IMGS-616-20141 Solution Set #8

1. The function $f[x] = SINC^2[100x]$ is sampled by the ideal COMB function with period Δx and therefore sampling "rate" $\xi_s = (\Delta x)^{-1}$. The sampled signal is passed through an ideal lowpass filter that "cuts off" at the sampling frequency.

   (a) Find the minimum sampling rate ξ_0 that will permit exact recovery of $f[x]$ from $f_s[x]$, so that $g[x] \propto f[x]$ (hint, you've done this already :-)

   Solution:

$$\begin{aligned} f[x] &= SINC^2[100x] = SINC^2\left[\frac{x}{0.01}\right] \\ g[x] &= f[x] \cdot (\xi_s \cdot COMB[\xi_s x]) \\ H[\xi] &= RECT\left[\frac{\xi}{\xi_s}\right] \end{aligned}$$

$$F[\xi] = \mathcal{F}\left\{SINC^2\left[\frac{x}{0.01}\right]\right\} = (0.01) \cdot TRI[0.01 \cdot \xi] = \frac{1}{100} \cdot TRI\left[\frac{\xi}{100}\right]$$

The maximum spatial frequency is $\xi_{\max} = 100$ *cycles per unit length*

The Nyquist frequency is $\xi_{Nyquist} = 2 \cdot \xi_{\max} = 200$ *cycles per unit length*

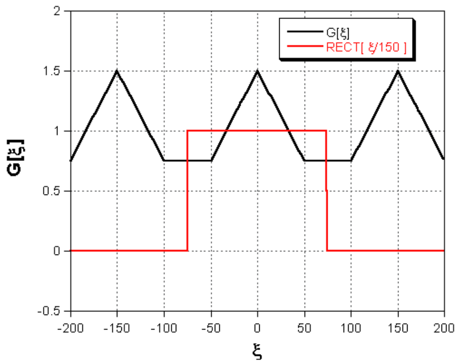
   (b) Find $g[x]$ if $\xi_s = 0.75 \cdot \xi_0$ and sketch.

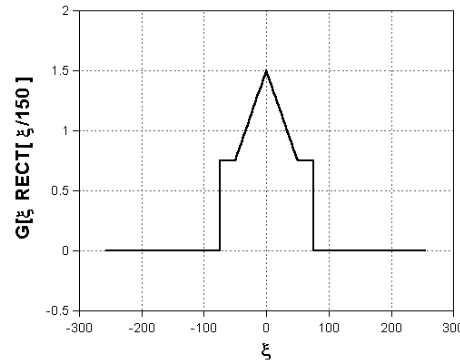
   SOLUTION:

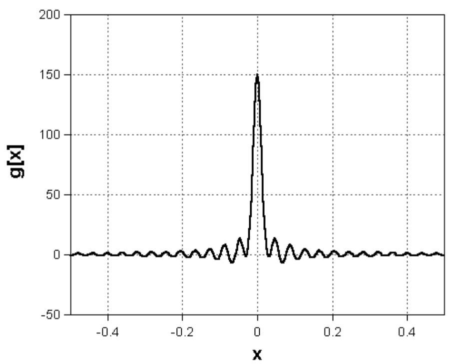
$$\begin{aligned} \xi_{Nyquist} &= 2 \cdot \xi_{\max} = 200 \Longrightarrow \xi_s = 0.75 \cdot \xi_{Nyquist} = 150 \\ \Delta x &= \frac{1}{\xi_s} = \frac{1}{150} \\ \mathcal{F}\{SAMP[x]\} &= \mathcal{F}\{150 \cdot COMB[150x]\} = COMB\left[\frac{\xi}{150}\right] \\ &= \sum_{k=-\infty}^{+\infty} \delta\left[\frac{\xi}{150} - k\right] = 150 \sum_{k=-\infty}^{+\infty} \delta[\xi - 150k] \end{aligned}$$

$$\begin{aligned} F[\xi] * COMB\left[\frac{\xi}{150}\right] &= \left(\frac{1}{100} \cdot TRI\left[\frac{\xi}{100}\right] * 150 \sum_{k=-\infty}^{+\infty} \delta[\xi - 150k]\right) \\ G[\xi] &= \left(F[\xi] * COMB\left[\frac{\xi}{150}\right]\right) \cdot \frac{1}{150} RECT\left[\frac{\xi}{150}\right] \\ &= \left(\frac{3}{2} TRI\left[\frac{\xi}{100}\right] * \sum_{k=-\infty}^{+\infty} \delta[\xi - 150k]\right) \cdot \frac{1}{150} RECT\left[\frac{\xi}{150}\right] \\ &= \frac{1}{200} \sum_{k=-\infty}^{\infty} TRI\left[\frac{\xi - 150k}{100}\right] \cdot RECT\left[\frac{\xi}{150}\right] \end{aligned}$$

This function consists of replicas of the triangle of "width" 100 separated by 150 cycles per unit length, as shown:





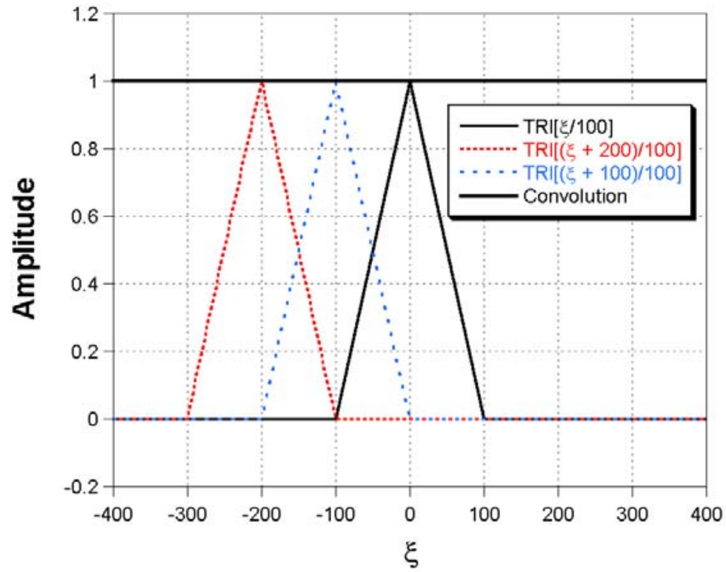


The segmented spectrum may be written in several forms, including:

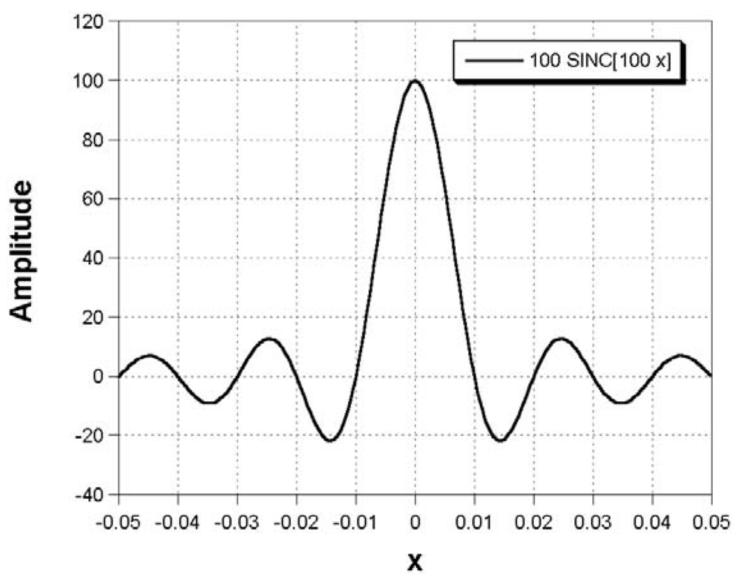
$$
\begin{aligned}
G[\xi] &= \frac{1}{200}\cdot\left(\frac{1}{2}\cdot RECT\left[\frac{\xi}{150}\right]+\frac{1}{2}\cdot TRI\left[\frac{\xi}{50}\right]\right)\\
g[x] &= \frac{1}{200}\cdot\left(75\cdot SINC[150x]+25\cdot SINC^2[50x]\right)\\
&= \frac{3}{4}\cdot SINC\left[\frac{x}{\left(\frac{1}{150}\right)}\right]+\frac{1}{4}\cdot SINC^2\left[\frac{x}{\left(\frac{1}{50}\right)}\right]
\end{aligned}
$$

(c) Find $g[x]$ if $\xi_s = 0.50\cdot\xi_0$ and sketch.
SOLUTION:

$$
\begin{aligned}
\xi_s &= 0.5\cdot\xi_{\text{Nyquist}} = 100\ \frac{\text{cycles}}{\text{unit length}}\\
f_s[x] &= SINC^2[100x]\times 100\cdot COMB[100x]\\
F_s[\xi] &= \frac{1}{100}\cdot TRI\left[\frac{\xi}{100}\right] * COMB\left[\frac{\xi}{100}\right] = \frac{1}{100}\cdot TRI\left[\frac{\xi}{100}\right] * \sum_{n=-\infty}^{+\infty}\delta\left[\frac{\xi}{100}-n\right]\\
&= \frac{1}{100}\cdot TRI\left[\frac{\xi}{100}\right] * 100\cdot\sum_{n=-\infty}^{+\infty}\delta[\xi-100n] = \sum_{n=-\infty}^{+\infty} TRI\left[\frac{\xi-100n}{100}\right] = 1[\xi]
\end{aligned}
$$



$$G_s[\xi] = 1[\xi] \cdot RECT\left[\frac{\xi}{100}\right] = RECT\left[\frac{\xi}{100}\right]$$
$$g_s[x] = 100 \cdot SINC[100x] = 100 \cdot SINC\left[\frac{x}{100^{-1}}\right]$$



2. A 1-D "50%" square wave grating (half "on" and half "off") may be written:

$$f[x] = COMB[x] * RECT[2x]$$

(a) Evaluate and sketch the spectrum $F[\xi]$; as a side comment, this is a scaled replica of the diffraction pattern from a 50% grating if viewed in the Fraunhofer diffraction region.

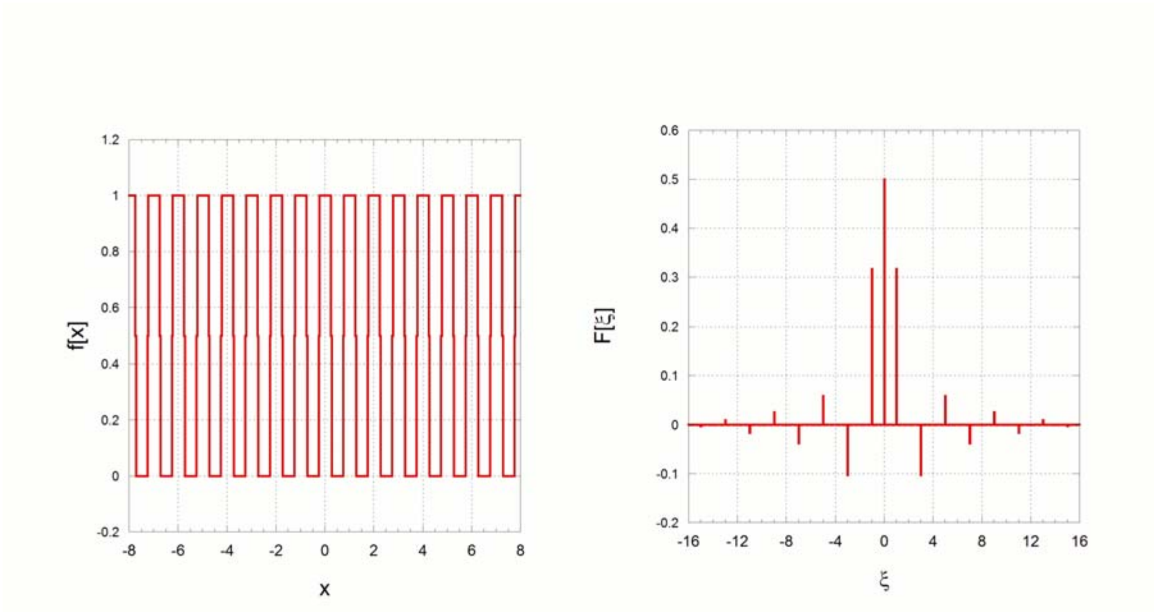
For each of the system functions $H[\xi]$ or $h[x]$ listed below, evaluate and sketch the representation, the "other representation" (i.e., $h[x]$ if $H[\xi]$ is given, or vice versa), and the output function $g[x] = f[x] * h[x]$. Classify the filters as lowpass, highpass, phase, etc.

$$\begin{aligned} f[x] &= COMB[x] * RECT[2x] \\ &= \sum_{n=-\infty}^{+\infty} \delta[x-n] * RECT\left[\frac{x}{\left(\frac{1}{2}\right)}\right] \\ &= \sum_{n=-\infty}^{+\infty} RECT\left[\frac{x-n}{\left(\frac{1}{2}\right)}\right] \end{aligned}$$

$$\begin{aligned} F[\xi] &= COMB[\xi] \cdot \frac{1}{2} SINC\left[\frac{\xi}{2}\right] \\ &= \sum_{k=-\infty}^{+\infty} \delta[\xi-k] \cdot \frac{1}{2} SINC\left[\frac{\xi}{2}\right] \\ &= \sum_{k=-\infty}^{+\infty} \delta[\xi-k] \cdot \frac{1}{2} \cdot SINC\left[\frac{k}{2}\right] \end{aligned}$$

$$\begin{aligned} k &= 0 \Longrightarrow \frac{1}{2} \cdot SINC\left[\frac{0}{2}\right] = \frac{1}{2} \\ k &= \pm 1 \Longrightarrow \frac{1}{2} \cdot SINC\left[\pm\frac{1}{2}\right] = \frac{1}{2} \cdot \frac{2}{\pi} = \frac{1}{\pi} \\ k &= \pm 2 \Longrightarrow \frac{1}{2} \cdot SINC[\pm 1] = 0 \\ k &= \pm 3 \Longrightarrow \frac{1}{2} \cdot SINC\left[\pm\frac{3}{2}\right] = \frac{1}{2} \cdot \frac{2}{3\pi} = \frac{1}{3\pi} \end{aligned}$$

$$F[\xi] = \frac{1}{2} \cdot \begin{cases} 1 & \text{if} \quad k = 0 \\ \frac{2}{k\pi} & \text{if} \quad \text{odd } k \\ 0 & \text{if} \quad \text{even } k \neq 0 \end{cases}$$

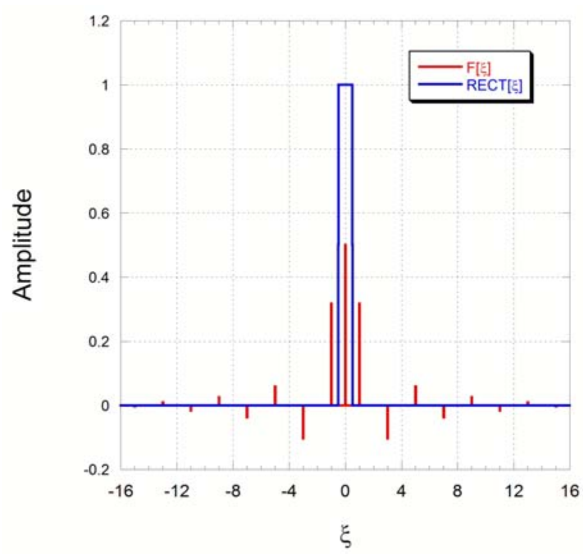


(a) periodic square-wave grating $f[x]$; (b) discrete spectrum $F[\xi]$ of square-wave grating, showing that the only nonzero terms have $k=0$ and odd values of k.

(b) $H[\xi] = RECT[\xi]$

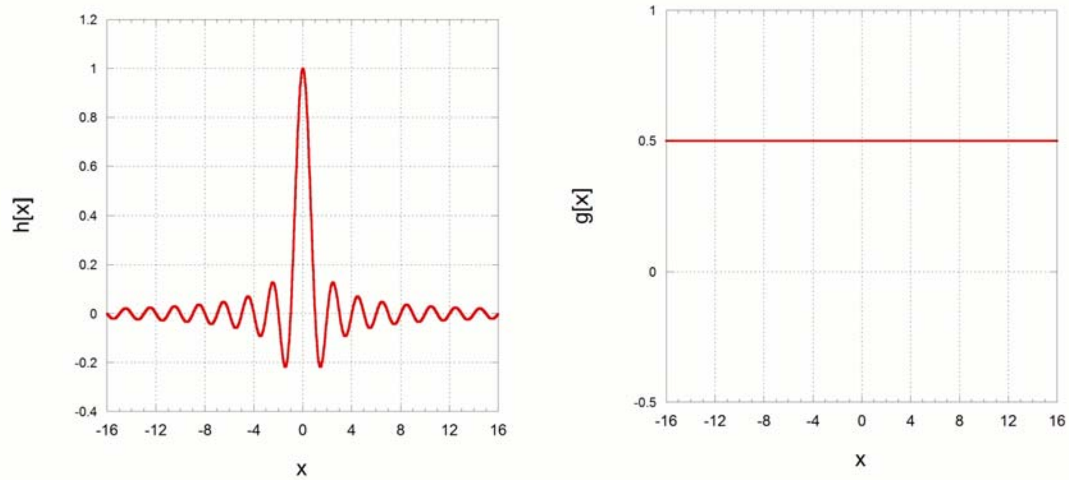
$$\begin{aligned} G[\xi] &= F[\xi]\cdot H[\xi] = COMB[\xi]\cdot\frac{1}{2}SINC\left[\frac{\xi}{2}\right]\cdot RECT[\xi] \\ h[x] &= \mathcal{F}^{-1}\{RECT[\xi]\} = SINC[x] \end{aligned}$$

Look at the plot of the frequency domain:



This shows that only the "central order" ($k=0$) is passed through the filter, so the output spectrum

$$G[\xi] = \frac{1}{2}\cdot\delta[\xi] \implies \boxed{g[x] = \frac{1}{2}\cdot 1[x]}$$


h[x]
x
g[x]
x

(c) $h[x] = RECT[x]$

$$h[x] = RECT[x] \implies H[\xi] = SINC[\xi]$$

Again, look at the plots: the SINC function evaluates to zero at the integer frequencies where the amplitude of $F[\xi]$ exists, so again, only the zero-order term gets "through" the filter:

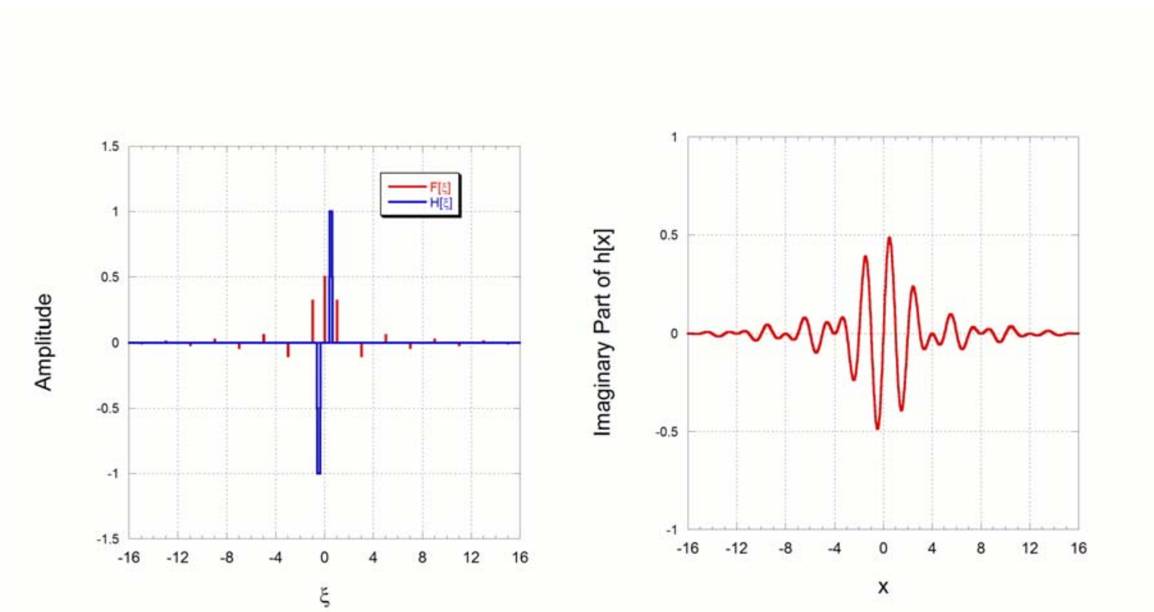
$$G[\xi] = \frac{1}{2} \cdot \delta[\xi] \implies g[x] = \frac{1}{2} \cdot 1[x]$$

(d) $H[\xi] = -RECT[4\xi + 2] + RECT[4\xi - 2]$

$$\begin{aligned} H[\xi] &= -RECT[4\xi + 2] + RECT[4\xi - 2] \\ &= -RECT\left[4 \cdot \left(\xi + \frac{1}{2}\right)\right] + RECT\left[4 \cdot \left(\xi - \frac{1}{2}\right)\right] \\ &= -RECT\left[\frac{\xi + \frac{1}{2}}{\frac{1}{4}}\right] + RECT\left[\frac{\xi - \frac{1}{2}}{\frac{1}{4}}\right] \\ &= RECT\left[\frac{\xi}{\left(\frac{1}{4}\right)}\right] * \left(-\delta\left[\xi + \frac{1}{2}\right] + \delta\left[\xi - \frac{1}{2}\right]\right) \\ &= RECT\left[\frac{\xi}{\left(\frac{1}{4}\right)}\right] * \left[-\left(\delta\left[\xi + \frac{1}{2}\right] - \delta\left[\xi - \frac{1}{2}\right]\right)\right] \end{aligned}$$

The transfer function consists of two "skinny" rectangles centered at $\xi = \pm\frac{1}{2}$, which do not pass any of the amplitude of the spectrum

$$\begin{aligned} h[x] &= \mathcal{F}^{-1}\{H[\xi]\} = \frac{1}{4} \cdot SINC\left[\frac{x}{4}\right] \cdot i \cdot \sin\left[2\pi \cdot \frac{1}{2} \cdot x\right] \\ g[x] &= f[x] * h[x] = 0[x] \end{aligned}$$



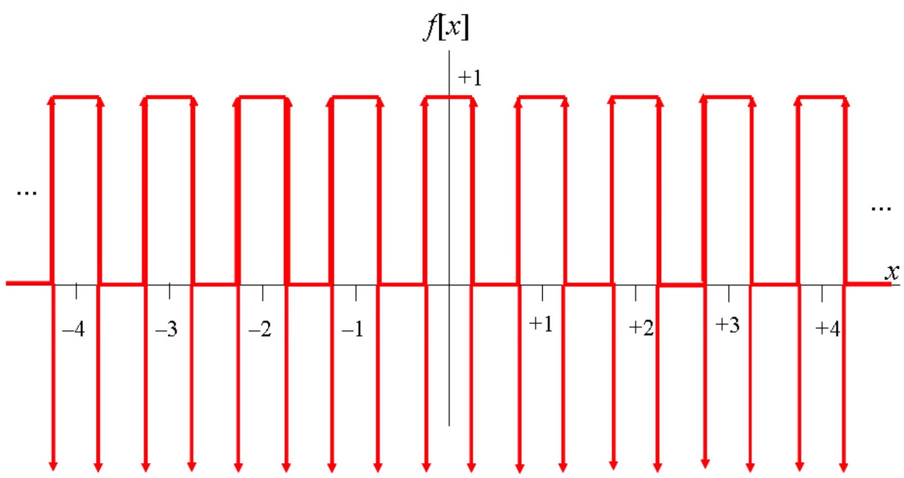
(e) $H[\xi] = 1 + \xi^2$

To find the impulse response, we need to evaluate $\mathcal{F}^{-1}\left\{\xi^2\right\}$*; use the derivative theorem:*

$$\begin{aligned}
\mathcal{F}\left\{\delta'[x] * f[x]\right\} &= (+i\cdot 2\pi\cdot\xi)\cdot F[\xi] \\
\mathcal{F}\left\{\delta''[x] * f[x]\right\} &= (+i\cdot 2\pi\cdot\xi)^2\cdot F[\xi] = -4\pi^2\xi^2\cdot F[\xi] \\
\mathcal{F}\left\{\delta''[x] * \delta[x]\right\} &= (+i\cdot 2\pi\cdot\xi)^2\cdot 1[\xi] = -4\pi^2\xi^2 \\
&\Longrightarrow \mathcal{F}\left\{-\left(\frac{1}{2\pi}\right)^2\cdot\delta''[x]\right\} = \xi^2
\end{aligned}$$

$$\begin{aligned}
h[x] &= \mathcal{F}^{-1}\left\{1+\xi^2\right\} = \delta[x] - \left(\frac{1}{2\pi}\right)^2\cdot\delta''[x] \\
g[x] &= f[x] * \left(\delta[x] - \left(\frac{1}{2\pi}\right)^2\cdot\delta''[x]\right) \\
&= f[x] * \delta[x] - \left(\frac{1}{2\pi}\right)^2\cdot\left(f[x] * \delta''[x]\right) \\
&= f[x] - \left(\frac{1}{2\pi}\right)^2\cdot f''[x]
\end{aligned}$$

so the output is the difference of the original function and a scaled second derivative, so it produces an "edge-enhanced" square wave, which means that there are "overshoots" at the edges of the square wave.



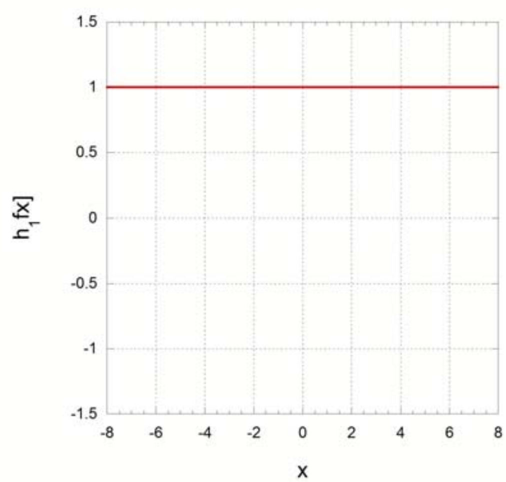
3. Sketch each of the transfer functions listed and evaluate and sketch the corresponding impulse response $h_n[x]$.

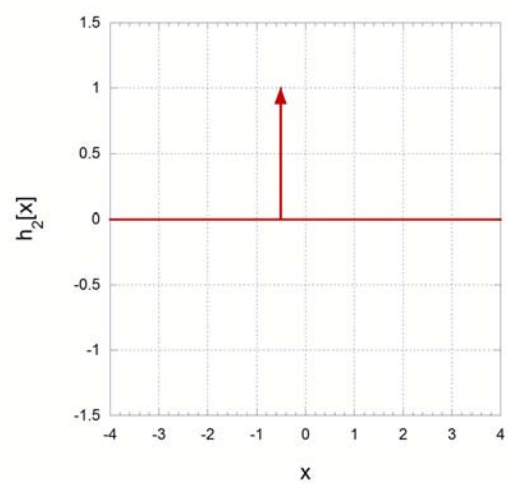
(a) $H_1[\xi] = \exp[+i\pi]$

$$\begin{aligned} H_1[\xi] &= \exp[+i\pi] = \cos[\pi] + i \cdot \sin[\pi] = -1 + i \cdot 0 \\ &= -1 \cdot 1[\xi] \\ h_1[x] &= -1 \cdot \delta[x] \end{aligned}$$

(b) $H_2[\xi] = \exp[+i\pi\xi]$

$$\begin{aligned} H_1[\xi] &= \exp[+i\pi\xi] = \exp\left[+i \cdot 2\pi \cdot \xi \cdot \frac{1}{2}\right] \\ h_1[x] &= \delta\left[x + \frac{1}{2}\right] \end{aligned}$$



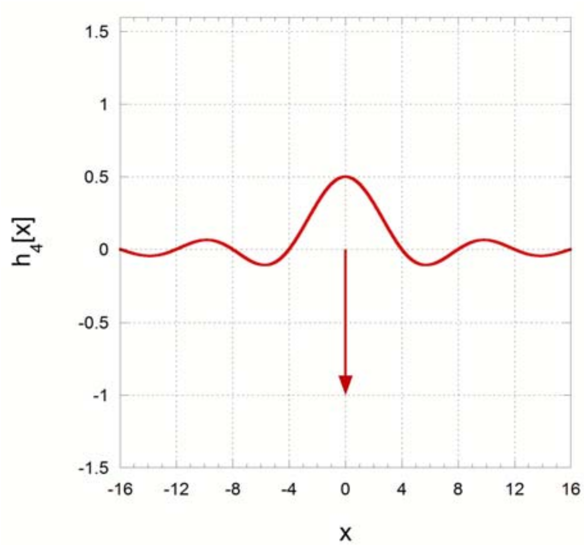


(c) $H_3[\xi] = \exp[+i\pi \cdot (1 - RECT[2\xi])]$

$$\begin{aligned}
H_3[\xi] &= \exp[+i\pi \cdot (1 - RECT[2\xi])] \\
\pi \cdot (1 - RECT[2\xi]) &= \pi \cdot \begin{cases} 0 & \text{if} \quad |\xi| < \frac{1}{4} \\ \frac{1}{2} & \text{if} \quad |\xi| = \frac{1}{4} \\ 1 & \text{if} \quad |\xi| > \frac{1}{4} \end{cases} \\
H_3[\xi] &= \begin{cases} \exp[+i \cdot 0] = 1 & \text{if} \quad |\xi| < \frac{1}{4} \\ \exp\left[+i \cdot \frac{\pi}{2}\right] = +i & \text{if} \quad |\xi| = \frac{1}{4} \\ \exp[+i \cdot \pi] = -1 & \text{if} \quad |\xi| > \frac{1}{4} \end{cases} \\
&= -1 + 2 \cdot RECT\left[\frac{\xi}{\left(\frac{1}{4}\right)}\right] + \textit{ two isolated points with imaginary amplitude}
\end{aligned}$$

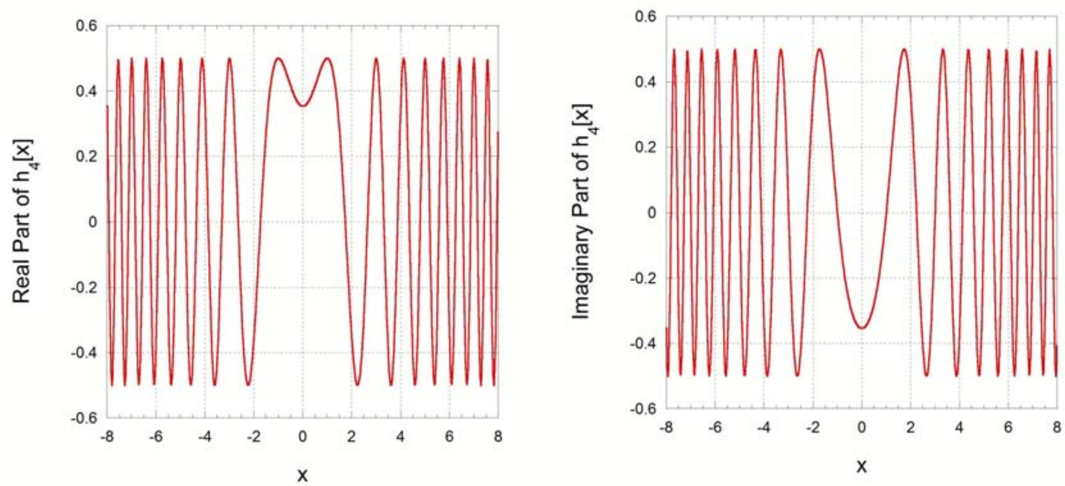
The isolated single points have no area and therefore no impact on the evaluation of the impulse response:

$$h_3[x] = -\delta[x] + \frac{1}{2} \cdot SINC\left[\frac{x}{4}\right]$$



(d) $H_4[\xi] = \exp\left[-i\pi \cdot (2\xi)^2\right]$

$$H_4[\xi] = \exp\left[-i\pi \cdot (2\xi)^2\right] = \exp\left[-i \cdot \pi \cdot \left(\frac{\xi}{\left(\frac{1}{2}\right)}\right)^2\right]$$
$$h_4[x] = \frac{1}{2} \cdot \exp\left[-i\frac{\pi}{4}\right] \cdot \exp\left[+i \cdot \pi \cdot \left(\frac{x}{2}\right)^2\right]$$



(e) (OPTIONAL BONUS) $H_5[\xi] = SINC[2\xi - 4] \cdot \exp\left[-i\pi \cdot (2\xi)^2\right]$ (HINT: stationary phase)

Solution: *From §13:*

$$f[x] = r[x] \cdot e^{+i \cdot \mathbf{\Phi}[x]}$$
$$\Longrightarrow \hat{F}[|\xi| \gg 0] \cong r[x_0] \cdot \sqrt{\frac{2\pi}{\xi \cdot \mu''[x_0]}} \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[+i\xi \cdot \mu[x_0]\right] \tag{13.67}$$

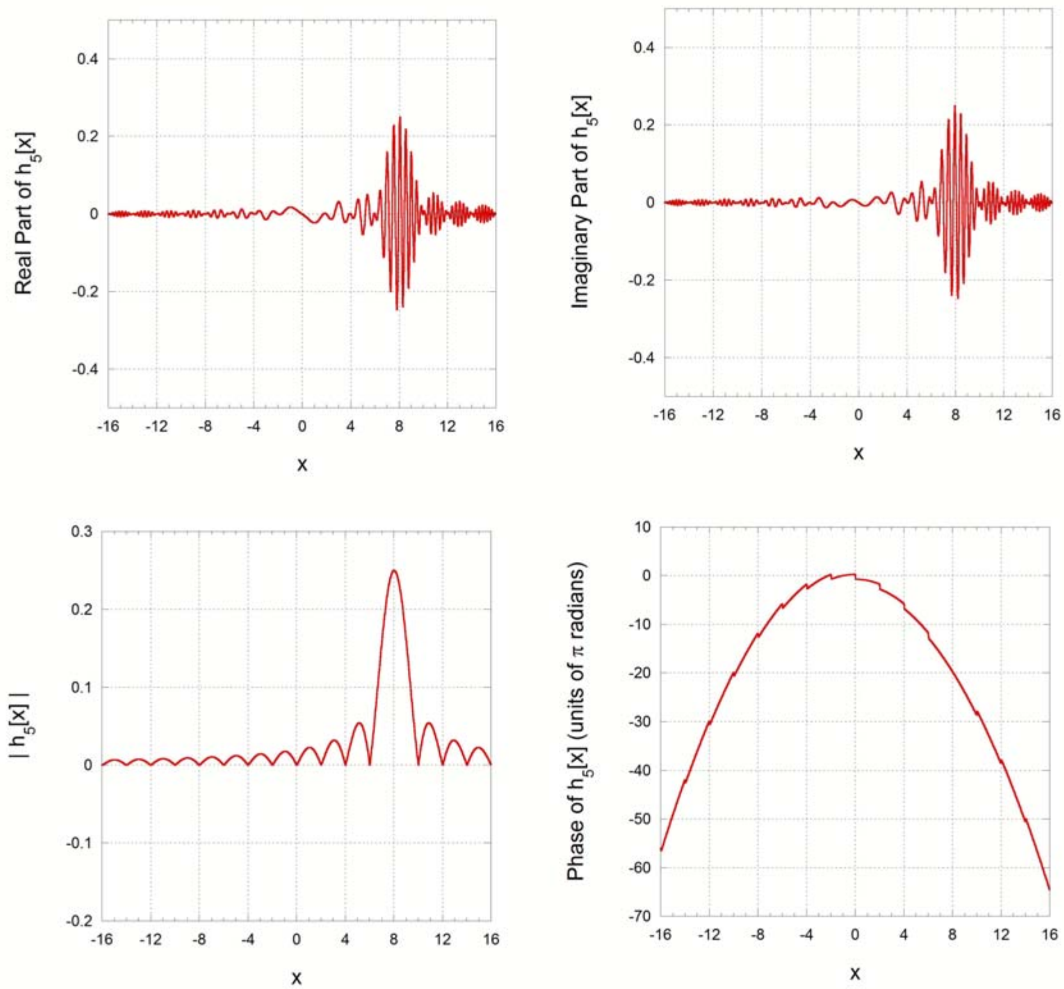
From which we can derive the inverse transform:

$$\begin{aligned} h[x] &= \int_{-\infty}^{+\infty} H[\xi] \cdot \exp[+i \cdot 2\pi \cdot \xi x] \; d\xi \\ &= \int_{-\infty}^{+\infty} R[\xi] \cdot \exp[+i \cdot \Phi[\xi]] \cdot \exp[+i \cdot 2\pi \cdot \xi x] \; d\xi \\ &\cong R[\xi_0] \cdot \sqrt{\frac{2\pi}{x \cdot \mu''[\xi_0]}} \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp[+ix \cdot \mu[\xi_0]] \end{aligned}$$

$$\begin{aligned} R[\xi] &= SINC[2\xi - 4] = SINC[2 \cdot (\xi - 2)] = SINC\left[\frac{\xi - 2}{\left(\frac{1}{2}\right)}\right] \\ \mathbf{\Phi}[\xi] &= -\pi \cdot (2\xi)^2 + 2\pi\xi x = -\pi \cdot \left(4\xi^2 - 2\xi x\right) \\ \mu[\xi] &= \frac{\mathbf{\Phi}[\xi]}{x} = -\pi \cdot \left(4\frac{\xi^2}{x} - 2\xi\right) \\ \mu'[\xi] &= -\pi \cdot \left(8\frac{\xi}{x} - 2\right) = -2\pi \cdot \left(4\frac{\xi}{x} - 1\right) \\ \mu'[\xi_0] &= 0 = -2\pi \cdot \left(4\frac{\xi_0}{x} - 1\right) \Longrightarrow 4\frac{\xi_0}{x} = 1 \Longrightarrow \boxed{\xi_0 = \frac{x}{4}} \\ \mu''[\xi] &= -2\pi \cdot \frac{4}{x} \Longrightarrow \boxed{\mu''[\xi_0] = -\frac{8\pi}{x}} \\ \mu[\xi_0] &= -\pi \cdot \left(4\frac{\left(\frac{x}{4}\right)^2}{x} - 2\frac{x}{4}\right) = \boxed{\mu[\xi_0] = +\frac{\pi x}{4}} \end{aligned}$$

$$\begin{aligned} h_5[x] &\cong R[\xi_0] \cdot \sqrt{\frac{2\pi}{x \cdot \mu''[\xi_0]}} \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp[+ix \cdot \mu[\xi_0]] \\ &= SINC\left[2 \cdot \frac{x}{4} - 4\right] \cdot \sqrt{\frac{2\pi}{x \cdot \frac{8\pi}{x}}} \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[+ix \cdot \left(-\frac{\pi x}{4}\right)\right] \\ &= SINC\left[\frac{x}{2} - 4\right] \cdot \frac{1}{4} \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi \cdot \left(\frac{x}{2}\right)^2\right] \\ &= SINC\left[\frac{x-8}{2}\right] \cdot \frac{1}{4} \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi \cdot \left(\frac{x}{2}\right)^2\right] \end{aligned}$$

$$\begin{aligned} h_5[x] &\cong \left(\frac{1}{4} \cdot \exp\left[+i\frac{\pi}{4}\right]\right) \cdot SINC\left[\frac{x-8}{2}\right] \cdot \exp\left[-i\pi \cdot \left(\frac{x}{2}\right)^2\right] \\ &= \frac{\sqrt{2}}{8} \cdot (1+i) \cdot SINC\left[\frac{x-8}{2}\right] \cdot \exp\left[-i\pi \cdot \left(\frac{x}{2}\right)^2\right] \end{aligned}$$


Real Part of h5[x]
Imaginary Part of h5[x]
| h5[x] |
Phase of h5[x] (units of π radians)
x

4. The following signals are applied separately to LSI systems with impulse response $h[x]$ to be determined:

$$\begin{aligned} s_1[x] &= e^{-x} \cdot STEP[x] \\ s_2[x] &= RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9]) \end{aligned}$$

For each case:

(a) Describe the impulse response $h[x]$ and transfer function $H[\xi]$ of the matched filter that will maximize the output at $x = 2$. (assume that $H[0] = 1$). Sketch the output.

Solution for $s_1[x]$

$$a_1 : \textit{maximum output at } x = 2$$

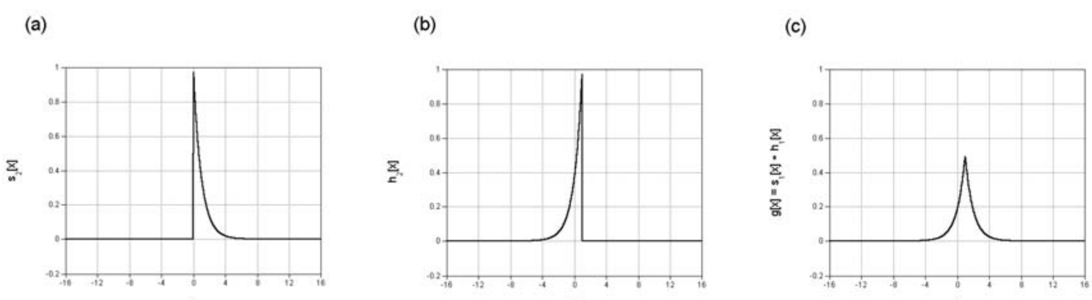
$$\textit{HINT: find a function that produces maximum at } x = 0 \textit{ and then translate via } \delta[x-2]$$

$$s_1[x] \bigstar s_1[x] = s_1[x] * s_1^*[-x] \;\; \textit{is hermitian with maximum at origin}$$

$$\begin{aligned} h[x] = s_1^*[-x] * \delta[x-2] &= \left(e^{-(-x)} \cdot STEP[-x]\right)^* * \delta[x-2] \\ &= e^{+x} \cdot STEP[-x] * \delta[x-2] \\ = e^{+x-2} \cdot STEP[-(x-2)] &= \boxed{e^{x-2} \cdot STEP[-x+2] = h[x]} \end{aligned}$$

$$\begin{aligned} H[\xi] &= \mathcal{F}_1\left\{e^{x-2} \cdot STEP[-x+2]\right\} = \left(\left(\mathcal{F}_1\left\{e^{-x} \cdot STEP[x]\right\}\right)\Big|_{\xi \to -\xi}\right) \cdot \mathcal{F}_1\{\delta[x-2]\} \\ &= \frac{1}{1+2\pi i(-\xi)} \cdot \exp[-2\pi i \cdot 2 \cdot \xi] = \boxed{\frac{\exp[-4\pi i\xi]}{1-2\pi i\xi} = H[\xi]} \end{aligned}$$

$$\begin{aligned} (s_1[x] \bigstar s_1[x]) * \delta[x-2] &= \mathcal{F}_1^{-1}\left\{|S_1[\xi]|^2\right\} * \delta[x-2] \\ &= \mathcal{F}_1^{-1}\left\{\left|\frac{1}{1+2\pi i\xi}\right|^2\right\} * \delta[x-2] \\ &= \mathcal{F}_1^{-1}\left\{\frac{1}{1+(2\pi\xi)^2}\right\} * \delta[x-2] = \frac{1}{2}e^{-|x-2|} \end{aligned}$$

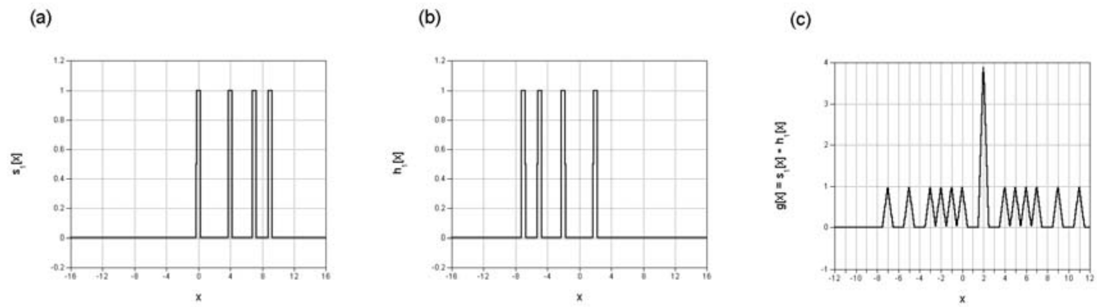


Solution for $s_2[x]$:

$$\begin{aligned} a_2 &: s_2[x] = RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9]) \\ (s_2[-x])^* &= RECT[2x] * (\delta[x] + \delta[x+4] + \delta[x+7] + \delta[x+9]) \end{aligned}$$

$$\boxed{h_2[x] = (s_2[-x])^* * \delta[x-2] = RECT[2x] + (\delta[x-2] + \delta[x+2] + \delta[x+5] + \delta[x+7])}$$

$$
\begin{aligned}
s_2[x] * h_2[x] &= (RECT[2x] * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9])) \\
&* RECT[2x] + (\delta[x-2] + \delta[x+2] + \delta[x+5] + \delta[x+7]) \\
&= (RECT[2x] * RECT[2x]) * (\delta[x] + \delta[x-4] + \delta[x-7] + \delta[x-9]) \\
&* (\delta[x-2] + \delta[x+2] + \delta[x+5] + \delta[x+7]) \\
&= \frac{1}{2} TRI[2x] \\
&* \delta[x+7] + \delta[x+5] + \delta[x+3] + \delta[x+2] + \delta[x+1] + \delta[x] + 4 \cdot \delta[x-2] \\
&+ \delta[x-4] + \delta[x-5] + \delta[x-6] + \delta[x-7] + \delta[x-9] + \delta[x-11]
\end{aligned}
$$



(b) Is it possible to construct a transfer function $H[\xi]$ that, when applied to $s[x]$, will produce $g[x] = \delta[x-2]$? Explain your reasoning.

Solution: *since the first function has no zeros in its spectrum (except at $\xi = \pm\infty$), an "inverse" matched filter exists, whereas the second function has zeros in its spectrum due to the rectangle function, and thus the "inverse matched" filter may not be constructed.*

5. Given an LSI system and a "phase-function" input:

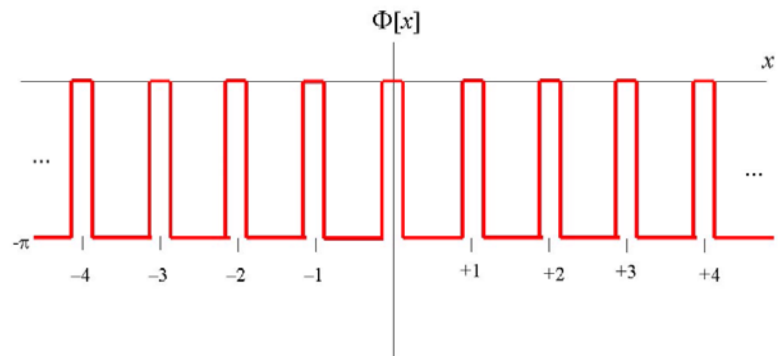
$$f[x] = 1[x] \cdot \exp[+i\Phi[x]]$$
$$\Phi[x] = -\pi + \pi\left(COMB[x] \cdot RECT\left[\frac{x}{101}\right]\right) * RECT[4x]$$

The unit magnitude indicates that $f[x]$ has infinite support.

(a) Sketch the phase function $\Phi[x]$ and the input function $f[x]$

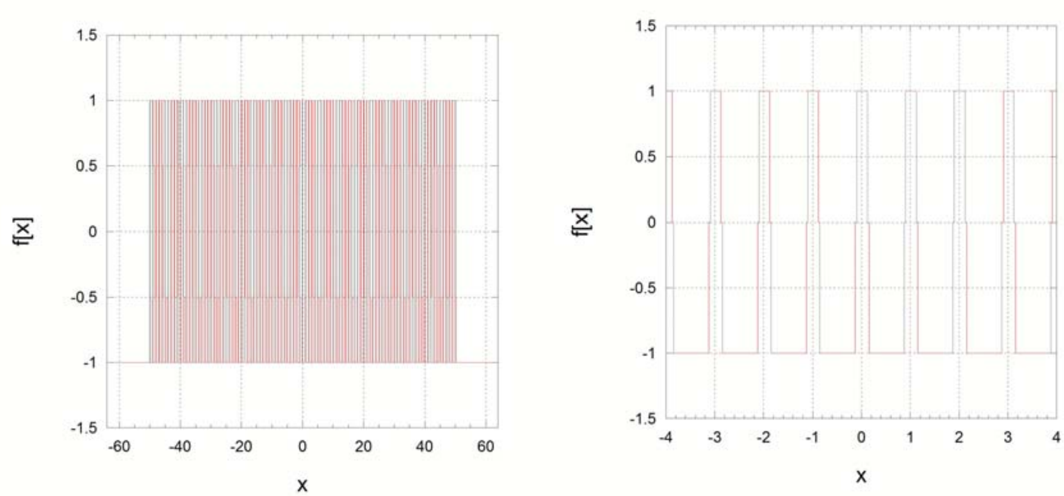
$$\Phi[x] = -\pi \cdot 1[x] + \pi\left(COMB[x] \cdot RECT\left[\frac{x}{101}\right]\right) * RECT\left[\frac{x}{\left(\frac{1}{4}\right)}\right]$$

The phase function includes a square-wave grating with a 25% duty cycle (at the upper level 25% of the distance and at the lower level 75%). The rectangle constrains the COMB so that there are 101 copies of the Dirac delta function at unit spacing centered at $-50 \leq x \leq +50$. This is convolved with a rectangle of width $\frac{1}{4}$ to make the grating and then is scaled by π to values of $+\pi$ and 0, *except at the endpoints. The subtraction of π limits the values to 0 where the rectangle is "on" and to $-\pi$ where the rectangle is "off".*



The prescription for the input function is:

$$f[x] = \begin{cases} \exp[-i \cdot 0] = +1 & \text{if} \quad RECT[4x] = 1 \\ \exp\left[-i \cdot \frac{\pi}{2}\right] = -i & \text{if} \quad RECT[4x] = \frac{1}{2} \\ \exp[-i \cdot \pi] = -1 & \text{if} \quad RECT[4x] = 0 \end{cases}$$



(a) $f[x]$ over domain $-64 \leq x \leq +64$, showing the region where the function oscillates between $f = \pm 1$.

(b) Show that the input may be written as:

$$f[x] = 2 \cdot \left[COMB[x] \cdot RECT\left[\frac{x}{101}\right]\right] * RECT[4x] - 1$$

This is apparent from the graph in part (a)

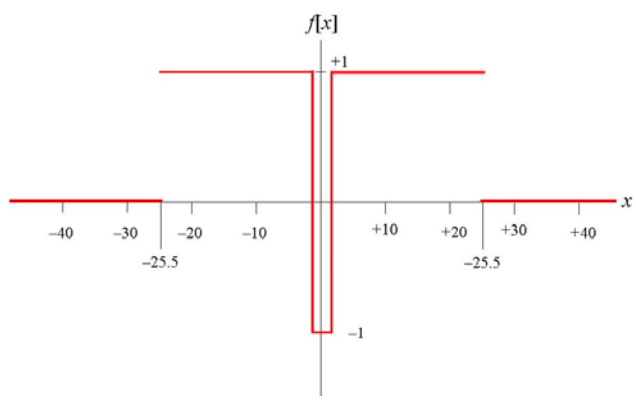
(c) Sketch the transfer function:

$$H[\xi] = RECT\left[\frac{\xi}{51}\right] \cdot \exp[+i\pi \cdot RECT[8\xi]]$$

$$\exp[+i\pi \cdot RECT[8\xi]] = \begin{cases} \exp[+i\pi \cdot 1] = -1 & \text{if} \quad RECT[8\xi] = 1 \implies |\xi| < \frac{1}{16} \\ \exp\left[+i\pi \cdot \frac{1}{2}\right] = i & \text{if} \quad RECT[8\xi] = \frac{1}{2} \implies |\xi| = \frac{1}{16} \\ \exp[+i\pi \cdot 0] = +1 & \text{if} \quad RECT[8\xi] = 0 \implies |\xi| > \frac{1}{16} \end{cases}$$

$$H[\xi] = \begin{cases} 0 & \text{if} \quad \xi < -25.5 \\ +1 & \text{if} \quad -25.5 < \xi < -\frac{1}{16} \\ i & \text{if} \quad \xi = -\frac{1}{16} \\ -1 & \text{if} \quad -\frac{1}{16} < \xi < +\frac{1}{16} \\ i & \text{if} \quad \xi = +\frac{1}{16} \\ +1 & \text{if} \quad +\frac{1}{16} < \xi < +25.5 \\ 0 & \text{if} \quad \xi > +25.5 \end{cases}$$

$$= +RECT\left[\frac{\xi}{51}\right] - 2 \cdot RECT[8\xi] + \textit{isolated points}$$



$$h[x] = +51 \cdot SINC\left[\frac{x}{\left(\frac{1}{51}\right)}\right] - 2 \cdot \frac{1}{8} \cdot SINC\left[\frac{x}{8}\right]$$

$$= \left(51 \cdot SINC\left[\frac{x}{\left(\frac{1}{51}\right)}\right]\right) - \left(\frac{1}{4} \cdot SINC\left[\frac{x}{8}\right]\right)$$

$$G[\xi] \cong F[\xi] - 2 \cdot F[0]$$

(d) Show that the output $g[x] = f[x] * h[x]$ is approximately equal to:

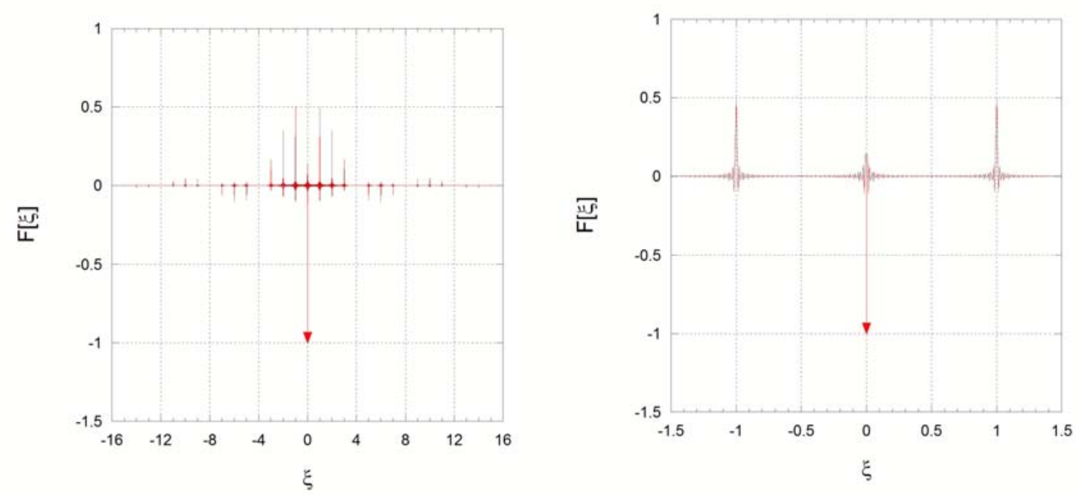
$$g[x] \cong 2\left[COMB[x] \cdot RECT\left[\frac{x}{101}\right]\right] * RECT[4x]$$

To determine the impact of the transfer function, we need to evaluate $F[\xi]$*:*

$$f[x] = 2 \cdot \left[COMB[x] \cdot RECT\left[\frac{x}{101}\right]\right] * RECT[4x] - 1$$

$$F[\xi] = 2\cdot\left(COMB[\xi] * 101\cdot SINC\left[\frac{\xi}{\left(\frac{1}{101}\right)}\right]\right)\cdot\frac{1}{4}SINC\left[\frac{\xi}{4}\right] - \delta[\xi]$$
$$= \frac{1}{2}\cdot\left(COMB[\xi] * 101\cdot SINC\left[\frac{\xi}{\left(\frac{1}{101}\right)}\right]\right)\cdot SINC\left[\frac{\xi}{4}\right] - \delta[\xi]$$

In the parentheses, the Dirac delta functions in the COMB are replaced with "tall and skinny" SINC functions; the entire set is modulated by the "wider" SINC. An approximate sketch is shown, with a detail (magnified) view.

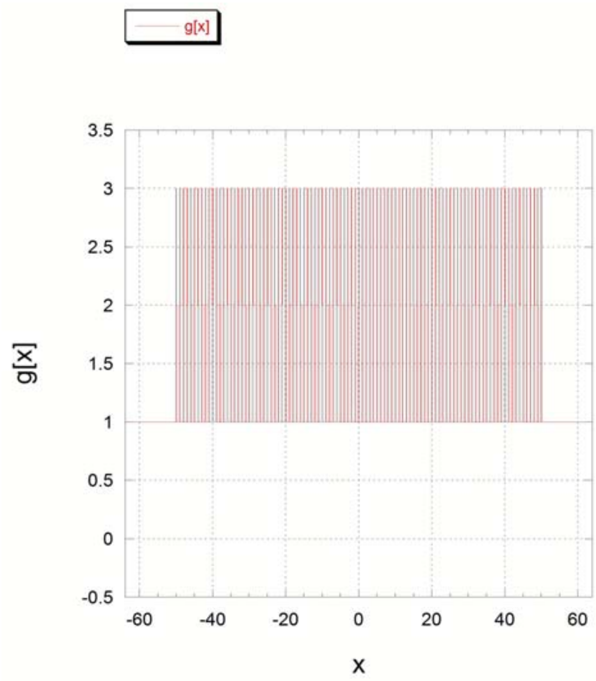


The transfer function $H[\xi]$ multiplies the spectrum $F[\xi]$ and "inverts" the DC component ($\xi = 0$) and "cuts off" the components at large spatial frequency ($|\xi| > 25.5$). The amplitude of the "wider" SINC function in $F[\xi]$ (i.e., the term $SINC\left[\frac{\xi}{4}\right]$), has already attenuated these high-frequency components, so there is little effect of the high-frequency cutoff of the transfer function; to a good approximation, $H[\xi]$ only "inverts" the DC component of $F[\xi]$. The output spectrum therefore is:

$$G[\xi] \cong \frac{1}{2}\cdot\left(COMB[\xi] * 101\cdot SINC\left[\frac{\xi}{\left(\frac{1}{101}\right)}\right]\right)\cdot SINC\left[\frac{\xi}{4}\right] - (-\delta[\xi])$$
$$= \frac{1}{2}\cdot\left(COMB[\xi] * 101\cdot SINC\left[\frac{\xi}{\left(\frac{1}{101}\right)}\right]\right)\cdot SINC\left[\frac{\xi}{4}\right] + \delta[\xi]$$

$$g[x] \cong f[x] + 2$$
$$= 2\cdot\left[COMB[x]\cdot RECT\left[\frac{x}{101}\right]\right] * RECT[4x] + 1$$

(e) Sketch $g[x]$


g[x]
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-60
-40
-20
0
20
40
60
x

6. For $f[x] = \delta[x+2]$

(a) evaluate the *M-C-M* chirp Fourier transform, showing the output after each M or C operation

Solution: *First, note that the spectrum is:*

$$\begin{aligned} F[\xi] &= \exp[-i \cdot 2\pi \cdot (-2) \cdot \xi] = \exp[+i \cdot 2\pi \cdot 2\xi] \\ F\left[\frac{x}{\alpha_0^2}\right] &= \exp\left[+i \cdot 2\pi \cdot 2\frac{x}{\alpha_0^2}\right] \end{aligned}$$

$$F\left[\frac{x}{\alpha_0^2}\right] = \left(\left(f[x] \cdot \exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right) * \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right) \cdot \exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]$$

After first multiplication:

$$\begin{aligned} \delta[x+2] \cdot \exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right] &= \delta[x+2] \cdot \exp\left[-i\pi\left(\frac{-2}{\alpha_0}\right)^2\right] \\ &= \delta[x+2] \cdot \exp\left[-i\pi\left(\frac{2}{\alpha_0}\right)^2\right] \end{aligned}$$

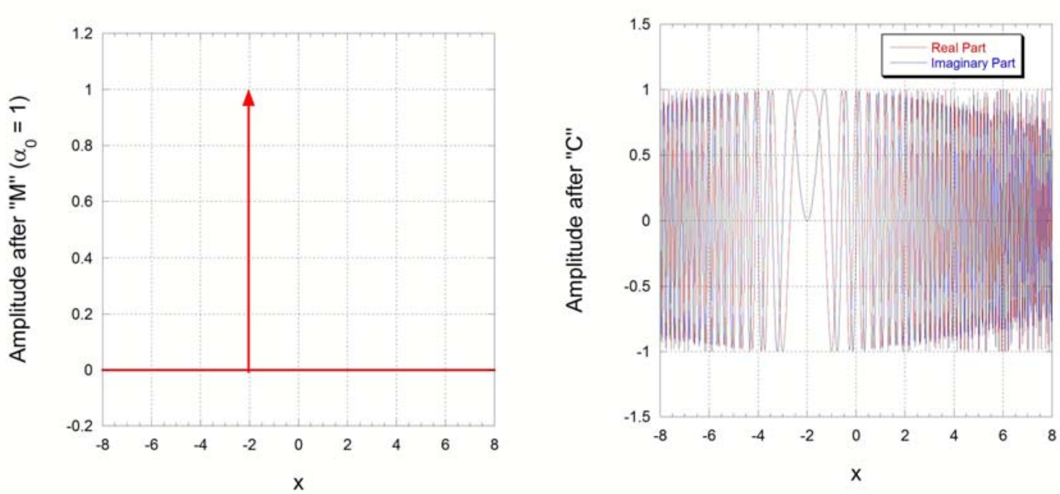
which is an "area-weighted" Dirac delta function, where the weighting is a complex-valued numerical constant that depends on α_0. In the graph, $\alpha_0 = 1$, which means that the constant evaluates to:

$$\exp\left[-i\pi\left(\frac{2}{1}\right)^2\right] = 1$$

After convolution:

$$\begin{aligned} &\left(\delta[x+2] \cdot \exp\left[-i\pi\left(\frac{2}{\alpha_0}\right)^2\right]\right) * \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \\ &= \exp\left[-i\pi\left(\frac{2}{\alpha_0}\right)^2\right] \cdot \left(\delta[x+2] * \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right) \\ &= \exp\left[-i\pi\left(\frac{2}{\alpha_0}\right)^2\right] \cdot \exp\left[+i\pi\left(\frac{x+2}{\alpha_0}\right)^2\right] \end{aligned}$$

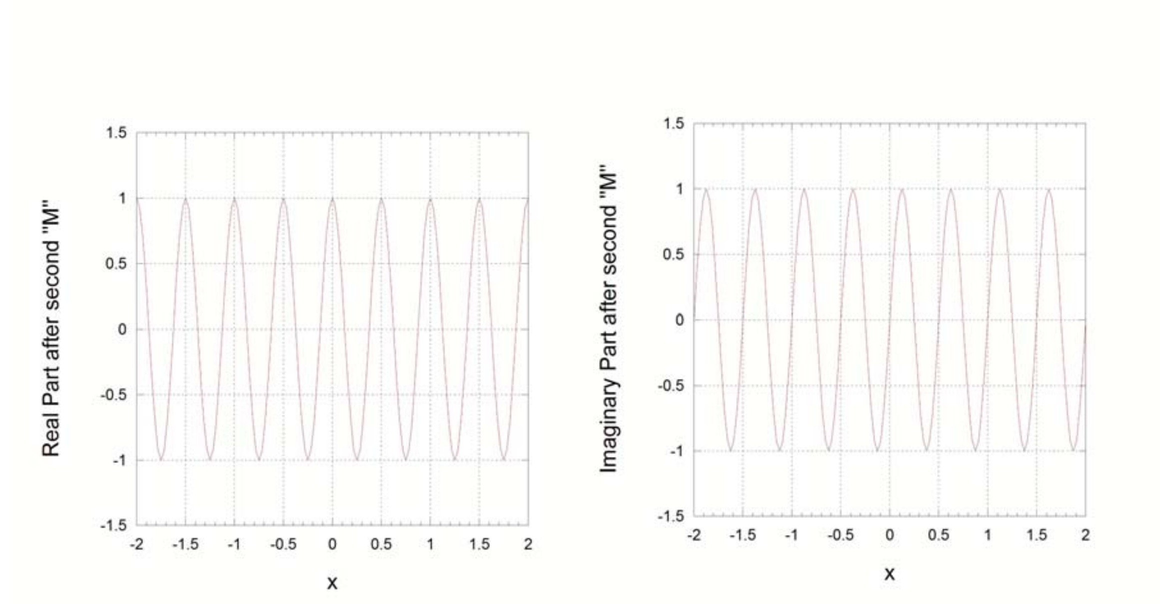
which is a translated "upchirp" scaled by the leading constant.



The second multiplication is the product of a centered with a "translated" chirp; you have already done this where you evaluated the convolution of a pair of impulses with a quadratic-phase function. The result is:

$$\begin{aligned}
&\exp\left[-i\pi\left(\frac{2}{\alpha_0}\right)^2\right]\cdot\exp\left[+i\pi\left(\frac{x+2}{\alpha_0}\right)^2\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\\
=\;&\exp\left[-i\pi\left(\frac{2}{\alpha_0}\right)^2\right]\cdot\exp\left[+i\pi\left(\frac{x^2+4+4x}{\alpha_0^2}\right)\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\\
=\;&\exp\left[-i\pi\left(\frac{2}{\alpha_0}\right)^2\right]\cdot\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\cdot\exp\left[+i\pi\left(\frac{2}{\alpha_0}\right)^2\right]\cdot\exp\left[+i\pi\left(\frac{4x}{\alpha_0^2}\right)\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\\
=\;&\exp\left[-i\pi\left(\frac{2}{\alpha_0}\right)^2\right]\cdot\exp\left[+i\pi\left(\frac{2}{\alpha_0}\right)^2\right]\cdot\exp\left[+i\cdot 2\pi\left(x\cdot\frac{2}{\alpha_0^2}\right)\right]\\
=\;&\boxed{\exp\left[+i\cdot 2\pi\left(2\cdot\frac{x}{\alpha_0^2}\right)\right]=F\left[\frac{x}{\alpha_0^2}\right]}
\end{aligned}$$

The output of the M-C-M is the appropriate linear-phase function that is a scaled replica of the Fourier transform, though in the space domain as a function of x.

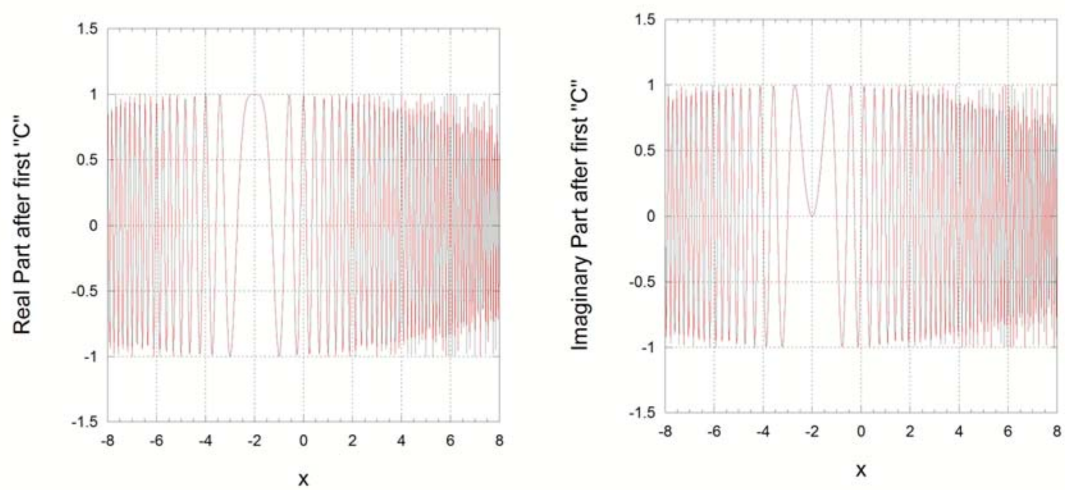


(b) evaluate the *C-M-C* chirp Fourier transform, showing the output after each M or C operation.

$$F\left[\frac{x}{\alpha_0^2}\right]=\frac{1}{|\alpha_0|}\exp\left[-i\frac{\pi}{4}\right]\cdot\left(\left\{\left(f[x]*\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right\}*\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right)$$

After first convolution, the result is a translated chirp function centered about $x=-2$

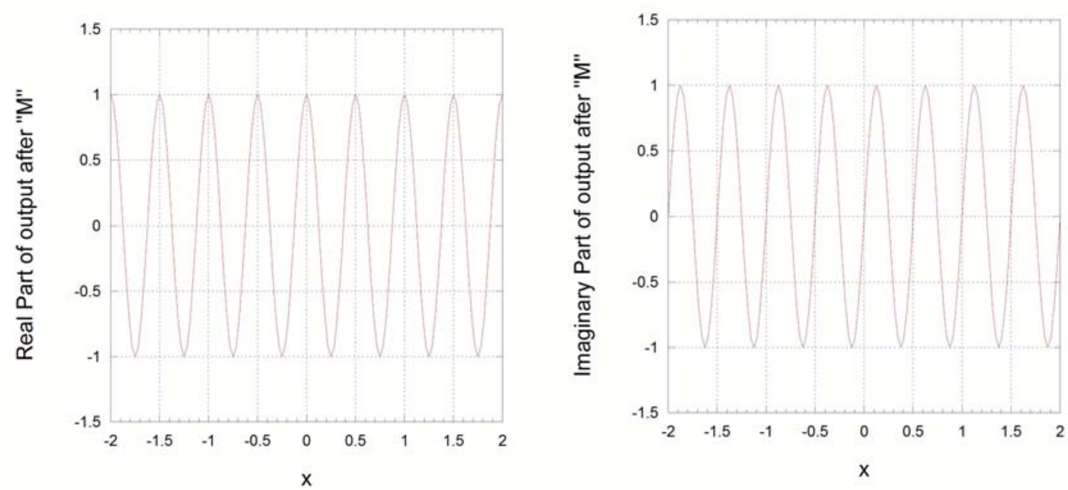
$$\begin{aligned}&\delta[x+2]*\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\\ =\;&\exp\left[+i\pi\left(\frac{x+2}{\alpha_0}\right)^2\right]\\ =\;&\exp\left[+i\pi\left(\frac{x^2+4+4x}{\alpha_0^2}\right)\right]\\ =\;&\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\cdot\exp\left[+i\cdot 2\pi\cdot\left(\frac{2}{\alpha_0^2}\right)\right]\cdot\exp\left[+i\cdot 2\pi\cdot\left(\frac{2x}{\alpha_0^2}\right)\right]\end{aligned}$$



After multiplication:

$$\left(\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\cdot\exp\left[+i\cdot 2\pi\cdot\left(\frac{2}{\alpha_0^2}\right)\right]\cdot\exp\left[+i\cdot 2\pi\cdot\left(\frac{2x}{\alpha_0^2}\right)\right]\right)\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]$$
$$=\exp\left[+i\cdot 2\pi\cdot\left(\frac{2}{\alpha_0^2}\right)\right]\cdot\exp\left[+i\cdot 2\pi\cdot\left(\frac{x}{\left(\frac{\alpha_0^2}{2}\right)}\right)\right]$$

which is the product of a complex-valued constant and a linear phase term with period $\frac{\alpha_0^2}{2}$.\



The second convolution has the form:

$$\boxed{\exp\left[+i\cdot 2\pi\cdot\left(\frac{2}{\alpha_0^2}\right)\right]}\cdot\left(\exp\left[+i\pi\left(\frac{4x}{\alpha_0^2}\right)\right]*\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right)$$

where the "boxed" term is the complex-valued constant that will be included later. We can evaluate convolution in the frequency domain via filter theorem:

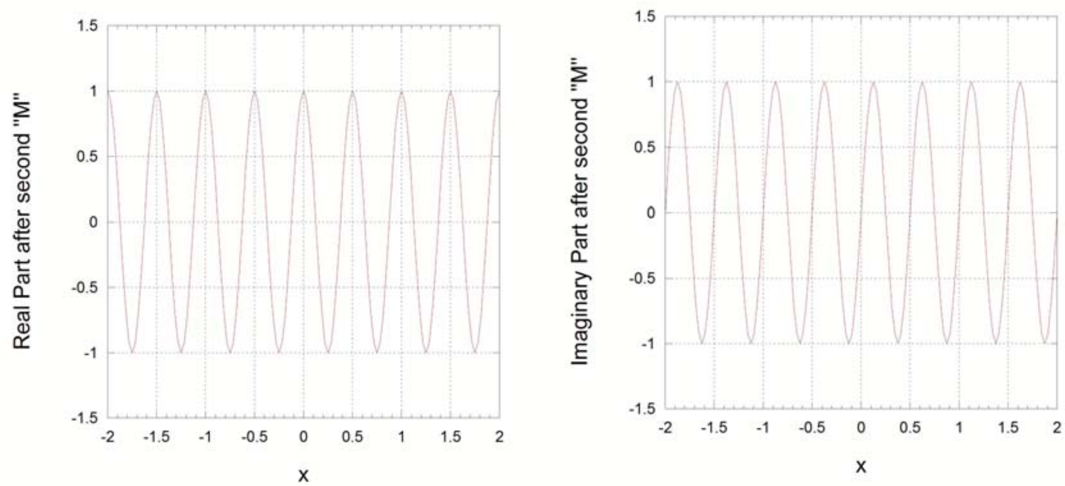
$$\begin{aligned}
&\exp\left[+i\pi\left(\frac{4x}{\alpha_0^2}\right)\right]*\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\\
=\;&\mathcal{F}^{-1}\left\{\mathcal{F}\left\{\exp\left[+i\cdot 2\pi\cdot x\cdot\left(\frac{2}{\alpha_0^2}\right)\right]\right\}\cdot\mathcal{F}\left\{\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right\}\right\}\\
=\;&\mathcal{F}^{-1}\left\{\delta\left[\xi-\frac{2}{\alpha_0^2}\right]\cdot|\alpha_0|\cdot\exp\left[+i\cdot\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\cdot(\alpha_0\xi)^2\right]\right\}\\
=\;&|\alpha_0|\cdot\exp\left[+i\cdot\frac{\pi}{4}\right]\cdot\mathcal{F}^{-1}\left\{\delta\left[\xi-\frac{2}{\alpha_0^2}\right]\cdot\exp\left[-i\pi\cdot(\alpha_0\xi)^2\right]\right\}\\
=\;&|\alpha_0|\cdot\exp\left[+i\cdot\frac{\pi}{4}\right]\cdot\mathcal{F}^{-1}\left\{\delta\left[\xi-\frac{2}{\alpha_0^2}\right]\cdot\exp\left[-i\pi\cdot\left(\alpha_0\cdot\frac{2}{\alpha_0^2}\right)^2\right]\right\}\\
=\;&|\alpha_0|\cdot\exp\left[+i\cdot\frac{\pi}{4}\right]\cdot\mathcal{F}^{-1}\left\{\delta\left[\xi-\frac{2}{\alpha_0^2}\right]\cdot\exp\left[-i\pi\cdot\frac{4}{\alpha_0^2}\right]\right\}\\
=\;&|\alpha_0|\cdot\exp\left[+i\cdot\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\cdot\frac{4}{\alpha_0^2}\right]\cdot\exp\left[+i\cdot 2\pi\cdot x\cdot\frac{2}{\alpha_0^2}\right]
\end{aligned}$$

Now multiply by the constant term in the box above:

$$\begin{aligned}
&\exp\left[+i\cdot 2\pi\cdot\left(\frac{2}{\alpha_0^2}\right)\right]\cdot|\alpha_0|\cdot\exp\left[+i\cdot\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\cdot\frac{4}{\alpha_0^2}\right]\cdot\exp\left[+i\cdot 2\pi\cdot x\cdot\frac{2}{\alpha_0^2}\right]\\
=\;&|\alpha_0|\cdot\exp\left[+i\cdot\frac{\pi}{4}\right]\cdot\exp\left[+i\cdot 2\pi\cdot x\cdot\frac{2}{\alpha_0^2}\right]
\end{aligned}$$

The leading scale factors in the C-M-C cancel the leading constant:

$$\left(\frac{1}{|\alpha_0|}\exp\left[-i\frac{\pi}{4}\right]\right)\cdot\left(|\alpha_0|\cdot\exp\left[+i\cdot\frac{\pi}{4}\right]\right)\cdot\exp\left[+i\cdot 2\pi\cdot x\cdot\frac{2}{\alpha_0^2}\right] = \exp\left[+i\cdot 2\pi\cdot x\cdot\frac{2}{\alpha_0^2}\right]$$
$$= F\left[\frac{x}{\alpha_0^2}\right]$$



(c) (OPTIONAL BONUS) evaluate the *M-C-M* AND *C-M-C* chirp Fourier transforms and sketch after each step for $f[x] = \cos\left[\pi\left(\frac{x}{\alpha_0}\right)^2\right]$

First, evaluate the transform:

$$\cos\left[\pi\left(\frac{x}{\alpha_0}\right)^2\right] = \frac{1}{2}\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] + \frac{1}{2}\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]$$

$$\mathcal{F}\left\{\cos\left[\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right\} = \frac{1}{2}\cdot\mathcal{F}\left\{\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right\} + \frac{1}{2}\cdot\mathcal{F}\left\{\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right\}$$

$$= \frac{1}{2}\cdot|\alpha_0|\cdot\left(\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi(\alpha_0\xi)^2\right] + \exp\left[ii\frac{\pi}{4}\right]\cdot\exp\left[+i\pi(\alpha_0\xi)^2\right]\right)$$

$$= \frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\left(\exp\left[-i\pi(\alpha_0\xi)^2\right] + i\cdot\exp\left[+i\pi(\alpha_0\xi)^2\right]\right)$$

M-C-M, Apply first multiplication:

$$\cos\left[\pi\left(\frac{x}{\alpha_0}\right)^2\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]$$

$$= \frac{1}{2}\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right] + \frac{1}{2}\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]$$

$$= \frac{1}{2}\cdot 1[x] + \frac{1}{2}\exp\left[-i\cdot 2\pi\left(\frac{x}{\alpha_0}\right)^2\right]$$

$$= \frac{1}{2}\cdot 1[x] + \frac{1}{2}\exp\left[-i\cdot\pi\left(\frac{x}{\frac{\alpha_0}{\sqrt{2}}}\right)^2\right]$$

which is the sum of a constant and a quadratic phase with a shorter chirp rate
Apply the convolution:

$$\left(\frac{1}{2}\cdot 1[x] + \frac{1}{2}\exp\left[-i\cdot\pi\left(\frac{x}{\frac{\alpha_0}{\sqrt{2}}}\right)^2\right]\right) * \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]$$

$$= \frac{1}{2}\cdot\left(1[x] * \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right) + \frac{1}{2}\cdot\left(\exp\left[-i\cdot\pi\left(\frac{x}{\frac{\alpha_0}{\sqrt{2}}}\right)^2\right] * \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right)$$

Evaluate the two convolutions separately:

$$\frac{1}{2}\cdot 1[x] * \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] = \frac{1}{2}\cdot\mathcal{F}^{-1}\left\{\delta[\xi]\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi(\alpha_0\xi)^2\right]\right\}$$

$$= \left(\frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\right)\cdot\mathcal{F}^{-1}\left\{\delta[\xi-0]\cdot\exp\left[-i\pi(\alpha_0\xi)^2\right]\right\}$$

$$= \left(\frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\right)\cdot\mathcal{F}^{-1}\left\{\delta[\xi]\cdot\exp\left[-i\pi(\alpha_0\cdot 0)^2\right]\right\}$$

$$= \left(\frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\right)\cdot 1[x]$$

$$
\begin{aligned}
&\frac{1}{2}\exp\left[-i\cdot\pi\left(\frac{x}{\frac{\alpha_0}{\sqrt{2}}}\right)^2\right] * \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \\
= \; & \frac{1}{2}\cdot\mathcal{F}^{-1}\left\{\frac{|\alpha_0|}{\sqrt{2}}\cdot\exp\left[-i\frac{\pi}{4}\right]\cdot\exp\left[+i\pi\left(\frac{\alpha_0}{\sqrt{2}}\xi\right)^2\right]\right\} \\
& \cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\exp\left[-i\pi\left(\alpha_0\xi\right)^2\right] \\
= \; & \mathcal{F}^{-1}\left\{\left(\frac{1}{2}\frac{|\alpha_0|}{\sqrt{2}}\cdot\exp\left[-i\frac{\pi}{4}\right]\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\right)\cdot\exp\left[+i\pi\left(\frac{\alpha_0}{\sqrt{2}}\xi\right)^2\right]\cdot\exp\left[-i\pi\left(\alpha_0\xi\right)^2\right]\right\} \\
= \; & \frac{1}{2}\frac{|\alpha_0|^2}{\sqrt{2}}\cdot\mathcal{F}^{-1}\left\{\exp\left[+i\pi\left(\frac{\alpha_0^2}{2}-\alpha_0^2\right)\xi^2\right]\right\} \\
= \; & \frac{1}{2}\frac{|\alpha_0|^2}{\sqrt{2}}\cdot\mathcal{F}^{-1}\left\{\exp\left[-i\pi\frac{\alpha_0^2}{2}\xi^2\right]\right\} \\
= \; & \frac{1}{2}\frac{|\alpha_0|^2}{\sqrt{2}}\cdot\mathcal{F}^{-1}\left\{\exp\left[-i\pi\left(\frac{\alpha_0}{\sqrt{2}}\xi\right)^2\right]\right\} \\
= \; & \frac{1}{2}\frac{|\alpha_0|^2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{|\alpha_0|}\cdot\exp\left[-i\frac{\pi}{4}\right]\cdot\exp\left[+i\pi\left(\frac{x}{\left(\frac{\alpha_0}{\sqrt{2}}\right)}\right)^2\right] \\
= \; & \frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[-i\frac{\pi}{4}\right]\cdot\exp\left[+i\pi\left(\frac{x}{\left(\frac{\alpha_0}{\sqrt{2}}\right)}\right)^2\right]
\end{aligned}
$$

So the output after the convolution is:

$$
\begin{aligned}
& \left(\frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\right)+\frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[-i\frac{\pi}{4}\right]\cdot\exp\left[+i\pi\left(\frac{x}{\left(\frac{\alpha_0}{\sqrt{2}}\right)}\right)^2\right] \\
= \; & \frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\left(1+\exp\left[-i\frac{\pi}{2}\right]\cdot\exp\left[+i\pi\left(\frac{x}{\left(\frac{\alpha_0}{\sqrt{2}}\right)}\right)^2\right]\right)
\end{aligned}
$$

After the second multiplication:

$$
\begin{aligned}
& \left[\frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\left(1-i\cdot\exp\left[+i\pi\left(\frac{x}{\left(\frac{\alpha_0}{\sqrt{2}}\right)}\right)^2\right]\right)\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \\
= \; & \frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\left(\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]-i\cdot\exp\left[+i\pi\left(\frac{x}{\left(\frac{\alpha_0}{\sqrt{2}}\right)}\right)^2\right]\cdot\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right) \\
= \; & \frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\left(\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]-i\cdot\exp\left[+i\pi\left(\frac{2x^2}{\alpha_0^2}+\frac{x^2}{\alpha_0^2}\right)\right]\right) \\
= \; & \frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\left(\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]-i\cdot\exp\left[+i\pi\frac{3x^2}{\alpha_0^2}\right]\right) \\
= \; & \frac{1}{2}\cdot|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\cdot\left(\exp\left[-i\pi\left(\frac{x}{\alpha_0}\right)^2\right]-i\cdot\exp\left[+i\pi\frac{x^2}{\left(\frac{\alpha_0}{\sqrt{3}}\right)^2}\right]\right)
\end{aligned}
$$

(d) (OPTIONAL BONUS) repeat (c) for $f\left[x\right] = \cos\left[\pi\left(\frac{x-1}{\alpha_0}\right)^2\right]$

I won't bother with solving this because of time and the fact that this is optional

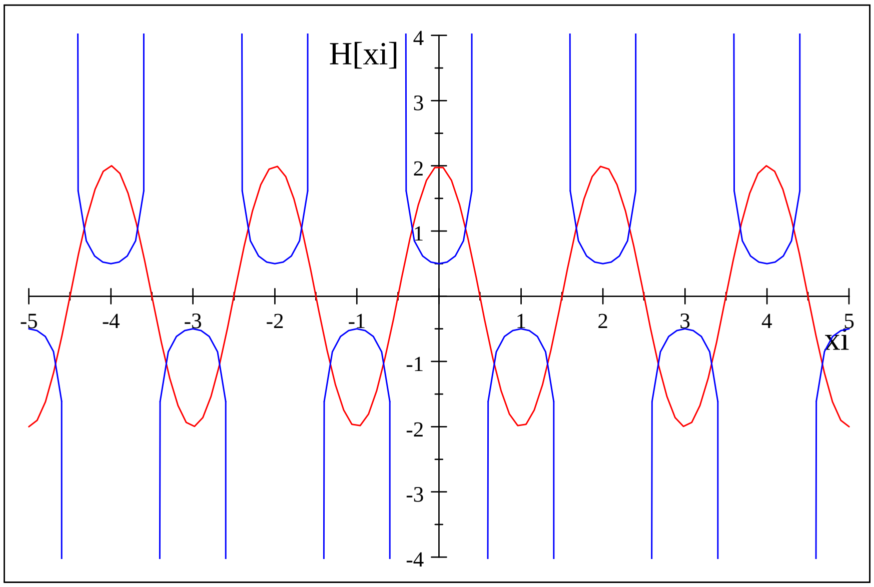
7. A 1-D image $g[x]$ has been created by a double exposure of the original object $f[x]$. The original scene was translated between the exposures by the known distance $+b_0$. The object was stationary during both exposures and the exposure time was the same in both cases.

I choose to put the origin midway between the impulses to remove any intrinsic linear phase:

$$h[x] = \delta\left[x+\frac{b_0}{2}\right]+\frac{1}{2}\delta\left[x-\frac{b_0}{2}\right]$$
$$= \frac{1}{2}\delta\left[x+\frac{b_0}{2}\right]+\left(\frac{1}{2}\delta\left[x+\frac{x_0}{2}\right]+\frac{1}{2}\delta\left[x-\frac{x_0}{2}\right]\right)$$

(a) Design the inverse filter for this system in the frequency domain. Comment about the potential of success of the deblurring process, particularly if noise is present.

$$h[x] = \delta\left[x+\frac{b_0}{2}\right]+\delta\left[x-\frac{b_0}{2}\right]$$
$$\Longrightarrow H[\xi] = \exp\left[+i\cdot 2\pi\cdot\xi\cdot\left(-\frac{b_0}{2}\right)\right]+\exp\left[+i\cdot 2\pi\cdot\xi\cdot\left(+\frac{b_0}{2}\right)\right]$$
$$= \exp\left[-i\cdot 2\pi\xi\cdot\frac{b_0}{2}\right]+\exp\left[+i\cdot 2\pi\xi\cdot\frac{b_0}{2}\right]$$
$$H[\xi] = 2\cdot\cos\left[2\pi\xi\cdot\frac{b_0}{2}\right]$$



$H[\xi]$ (red)and $W[\xi]$ (blue) for $b_0 = 1$

(b) (optional, bonus) Find an exact or approximate expression for the inverse filter in the space domain.

the reciprocal of the cosine is the secant:

$$W[\xi] = \frac{1}{H[\xi]} = \frac{1}{2\cdot\cos\left[2\pi\xi\cdot\frac{b_0}{2}\right]}$$
$$= \frac{1}{2}\cdot\sec\left[2\pi\xi\cdot\frac{b_0}{2}\right]$$
$$= \frac{1}{2}\cdot\sec[\pi\xi b_0]$$

We have not derived an analytical expression for the forward or inverse transform of a secant function (though it may exist); this may be coarsely approximated by a square wave with COMB functions to form the edges that asymptotically approach $\pm\infty$

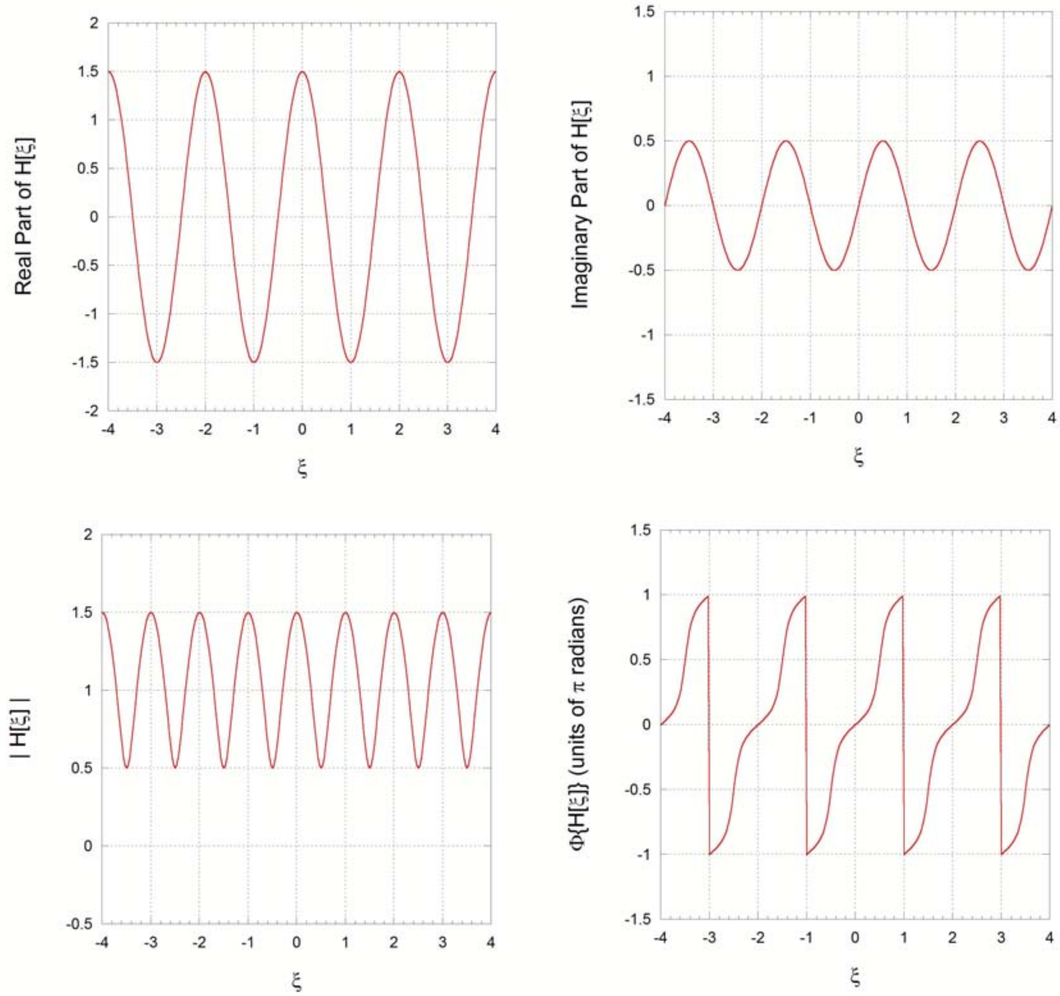
(c) Find the inverse filter for the case where the second exposure is only half as long as the first.

$$\begin{aligned} h[x] &= \delta\left[x+\frac{b_0}{2}\right]+\frac{1}{2}\delta\left[x-\frac{b_0}{2}\right] \\ &= \frac{1}{2}\delta\left[x+\frac{b_0}{2}\right]+\left(\frac{1}{2}\delta\left[x+\frac{x_0}{2}\right]+\frac{1}{2}\delta\left[x-\frac{x_0}{2}\right]\right) \end{aligned}$$

$$\begin{aligned} H[\xi] &= \frac{1}{2}\exp\left[-2\pi i\xi\frac{x_0}{2}\right]+\cos\left[2\pi\xi\left(\frac{x_0}{2}\right)\right] \\ &= \frac{3}{2}\cos\left[2\pi\xi\frac{x_0}{2}\right]-i\sin\left[2\pi\xi\frac{x_0}{2}\right] \end{aligned}$$

$$\begin{aligned} |H[\xi]| &= \sqrt{\frac{9}{4}\cos^2\left[2\pi\xi\frac{x_0}{2}\right]+\sin^2\left[2\pi\xi\frac{x_0}{2}\right]} \\ &= \sqrt{1+\frac{5}{4}\cos^2\left[2\pi\xi\frac{x_0}{2}\right]}>0 \\ \textit{since }\ \cos^2[\pi\xi x_0] &> 0,\ \textit{then}\ |H[\xi]|\neq 0 \\ &\Longrightarrow \textit{inverse filter exists! (no zeros in } H[\xi]\,) \end{aligned}$$

$$\Phi\{H[\xi]\}=\tan^{-1}\left[\frac{-\sin[\pi\xi x_0]}{\frac{3}{2}\cos[\pi\xi x_0]}\right]$$


Real Part of H[ξ]
ξ
Imaginary Part of H[ξ]
ξ
| H[ξ] |
ξ
Φ{H[ξ]} (units of π radians)
ξ

8. Design the transfer function of the Wiener or Wiener-Helstrom filter for the following combinations of input signal, impulse response, and noise power spectrum. In each case, sketch the spectrum $F[\xi]$ and the transfer function $W[\xi]$ on the same graph.

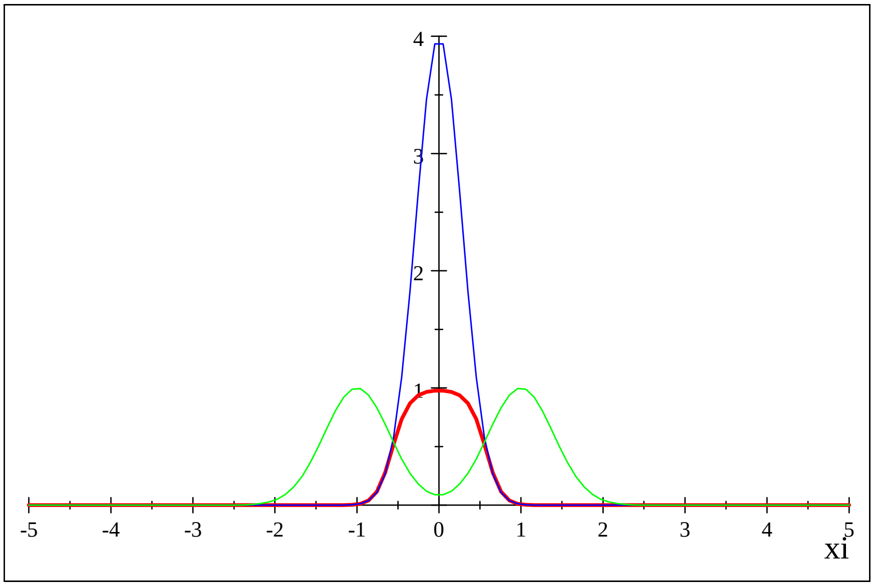
(a) $f[x] = 2 \cdot GAUS[x]$, $h[x] = \delta[x]$, $|N[\xi]|^2 = GAUS[\xi + \xi_0] + GAUS[\xi - \xi_0]$

$$f[x] = 2 \cdot GAUS[x] = 2 \cdot \exp\left[-\pi x^2\right] \implies F[\xi] = 2 \cdot \exp\left[-\pi \xi^2\right] = 2 \cdot GAUS[\xi]$$

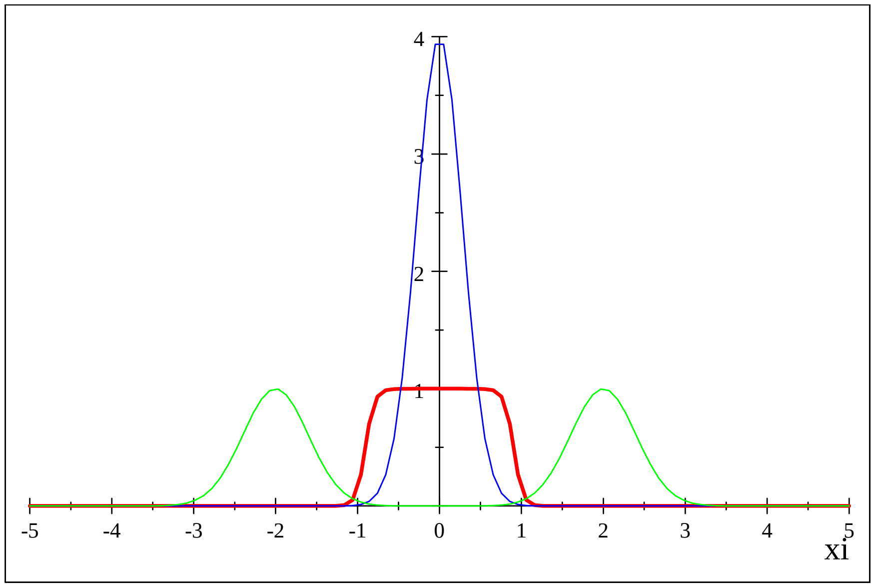
$$\implies |F[\xi]|^2 = \left|2 \cdot \exp\left[-\pi \xi^2\right]\right|^2 = 4 \cdot \left(\exp\left[-\pi \xi^2\right]\right)^2 = 4 \cdot \exp\left[-\pi 2\xi^2\right] = 4 \cdot GAUS\left[\frac{\xi}{\left(\frac{1}{\sqrt{2}}\right)}\right]$$

$$h[x] = \delta[x] \implies H[\xi] = 1[\xi] \implies H^*[\xi] = 1[\xi] \implies |H[\xi]|^2 = 1[\xi]$$

$$\begin{aligned} W[\xi] &= \frac{|F[\xi]|^2 \cdot H^*[\xi]}{|F[\xi]|^2 \cdot |H[\xi]|^2 + |N[\xi]|^2} \\ &= \frac{4 \cdot \exp\left[-\pi 2\xi^2\right] \cdot 1[\xi]}{4 \cdot \exp\left[-\pi 2\xi^2\right] \cdot 1[\xi] + \left(\exp\left[-\pi(\xi+\xi_0)^2\right] + \exp\left[-\pi(\xi-\xi_0)^2\right]\right)} \\ &= \frac{4 \cdot \exp\left[-\pi 2\xi^2\right]}{4 \cdot \exp\left[-\pi 2\xi^2\right] + \left(\exp\left[-\pi(\xi+\xi_0)^2\right] + \exp\left[-\pi(\xi-\xi_0)^2\right]\right)} \end{aligned}$$



Power spectra of signal (blue), noise (green), and transfer function of Wiener filter (red) for $\xi_0 = 1$



Power spectra of signal (blue), noise (green), and transfer function of Wiener filter (red) for $\xi_0 = 2$

(b) $f[x] = GAUS\left[\frac{x}{b_0}\right] \cdot \exp\left[+i\pi x^2\right]$, $h[x] = RECT[x]$, $|N[\xi]|^2 = GAUS[\xi+\xi_0] + GAUS[\xi-\xi_0]$

$$f[x] = GAUS\left[\frac{x}{b_0}\right] \cdot \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] = \exp\left[-\pi\left(\frac{x}{b_0}\right)^2\right] \cdot \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \text{ where } \alpha_0 = 1$$

$$\Longrightarrow \quad |F[\xi]|^2 = \left|\mathcal{F}\left\{GAUS\left[\frac{x}{b_0}\right] \cdot \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right\}\right|^2$$

Use stationary phase to approximate the spectrum:

$$F[\xi] \cong \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-\pi\left(\frac{\xi}{b_0/\alpha_0^2}\right)^2\right] \cdot \exp\left[-i\pi\alpha_0^2\xi^2\right] \to \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-\pi\left(\frac{\xi}{b_0}\right)^2\right] \cdot \exp\left[-i\pi\xi^2\right]$$

$$\Longrightarrow \quad |F[\xi]|^2 \cong \left(\exp\left[-\pi\left(\frac{\xi}{b_0}\right)^2\right]\right)^2 = \exp\left[-\pi\left(\frac{\xi}{b_0}\right)^2\right] \cdot \exp\left[-\pi\left(\frac{\xi}{b_0}\right)^2\right]$$

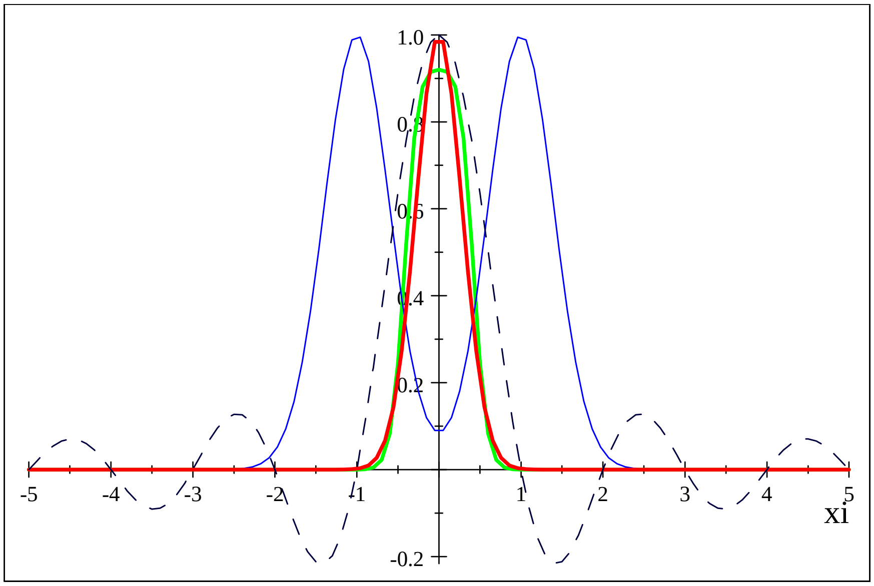
$$|F[\xi]|^2 \cong \exp\left[-\pi \cdot 2 \cdot \left(\frac{\xi}{b_0}\right)^2\right] = \exp\left[-\pi\left(\frac{\xi}{\left(\frac{b_0}{\sqrt{2}}\right)}\right)^2\right]$$

$$h[x] = RECT[x] \Longrightarrow H[\xi] = SINC[\xi]$$

$$\begin{aligned} |N[\xi]|^2 &= GAUS[\xi+\xi_0] + GAUS[\xi-\xi_0] \\ &= \exp\left[-\pi(\xi+\xi_0)^2\right] + \exp\left[-\pi(\xi-\xi_0)^2\right] \end{aligned}$$

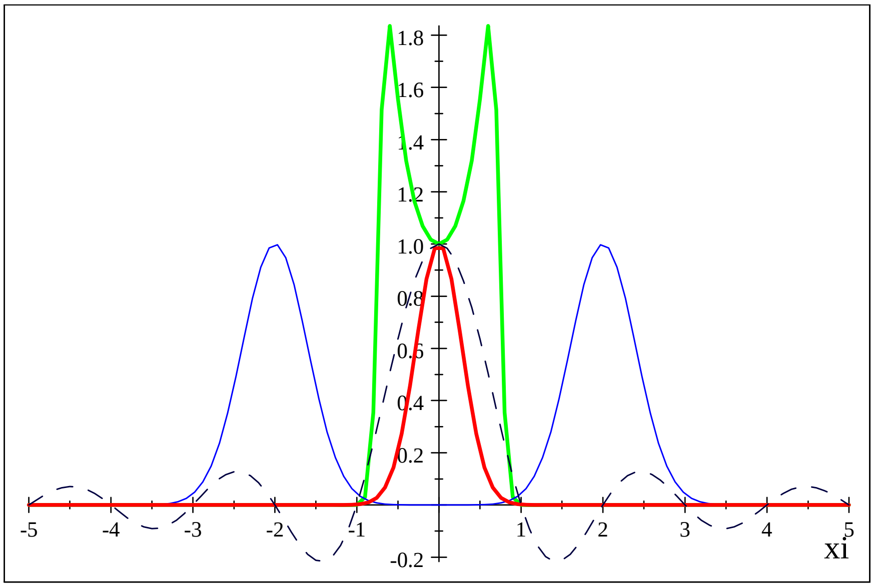
$$\begin{aligned} W[\xi] &= \frac{H^*[\xi] \cdot |F[\xi]|^2}{|H[\xi]|^2 \cdot |F[\xi]|^2 + |N[\xi]|^2} \\ &= \frac{SINC[\xi] \cdot |F[\xi]|^2}{|SINC[\xi]|^2 \cdot |F[\xi]|^2 + |N[\xi]|^2} \end{aligned}$$

$$\frac{\frac{\sin\pi\xi}{\pi\xi} \cdot \exp\left[-\pi\left(\frac{\xi}{\left(\frac{1}{\sqrt{2}}\right)}\right)^2\right]}{\left|\frac{\sin\pi\xi}{\pi\xi}\right|^2 \cdot \exp\left[-\pi\left(\frac{\xi}{\left(\frac{1}{\sqrt{2}}\right)}\right)^2\right] + \left(\exp\left[-\pi(\xi+1)^2\right] + \exp\left[-\pi(\xi-1)^2\right]\right)}$$

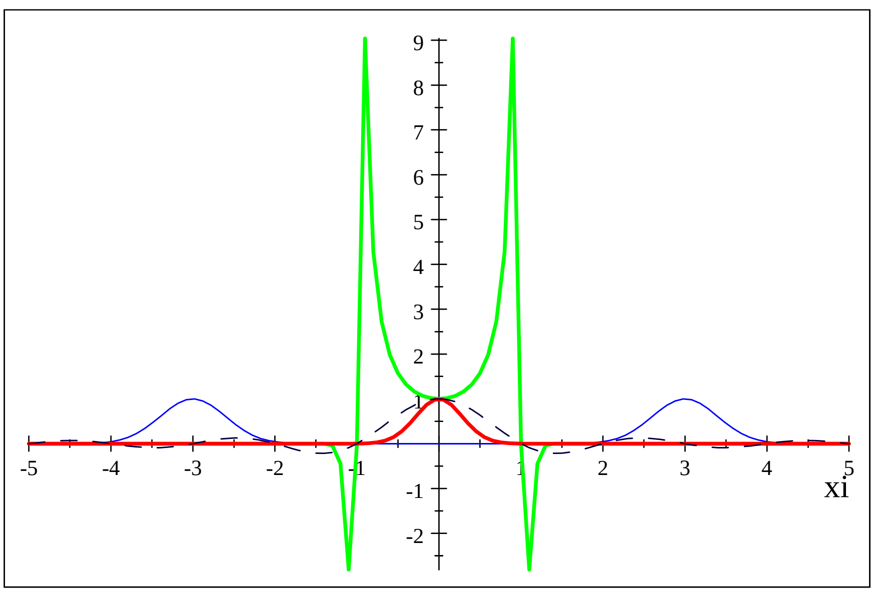


$|F[\xi]|^2$ *in red,* $|N[\xi]|^2$ *in blue,* $W[\xi]$ *in green, and* $H[\xi]$ *as dashed line for* $\xi_0 = 1$ *and* $b_0 = 1$

$$\frac{\frac{\sin \pi\xi}{\pi\xi} \cdot \exp\left[-\pi\left(\frac{\xi}{\left(\frac{1}{\sqrt{2}}\right)}\right)^2\right]}{\left|\frac{\sin \pi\xi}{\pi\xi}\right|^2 \cdot \exp\left[-\pi\left(\frac{\xi}{\left(\frac{1}{\sqrt{2}}\right)}\right)^2\right] + \left(\exp\left[-\pi\left(\xi+2\right)^2\right] + \exp\left[-\pi\left(\xi-2\right)^2\right]\right)}$$



$|F[\xi]|^2$ in red, $|N[\xi]|^2$ in blue, $W[\xi]$ in green, and $H[\xi]$ as dashed line for $\xi_0 = 2$ and $b_0 = 1$; in this case, the noise power is "farther out" (at larger frequencies), so the inverse filter has some impact for smaller frequencies.



$|F[\xi]|^2$ in red, $|N[\xi]|^2$ in blue, $W[\xi]$ in green, and $H[\xi]$ as dashed line for $\xi_0 = 3$ and $b_0 = 1$, so that the noise power is even "farther out" (at larger frequencies) and the impact of the inverse filter is seen at largerfrequencies.

(c) $f[x] = GAUS\left[\frac{x}{2}\right] \cdot \cos[10\pi x]$, $h[x] = \delta[x]$, $|N[\xi]|^2 = GAUS[\xi]$
$h[x] = \delta[x] \implies$ *Wiener filter*

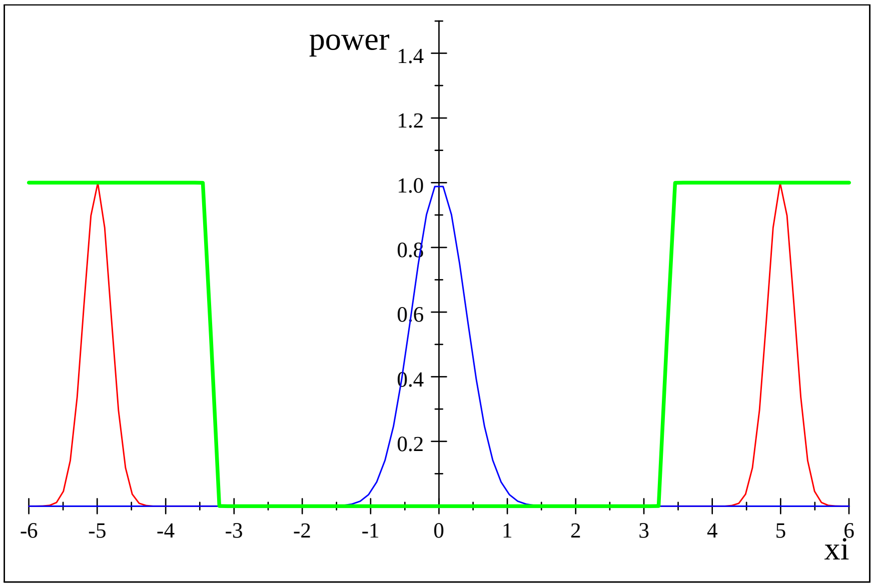
$$\begin{aligned} f[x] &= GAUS\left[\frac{x}{2}\right] \cdot \cos[10\pi x] = GAUS\left[\frac{x}{2}\right] \cdot \cos[2\pi \cdot 5 \cdot x] \\ F[\xi] &= 2 \cdot GAUS[2\xi] * \left(\frac{1}{2}\delta[\xi+5] + \frac{1}{2}\delta[\xi-5]\right) \\ &= GAUS[2(\xi+5)] + GAUS[2(\xi-5)] \\ &= GAUS\left[\frac{(\xi+5)}{\left(\frac{1}{2}\right)}\right] + GAUS\left[\frac{(\xi-5)}{\left(\frac{1}{2}\right)}\right] \end{aligned}$$

where the Gaussian width is much smaller than the separation, so these are approximately disjoint.

$$\begin{aligned} |F[\xi]|^2 &= \left|GAUS\left[\frac{(\xi+5)}{\left(\frac{1}{2}\right)}\right] + GAUS\left[\frac{(\xi-5)}{\left(\frac{1}{2}\right)}\right]\right|^2 \\ &\cong \left|GAUS\left[\frac{(\xi+5)}{\left(\frac{1}{2}\right)}\right]\right|^2 + \left|GAUS\left[\frac{(\xi-5)}{\left(\frac{1}{2}\right)}\right]\right|^2 \\ &= \exp\left[-\pi\left(\frac{(\xi+5)^2}{\left(\frac{1}{4}\right)}\right)\right] + \exp\left[-\pi\left(\frac{(\xi-5)^2}{\left(\frac{1}{4}\right)}\right)\right] \end{aligned}$$

$$|N[\xi]|^2 = GAUS[\xi] = \exp\left[-\pi\xi^2\right]$$

$$\begin{aligned} W[\xi] &= \frac{|F[\xi]|^2}{|F[\xi]|^2 + |N[\xi]|^2} \\ &= \frac{\exp\left[-\pi\left(\frac{(\xi+5)^2}{\left(\frac{1}{4}\right)}\right)\right] + \exp\left[-\pi\left(\frac{(\xi-5)^2}{\left(\frac{1}{4}\right)}\right)\right]}{\exp\left[-\pi\left(\frac{(\xi+5)^2}{\left(\frac{1}{4}\right)}\right)\right] + \exp\left[-\pi\left(\frac{(\xi-5)^2}{\left(\frac{1}{4}\right)}\right)\right] + \exp\left[-\pi\xi^2\right]} \end{aligned}$$



$|F[\xi]|^2$ *in red,* $|N[\xi]|^2$ *in blue, and* $W[\xi]$ *in green, showing that the filter blocks the noise power.*