determine the output for a "centered" impulse:*

$$f[x] = \delta[x-0] \implies P\left[\frac{x+x_0}{\alpha_0^2}\right] \to P\left[\frac{x+0}{\alpha_0^2}\right] = P\left[\frac{x}{\alpha_0^2}\right]$$

Since the outputs for both cases are scaled and translated replicas of the Fourier transform of the aperture, and since any input object may be expressed as a superposition of impulses, it is possible to express the output as a convolution of the input with scaled replicas of the transform of the aperture; this may be represented as a linear shift-invariant system.



Steps in the process: (a) input $f[x] = \delta[x - x_0]$; (b) after first propagation; (c) after multiplication by pupil function; (d) after second propagation to yield a lowpass-filtered and reversed replica of the input.

6. Modify the imaging system in the previous problem so that the explicit steps in the process are now:

   (a) evaluate $\mathcal{F}\{f[x]\}|_{\xi\to\frac{x}{(\alpha_0)^2}}$ where α_0 is a constant parameter with units of "length";

   (b) multiply by $p[x] = RECT\left[\frac{x}{b_0}\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_1}\right)^2\right]$, where both b_0 and α_1 are system parameters with units of length

   (c) convolve the product in step (b) with impulse response $h[x] = \exp\left[+i\pi\left(\frac{x}{\alpha_1}\right)^2\right]$

   In this system, steps (b) and (c) are the first two steps in an *M-C-M* chirp Fourier transformer. This is also a realistic imaging imaging system step (a) represents propagation from the object to a location in the Fraunhofer diffraction region, step (b) is the action of a lens, and step (c) is propagation to the focal plane of the lens. In this example, the constants $\alpha_1 = \sqrt{\lambda_0 z_1}$ and $\alpha_1 = \sqrt{\lambda_0 \mathbf{f}}$, where z_1 is the propagation distance from the object to the lens and $\mathbf{f}$ is the focal length of the lens.

   Same comment as in #5 about the choice of function: $f[x] = 1[x]$ supplies almost no information about the action of the system and therefore is NOT a useful input. A function with all frequencies is a MUCH better choice, such as an off-axis impulse:

$$\text{(a):}\quad f[x] = \delta[x - x_0] \implies \mathcal{F}\{f[x]\}|_{\xi\to\frac{x}{(\alpha_0)^2}} = 1\left[\frac{x}{\alpha_0^2}\right]\cdot\exp\left[-i\cdot 2\pi\cdot\frac{x}{\alpha_0^2}\cdot x_0\right]$$

$$= 1[x]\cdot\exp\left[-i\cdot 2\pi\cdot\left(\frac{x_0}{\alpha_0^2}\right)\cdot x\right]$$

$$\text{(b):}\quad 1[x]\cdot\exp\left[-i\cdot 2\pi\cdot\left(\frac{x_0}{\alpha_0^2}\right)\cdot x\right]\cdot RECT\left[\frac{x}{b_0}\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_1}\right)^2\right]$$

$$= \left(RECT\left[\frac{x}{b_0}\right]\cdot\exp\left[-i\cdot 2\pi\cdot\left(\frac{x_0}{\alpha_0^2}\right)\cdot x\right]\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_1}\right)^2\right]$$

$$\text{(c):}\quad \left(RECT\left[\frac{x}{b_0}\right]\cdot\exp\left[-i\cdot 2\pi\cdot\left(\frac{x_0}{\alpha_0^2}\right)\cdot x\right]\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_1}\right)^2\right] * \exp\left[+i\pi\left(\frac{x}{\alpha_1}\right)^2\right]$$

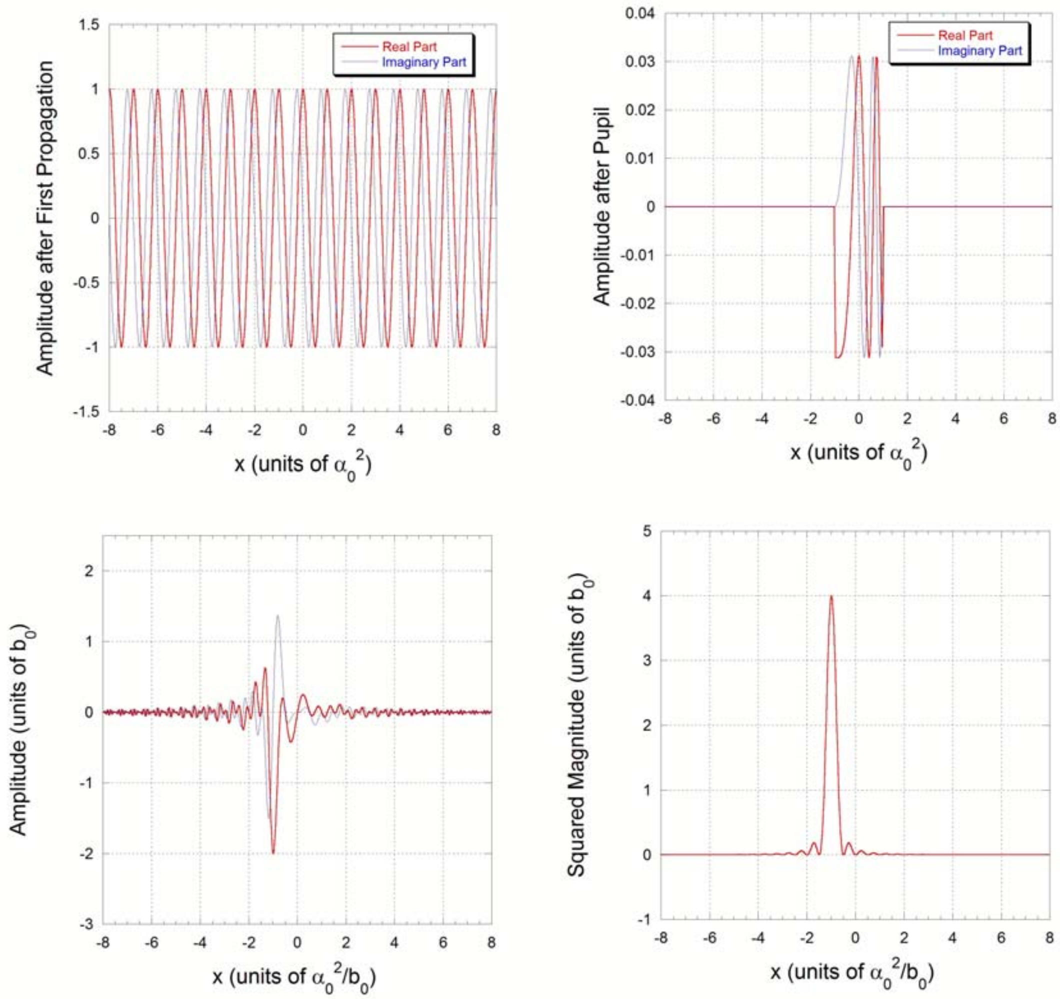
$$= \left(|b_0|\,SINC\left[b_0\frac{x}{\alpha_1^2}\right] * \delta\left[\frac{x}{\alpha_1^2} + \frac{x_0}{\alpha_0^2}\right]\right)\cdot\exp\left[+i\pi\left(\frac{x}{\alpha_1}\right)^2\right]$$

$$= \left(|b_0|\,SINC\left[b_0\frac{x}{\alpha_1^2}\right] * \delta\left[\frac{1}{\alpha_1^2}\left(x + x_0\frac{\alpha_1^2}{\alpha_0^2}\right)\right]\right)\cdot\exp\left[+i\pi\left(\frac{x}{\alpha_1}\right)^2\right]$$

$$= \alpha_1^2\left(|b_0|\,SINC\left[b_0\frac{x}{\alpha_1^2}\right] * \delta\left[x + x_0\frac{\alpha_1^2}{\alpha_0^2}\right]\right)\cdot\exp\left[+i\pi\left(\frac{x}{\alpha_1}\right)^2\right]$$

$$= \alpha_1^2\cdot|b_0|\cdot SINC\left[\frac{\left(x + x_0\frac{\alpha_1^2}{\alpha_0^2}\right)}{\left(\frac{\alpha_1^2}{b_0}\right)}\right]\cdot\exp\left[+i\pi\left(\frac{x}{\alpha_1}\right)^2\right]$$

   Again, the output is a SINC function (the Fourier transform of the rectangular pupil function) displaced in the OPPOSITE direction from the input. The displacement is proportional to the input location x_0, and so the output is a reversed, magnified, and "lowpass-filtered" version of the input.



Outputs after stages in the process for same object as used in previous problem: (a) amplitude after first propagation (same as (b) from previous images); (b) amplitude after pupil, showing effect of quadratic phase; (c) amplitude at output; (d) squared magnitude at output, showing that "image" is the squared magnitude of the lowpass-filtered and reversed replica of the object.

7. A measured signal $g[x]$ is the sum of a deterministic part $f[x]$ and a "stochastic" random part $n[x]$

   (a) Describe the constraints on $f[x]$ AND on $n[x]$ that would allow a filter $w[x]$ to be constructed such that the output of the filter is *exactly* equal to $f[x]$, i.e.,

$$\begin{aligned}(f[x] + n[x]) * w[x] &= f[x] \\ (F[\xi] + N[\xi]) \cdot W[\xi] &= F[\xi]\end{aligned}$$

$$\begin{aligned}F[\xi] \cdot W[\xi] &= F[\xi] \\ N[\xi] \cdot W[\xi] &= 0[\xi]\end{aligned}$$

The input and noise spectra must be "disjoint," i.e.,

$$|F[\xi]| \neq 0 \Longrightarrow |N[\xi]| = 0$$

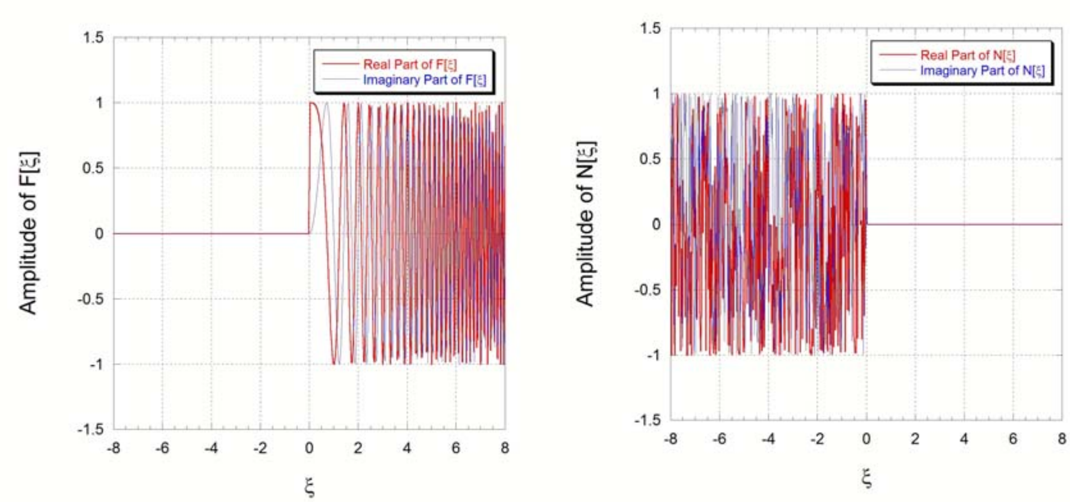
and

$$|N[\xi]| \neq 0 \Longrightarrow |F[\xi]| = 0$$

   (b) Specify AND SKETCH the transfer function AND impulse response of the filter that will "extract" $f[x]$ from $g[x]$ if the signal and noise spectra are the functions::

$$\begin{aligned}F[\xi] &= STEP[\xi] \cdot \exp\left[+i\pi\xi^2\right] \\ N[\xi] &= STEP[-\xi] \cdot \exp\left[+i\pi \cdot R[\xi]\right]\end{aligned}$$

where $R[\xi]$ is a random number selected from the interval $-\pi \leq R[\xi] < +\pi$



Since the spectra do not overlap (except for the zero frequency term which only changes the answer by a constant), then the transfer function of the filter that passes the signal and blocks the noise must be a step function:

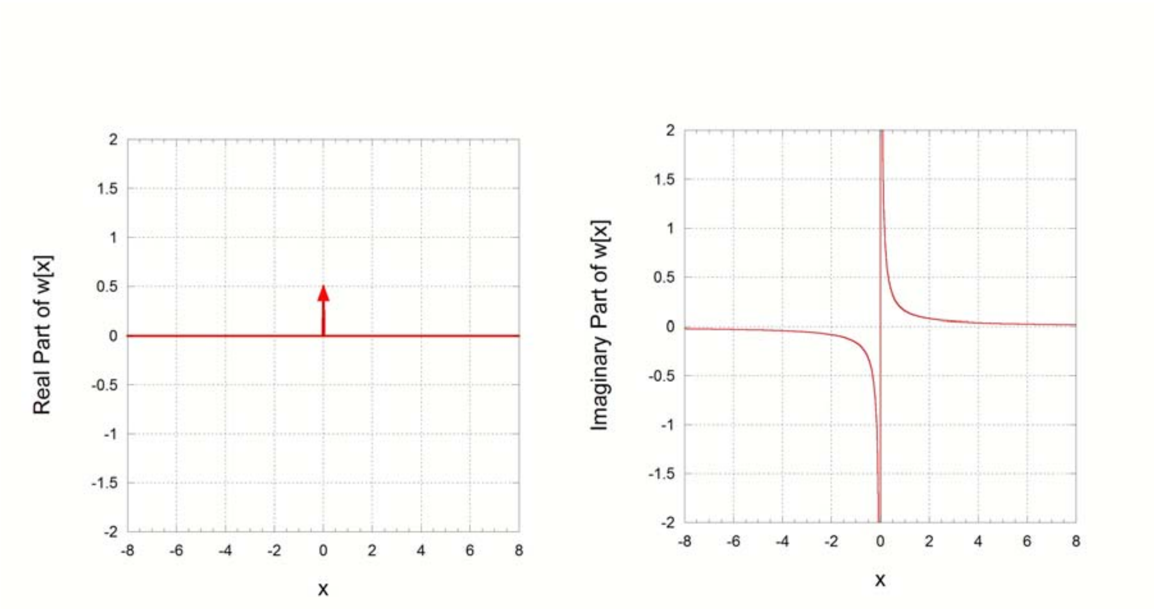
$$W[\xi] = STEP[\xi]$$

where the impulse response is obtained by the known transform for $STEP[x]$ and the "transform-

of-a-transform" theorem:

$$\begin{aligned}
\mathcal{F}\{STEP[x]\} &= \mathcal{F}\left\{\frac{1}{2}\cdot 1[x] + \frac{1}{2}\cdot SGN[x]\right\} \\
&= \frac{1}{2}\delta[\xi] + \frac{1}{2}\cdot\frac{1}{i\pi\xi} \\
&= \frac{1}{2}\delta[\xi] - i\cdot\frac{1}{2\pi\xi} \\
\mathcal{F}^{-1}\{STEP[\xi]\} &= \frac{1}{2}\delta[-x] - i\cdot\frac{1}{2\pi(-x)} \\
&= \frac{1}{2}\delta[x] + i\cdot\frac{1}{2\pi x}
\end{aligned}$$

$$\boxed{w[x] = \tfrac{1}{2}\delta[x] + i\cdot\frac{1}{2\pi x}}$$



Impulse response of filter that blocks $n[x]$ and passes $f[x]$

8. We already know that the output of a "perfect" imaging system presented with the input $f[x]$ is $f[x]$:

$$f[x] * \delta[x] = f[x]$$

The output of a second imaging system consists of the sum of two identical but translated replicas of the input, where the two replicas are displaced by the distance b_0.

(a) Write down the expression for the impulse response of the second system.
I choose to put the origin midway between the impulses to remove any intrinsic linear phase:

$$\boxed{h[x] = \delta\left[x + \frac{b_0}{2}\right] + \delta\left[x - \frac{b_0}{2}\right]}$$

though clearly you could also select

$$h[x] = \delta[x] + \delta[x \pm b_0]$$

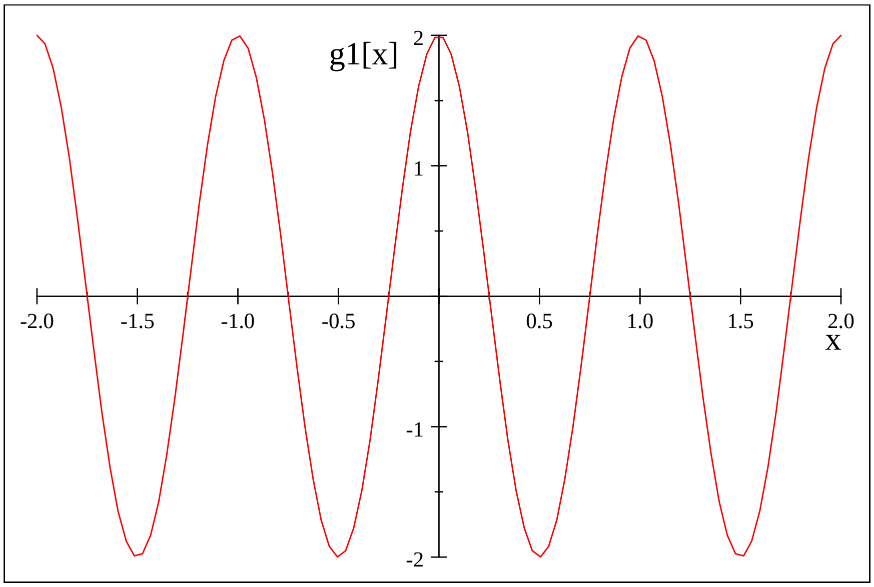
(b) Evaluate and sketch the outputs for the following inputs:

$$\begin{aligned} f_1[x] &= \cos\left[2\pi\left(\frac{x}{b_0} + \frac{1}{2}\right)\right] \\ f_2[x] &= \cos\left[2\pi\left(\frac{x}{2b_0}\right)\right] \end{aligned}$$

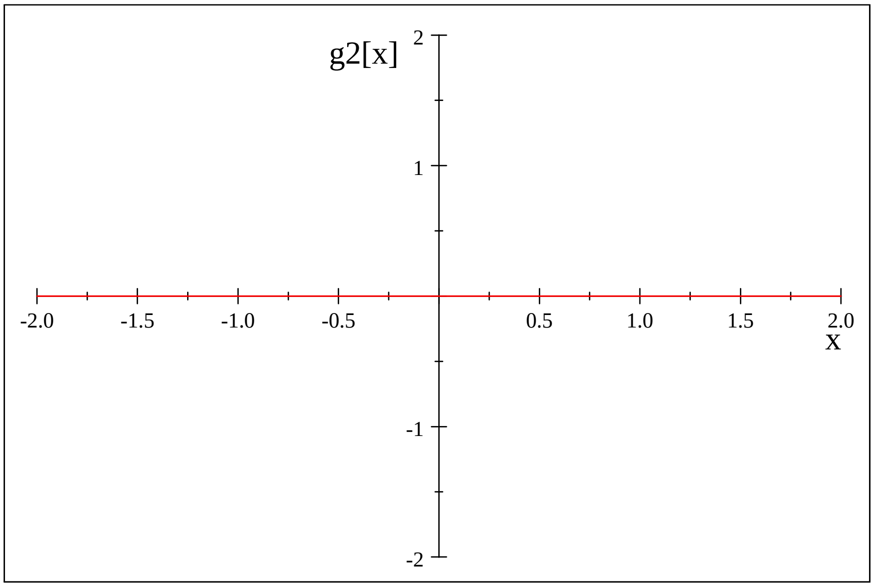
$$\begin{aligned} g_1[x] &= \cos\left[2\pi\left(\frac{x}{b_0} + \frac{1}{2}\right)\right] * \left(\delta\left[x + \frac{b_0}{2}\right] + \delta\left[x - \frac{b_0}{2}\right]\right) \\ &= \cos\left[2\pi\left(\frac{x + \frac{b_0}{2}}{b_0} + \frac{1}{2}\right)\right] + \cos\left[2\pi\left(\frac{x - \frac{b_0}{2}}{b_0} + \frac{1}{2}\right)\right] \\ &= \cos\left[2\pi\left(\frac{x}{b_0} + \frac{1}{2} + \frac{1}{2}\right)\right] + \cos\left[2\pi\left(\frac{x}{b_0} - \frac{1}{2} + \frac{1}{2}\right)\right] \\ &= \cos\left[2\pi\left(\frac{x}{b_0} + 1\right)\right] + \cos\left[2\pi\left(\frac{x}{b_0}\right)\right] \\ &= \cos\left[2\pi\frac{x}{b_0} + 2\pi\right] + \cos\left[2\pi\left(\frac{x}{b_0}\right)\right] \\ &= \cos\left[2\pi\frac{x}{b_0}\right] + \cos\left[2\pi\left(\frac{x}{b_0}\right)\right] \end{aligned}$$

$$\boxed{g_1[x] = 2 \cdot \cos\left[2\pi\left(\frac{x}{b_0}\right)\right]}$$

$$2 \cdot \cos[2\pi x]$$



$$
\begin{aligned}
g_2[x] &= \cos\left[2\pi\left(\frac{x}{2b_0}\right)\right] * \left(\delta\left[x+\frac{b_0}{2}\right]+\delta\left[x-\frac{b_0}{2}\right]\right) \\
&= \cos\left[2\pi\frac{\left(x+\frac{b_0}{2}\right)}{2b_0}\right] + \cos\left[2\pi\frac{\left(x-\frac{b_0}{2}\right)}{2b_0}\right] \\
&= \cos\left[2\pi\left(\frac{x}{2b_0}+\frac{1}{4}\right)\right] + \cos\left[2\pi\left(\frac{x}{2b_0}-\frac{1}{4}\right)\right] \\
&= \cos\left[2\pi\frac{x}{2b_0}+\frac{\pi}{2}\right] + \cos\left[2\pi\frac{x}{2b_0}-\frac{\pi}{2}\right] \\
&= \left(-\sin\left[2\pi\frac{x}{2b_0}\right]\right) + \sin\left[2\pi\frac{x}{2b_0}\right] \\
&\boxed{g_2[x] = 0[x]}
\end{aligned}
$$



(c) Evaluate and sketch the transfer function of the system:

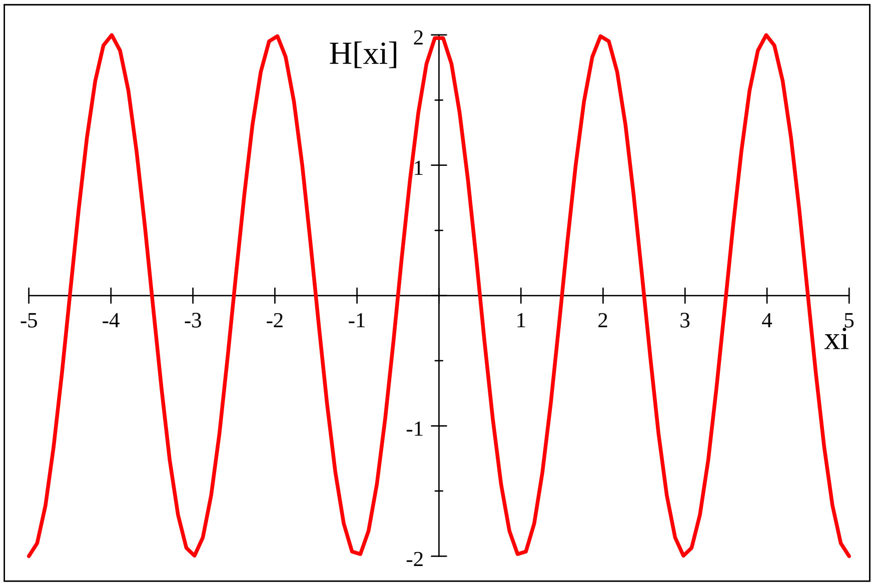
$$h[x] = \delta\left[x + \frac{b_0}{2}\right] + \delta\left[x - \frac{b_0}{2}\right]$$

$$\Longrightarrow H[\xi] = \exp\left[+i \cdot 2\pi \cdot \xi \cdot \left(-\frac{b_0}{2}\right)\right] + \exp\left[+i \cdot 2\pi \cdot \xi \cdot \left(+\frac{b_0}{2}\right)\right]$$

$$= \exp\left[-i \cdot 2\pi\xi \cdot \frac{b_0}{2}\right] + \exp\left[+i \cdot 2\pi\xi \cdot \frac{b_0}{2}\right]$$

$$H[\xi] = 2 \cdot \cos\left[2\pi\xi \cdot \frac{b_0}{2}\right]$$

The transfer function is a cosine in the frequency domain, so it has infinite support and isolated zeros.



Transfer function of the double-exposure system for $b_0 = 1$ *unit; the transfer function has zeros and infinite support.*

(d) Is it possible to extract the input $f[x]$ from the output $g[x]$ and the impulse response $h[x]$? Explain why or why not.

Since the transfer function has zeros, those frequencies may not be recovered, so the inverse filter does not exist. Note that if the input function does not include the "blocked" frequencies, then it is possible to recover that particular input signal.

9. Modify the previous problem so that the second exposure of the system is attenuated relative to the first by a factor of 2.

   (a) Write down the expression for the impulse response of the second system.
   Again, it is useful to center the impulse response:

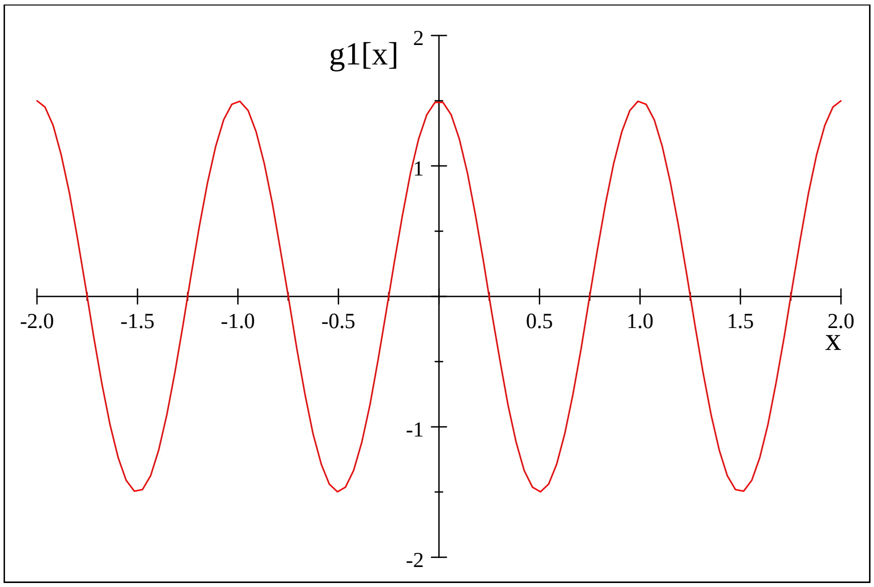
$$\begin{aligned} h[x] &= \delta\left[x+\frac{b_0}{2}\right]+\frac{1}{2}\delta\left[x-\frac{b_0}{2}\right] \\ &= \frac{1}{2}\delta\left[x+\frac{b_0}{2}\right]+\left(\frac{1}{2}\delta\left[x+\frac{x_0}{2}\right]+\frac{1}{2}\delta\left[x-\frac{x_0}{2}\right]\right) \end{aligned}$$

   (b) Evaluate and sketch the outputs for the following inputs:

$$\begin{aligned} f_1[x] &= \cos\left[2\pi\left(\frac{x}{b_0}+\frac{1}{2}\right)\right] \\ f_2[x] &= \cos\left[2\pi\left(\frac{x}{2b_0}\right)\right] \end{aligned}$$

$$\begin{aligned} g_1[x] &= \cos\left[2\pi\left(\frac{x}{b_0}+\frac{1}{2}\right)\right]*\left(\delta\left[x+\frac{b_0}{2}\right]+\frac{1}{2}\delta\left[x-\frac{b_0}{2}\right]\right) \\ &= \cos\left[2\pi\left(\frac{x+\frac{b_0}{2}}{b_0}+\frac{1}{2}\right)\right]+\frac{1}{2}\cos\left[2\pi\left(\frac{x-\frac{b_0}{2}}{b_0}+\frac{1}{2}\right)\right] \\ &= \cos\left[2\pi\left(\frac{x}{b_0}+\frac{1}{2}+\frac{1}{2}\right)\right]+\frac{1}{2}\cos\left[2\pi\left(\frac{x}{b_0}-\frac{1}{2}+\frac{1}{2}\right)\right] \\ &= \cos\left[2\pi\left(\frac{x}{b_0}+1\right)\right]+\frac{1}{2}\cos\left[2\pi\left(\frac{x}{b_0}\right)\right] \\ &= \cos\left[2\pi\frac{x}{b_0}+2\pi\right]+\frac{1}{2}\cos\left[2\pi\left(\frac{x}{b_0}\right)\right] \\ &= \cos\left[2\pi\frac{x}{b_0}\right]+\frac{1}{2}\cos\left[2\pi\left(\frac{x}{b_0}\right)\right] \end{aligned}$$

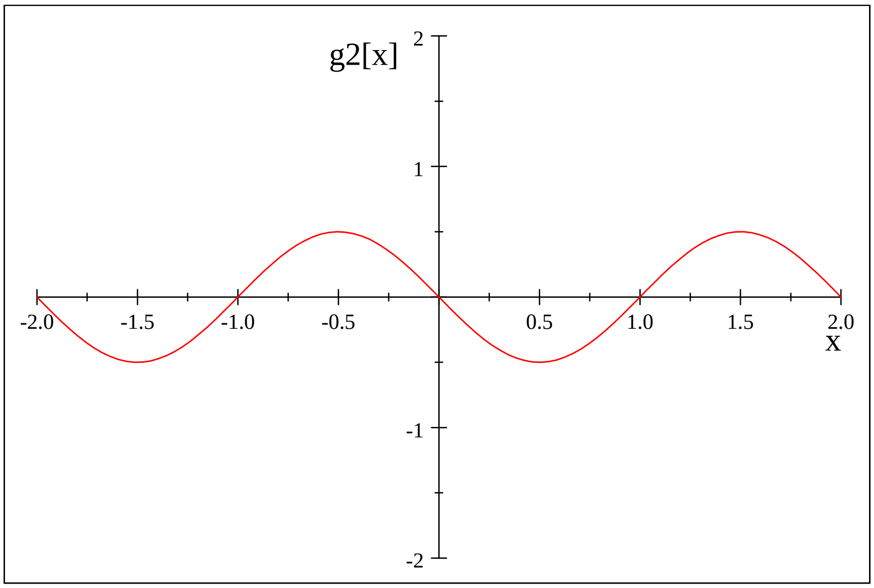
$$\boxed{g_1[x]=\frac{3}{2}\cdot\cos\left[2\pi\left(\frac{x}{b_0}\right)\right]}$$



$g_1[x]$ where the x axis is in units of b_0^{-1}

$$
\begin{aligned}
g_2[x] &= \cos\left[2\pi\left(\frac{x}{2b_0}\right)\right] * \left(\delta\left[x+\frac{b_0}{2}\right]+\frac{1}{2}\delta\left[x-\frac{b_0}{2}\right]\right)\\
&= \cos\left[2\pi\frac{\left(x+\frac{b_0}{2}\right)}{2b_0}\right]+\frac{1}{2}\cos\left[2\pi\frac{\left(x-\frac{b_0}{2}\right)}{2b_0}\right]\\
&= \cos\left[2\pi\left(\frac{x}{2b_0}+\frac{1}{4}\right)\right]+\frac{1}{2}\cos\left[2\pi\left(\frac{x}{2b_0}-\frac{1}{4}\right)\right]\\
&= \cos\left[2\pi\frac{x}{2b_0}+\frac{\pi}{2}\right]+\frac{1}{2}\cos\left[2\pi\frac{x}{2b_0}-\frac{\pi}{2}\right]\\
&= \left(-\sin\left[2\pi\frac{x}{2b_0}\right]\right)+\frac{1}{2}\sin\left[2\pi\frac{x}{2b_0}\right]
\end{aligned}
$$

$$
\boxed{g_2[x] = -\frac{1}{2}\sin\left[2\pi\frac{x}{2b_0}\right]}
$$



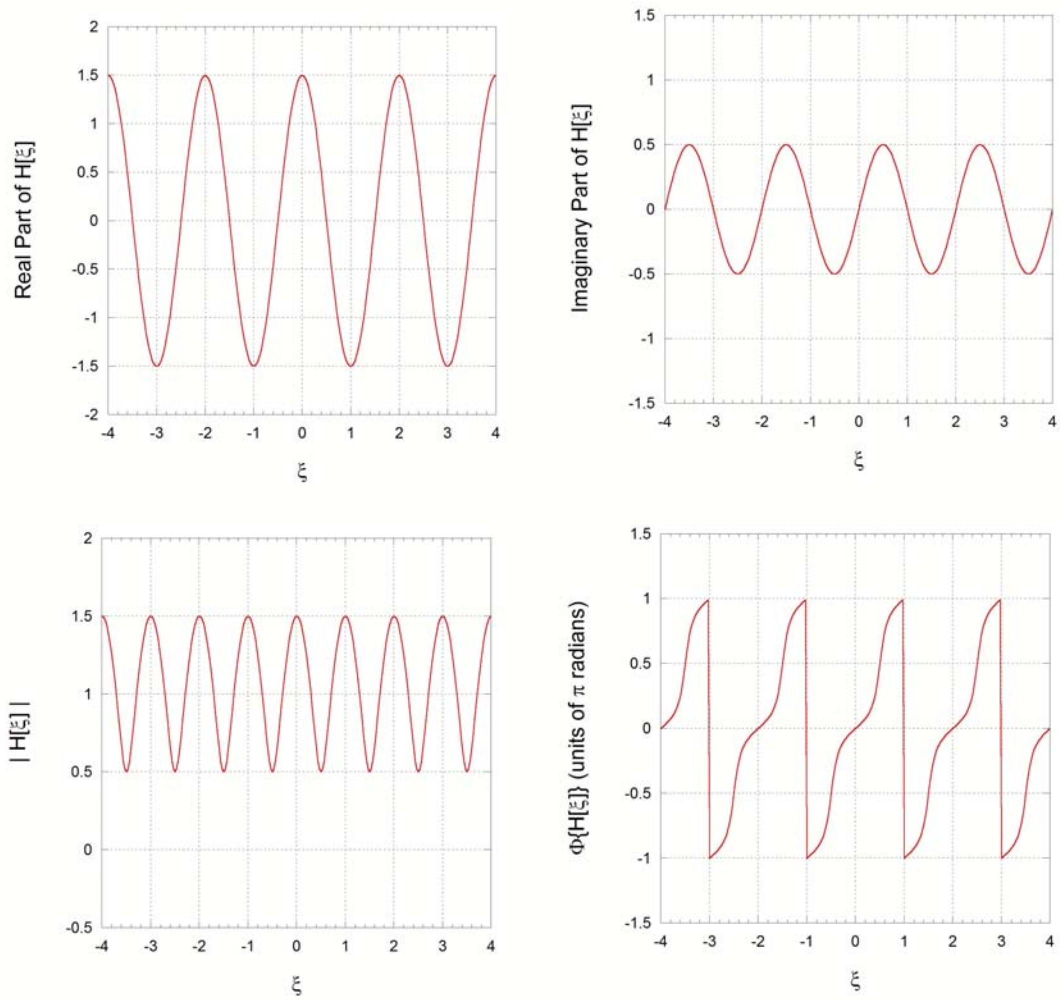
(c) Evaluate and sketch the transfer function of the system:

$$
\begin{aligned}
H[\xi] &= \frac{1}{2}\exp\left[-2\pi i\xi\frac{x_0}{2}\right]+\cos\left[2\pi\xi\left(\frac{x_0}{2}\right)\right]\\
&= \frac{3}{2}\cos\left[2\pi\xi\frac{x_0}{2}\right]-i\sin\left[2\pi\xi\frac{x_0}{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
|H[\xi]| &= \sqrt{\frac{9}{4}\cos^2\left[2\pi\xi\frac{x_0}{2}\right]+\sin^2\left[2\pi\xi\frac{x_0}{2}\right]}\\
&= \sqrt{1+\frac{5}{4}\cos^2\left[2\pi\xi\frac{x_0}{2}\right]} > 0\\
\textit{since}\ \cos^2[\pi\xi x_0] &> 0,\ \textit{then}\ |H[\xi]| \neq 0\\
&\Longrightarrow \textit{inverse filter exists! (no zeros in } H[\xi]\textit{)}
\end{aligned}
$$

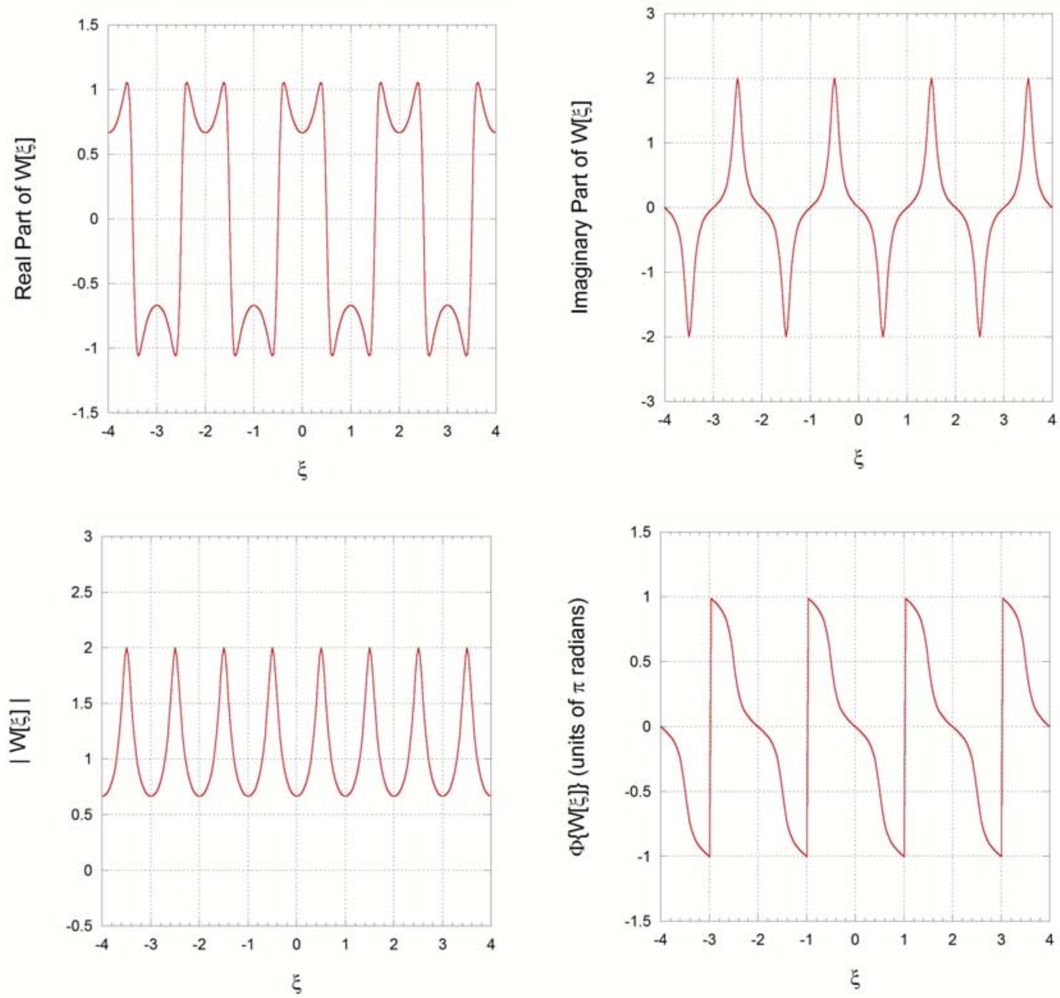
$$
\Phi\{H[\xi]\} = \tan^{-1}\left[\frac{-\sin[\pi\xi x_0]}{\frac{3}{2}\cos[\pi\xi x_0]}\right]
$$

not linear (has some wiggles in it)

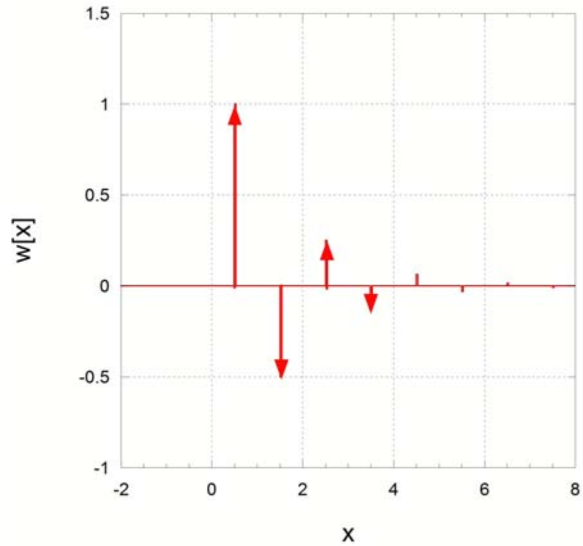


(d) Is it possible to extract the input $f[x]$ from the output $g[x]$ and the impulse response $h[x]$? Explain why or why not.

Since $|H[\xi]| \neq 0$, *it IS possible to recover the original signal. The transfer function of the inverse filter is shown:*



$W[\xi]$ for $h[x] = \delta\left[x+\frac{1}{2}\right] + \frac{1}{2}\cdot\delta\left[x-\frac{1}{2}\right]$, showing that the magnitude is the reciprocal of $|H[\xi]|$ and the phase is the negative of the phase of $H[\xi]$



Impulse response of the inverse filter, showing that it consists of an infinite set of Dirac delta functions arranged and weighted to cancel out the contributions of the two impulses in $h\,[x]$

10. Modify the problem so that the image is not created from the two exposures but rather from a single exposure made over the displacement by the distance b_0.

This is "linear-motion blur," where the object or the camera move during the exposure. The normalized impulse response is:

$$h[x] = \frac{1}{|b_0|} \cdot RECT\left[\frac{x}{b_0}\right]$$

so the transfer function is a SINC

$$H[\xi] = SINC[b_0 \cdot \xi] = SINC\left[\frac{\xi}{b_0^{-1}}\right]$$

which clearly has zeros and also is bipolar. The transfer function of the inverse filter would be the reciprocal of $H[\xi]$, which is undefined at nonzero integer multiples of b_0^{-1}, so the inverse filter does not exist. The transfer function of the pseudoinverse filter is:

$$\hat{W}[\xi] = \begin{cases} \frac{1}{SINC[b_0 \cdot \xi]} & \text{if} \quad SINC[b_0 \cdot \xi] \neq 0 \\ 0 & \text{if} \quad SINC[b_0 \cdot \xi] = 0 \end{cases}$$

$$\begin{aligned} f_1[x] &= \cos\left[2\pi\left(\frac{x}{b_0} + \frac{1}{2}\right)\right] \\ f_2[x] &= \cos\left[2\pi\left(\frac{x}{2b_0}\right)\right] \end{aligned}$$

$$\begin{aligned} g_1[x] &= \cos\left[2\pi\frac{x}{b_0} + \pi\right] * RECT\left[\frac{x}{b_0}\right] \\ &= -\cos\left[2\pi\frac{x}{b_0}\right] * RECT\left[\frac{x}{b_0}\right] = 0[x] \\ G_1[\xi] &= -\left(\frac{1}{2}\delta\left[\xi + \frac{1}{b_0}\right] + \frac{1}{2}\delta\left[\xi - \frac{1}{b_0}\right]\right) \cdot SINC[b_0 \cdot \xi] \\ &= -\frac{1}{2} \cdot \left(\delta\left[\xi + \frac{1}{b_0}\right] \cdot SINC[b_0 \cdot \xi] + \delta\left[\xi - \frac{1}{b_0}\right] \cdot SINC[b_0 \cdot \xi]\right) \\ &= -\frac{1}{2} \cdot \left(\delta\left[\xi + \frac{1}{b_0}\right] \cdot SINC\left[b_0 \cdot -\frac{1}{b_0}\right] + \delta\left[\xi - \frac{1}{b_0}\right] \cdot SINC\left[b_0 \cdot +\frac{1}{b_0}\right]\right) \\ &= -\frac{1}{2} \cdot \left(\delta\left[\xi + \frac{1}{b_0}\right] \cdot SINC[-1] + \delta\left[\xi - \frac{1}{b_0}\right] \cdot SINC[+1]\right) \\ &= -\frac{1}{2} \cdot \left(\delta\left[\xi + \frac{1}{b_0}\right] \cdot 0 + \delta\left[\xi - \frac{1}{b_0}\right] \cdot 0\right) \\ &= -\frac{1}{2} \cdot 0 = 0[\xi] \\ & \boxed{g_1[x] = 0[x]} \end{aligned}$$

$$
\begin{aligned}
g_2[x] &= \cos\left[2\pi\left(\frac{x}{2b_0}\right)\right] * RECT\left[\frac{x}{b_0}\right] \\
G_2[\xi] &= +\left(\frac{1}{2}\delta\left[\xi+\frac{1}{2b_0}\right]+\frac{1}{2}\delta\left[\xi-\frac{1}{2b_0}\right]\right)\cdot SINC\left[b_0\cdot\xi\right] \\
&= \frac{1}{2}\cdot\left(\delta\left[\xi+\frac{1}{2b_0}\right]\cdot SINC\left[b_0\cdot\xi\right]+\delta\left[\xi-\frac{1}{2b_0}\right]\cdot SINC\left[b_0\cdot\xi\right]\right) \\
&= \frac{1}{2}\cdot\left(\delta\left[\xi+\frac{1}{2b_0}\right]\cdot SINC\left[b_0\cdot-\frac{1}{2b_0}\right]+\delta\left[\xi-\frac{1}{2b_0}\right]\cdot SINC\left[b_0\cdot+\frac{1}{2b_0}\right]\right) \\
&= \frac{1}{2}\cdot\left(\delta\left[\xi+\frac{1}{2b_0}\right]\cdot SINC\left[-\frac{1}{2}\right]+\delta\left[\xi-\frac{1}{2b_0}\right]\cdot SINC\left[+\frac{1}{2}\right]\right) \\
&= \frac{1}{2}\cdot SINC\left[+\frac{1}{2}\right]\cdot\left(\delta\left[\xi+\frac{1}{2b_0}\right]+\delta\left[\xi-\frac{1}{2b_0}\right]\right) \\
&= \frac{1}{2}\cdot\frac{2}{\pi}\left(\delta\left[\xi+\frac{1}{2b_0}\right]+\delta\left[\xi-\frac{1}{2b_0}\right]\right) \\
&= \frac{1}{\pi}\cdot\left(\delta\left[\xi+\frac{1}{2b_0}\right]+\delta\left[\xi-\frac{1}{2b_0}\right]\right) \\
g_2[x] &= \frac{1}{\pi}\cdot 2\cdot\cos\left[2\pi\frac{x}{2b_0}\right] \\
&\boxed{g_1[x]=\frac{2}{\pi}\cdot\cos\left[2\pi\tfrac{x}{2b_0}\right]}
\end{aligned}
$$