HW # 7.

Fourier Methods IMGS-616-20141
Solution Set #7

1.

(1). $m_\ell = \int_{-\infty}^{+\infty} \delta(x-x_0)\, x^\ell \cdot dx = x_0^{\ell}$.

(2) $F(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} (-i2\pi)^n m_n \cdot \xi^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-i2\pi)^n \cdot x_0^n \cdot \xi^n$

$= e^{-i2\pi x_0 \xi}$

if $x_0 = 0$. $F(\xi) = 1$

$x_0 = 1$ $F(\xi) = e^{-i2\pi\xi}$ $x_0 = -1$ $F(\xi) = e^{i2\pi\xi}$

2. $\bar{x}_f' = \frac{m_1}{m_0} = \frac{\int_{-\infty}^{+\infty} f(x) \cdot x\, dx}{\int_{-\infty}^{+\infty} f(x)\, dx} = \frac{\int_{-\infty}^{+\infty} RECT(x-x_0) \cdot x\, dx}{\int_{-\infty}^{+\infty} RECT(x-x_0) \cdot dx} = \frac{\int_{-\frac{1}{2}+x_0}^{\frac{1}{2}+x_0} x\, dx}{1}$

$= x_0$

$\bar{x}_f' = -\frac{1}{2\pi i} \cdot \frac{F'(0)}{F(0)} = -\frac{1}{2\pi i} \frac{\left(e^{-i2\pi\xi x_0} SINC(\xi)\right)' \big|_{\xi=0}}{\left(e^{-i2\pi\xi x_0} SINC(\xi)\right) \big|_{\xi=0}}$

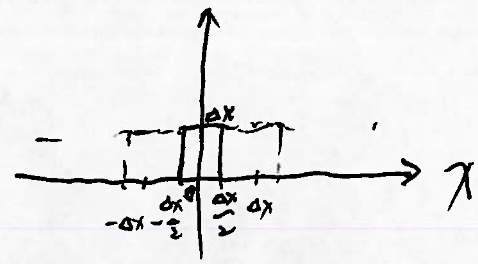
$= x_0$.

3.

(a) $COMB\left[\frac{x}{\Delta x}\right] * RECT\left[\frac{x}{\Delta x}\right]$

$= |\Delta x| \cdot \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) * RECT\left[\frac{x}{\Delta x}\right]$

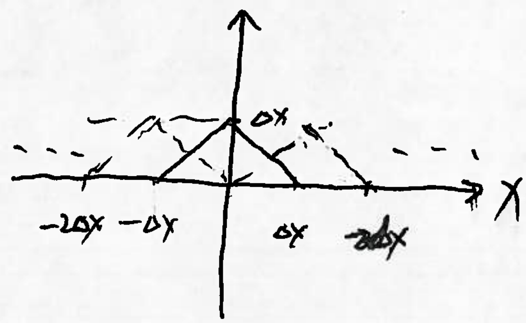
$= |\Delta x| \cdot [x]$



(b) $COMB\left[\frac{x}{\Delta x}\right] * TRI\left[\frac{x}{\Delta x}\right]$

$= |\Delta x| \sum_{n=-\infty}^{+\infty} \delta(x - n\Delta x) * TRI\left(\frac{x}{\Delta x}\right)$

$= |\Delta x| \, [x]$



(c). $COMB\left[\frac{x}{\Delta x}\right] * SINC\left[\frac{x}{\Delta x}\right]$

$\Downarrow$

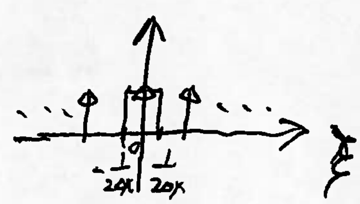
$F(\xi) = |\Delta x| \cdot COMB[\Delta x \cdot \xi] \cdot |\Delta x| \cdot RECT[\Delta x \cdot \xi]$

$= |\Delta x|^2 \cdot \frac{1}{|\Delta x|} \sum_{-\infty}^{+\infty} \delta\left(\xi - \frac{1}{\Delta x}\right) \cdot RECT[\Delta x \cdot \xi]$

$= |\Delta x| \cdot \delta(\xi)$

$\Downarrow$

$f(x) = |\Delta x| \cdot [x]$



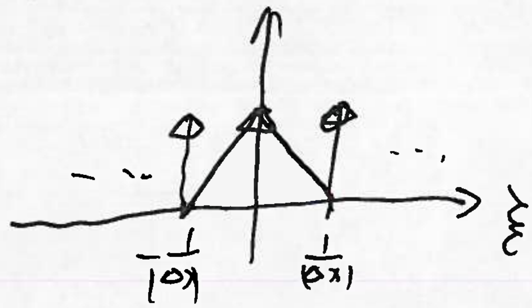
(d). $\mathrm{COMB}\left(\frac{x}{\Delta x}\right) * \mathrm{SINC}^2\left(\frac{x}{\Delta x}\right)$

$$\mathcal{F} \Downarrow$$

$$|\Delta x| \cdot \mathrm{COMB}[\Delta x \cdot \xi] \cdot \mathrm{TRI}[\Delta x \cdot \xi] \cdot |\Delta x|$$

$$|\Delta x|^2 \cdot \sum_{n=-\infty}^{\infty} \delta(\Delta x \cdot \xi - n) \cdot \mathrm{TRI}[\Delta x \cdot \xi]$$

$$|\Delta x| \cdot \sum_{n=-\infty}^{\infty} \delta\left(\xi - \frac{n}{\Delta x}\right) \cdot \mathrm{TRI}[\Delta x \cdot \xi]$$



$$= |\Delta x| \cdot \delta(\xi).$$

$$\mathcal{F}^{-1} \Downarrow$$

$$|\Delta x| \cdot 1[x].$$

4. for $SINC(x)$: $\mathcal{F}(SINC(x)) = RECT(\xi)$ $\xi_{max} = \frac{1}{2}$

$\Delta x = \frac{1}{2\xi_{max}} = \frac{1}{1} = 1.$ Nyquist frequence is 1.

for $SINC^2(x)$: $SINC^2(x) \xrightarrow{\mathcal{F}} TRI(\xi)$ $\xi_{max} = 1.$

$\xi_{nyquist} = 2 \cdot \xi_{max} = 2.$ $\Delta x = \frac{1}{\xi_{nyquist}} = \frac{1}{2}$

Then: for amplitude of samples

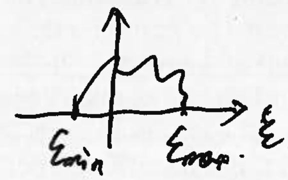
for $SINC(x)$: $SINC(x) \cdot COMB(\frac{x}{1}).$

for $SINC^2(x)$: $SINC(x) \cdot \frac{1}{\Delta x} \cdot COMB(\frac{x}{\Delta x})$

$= SINC(x) \cdot 2 \cdot COMB(\frac{x}{1/2}).$

5. for asymmetric situation.

$$\frac{1}{\Delta x} = \xi_{max} - \xi_{min}.$$



usually. $\xi_{min} < 0$, $\xi_{max} > 0$,

for $|\xi_{max}| = |\xi_{min}|$: $\frac{1}{\Delta x} = 2\,\xi_{max}$,

this is usual symmetric problem.

6.

$$\xi = \frac{1}{2\pi} \cdot \frac{\Delta\phi}{\Delta x} = \frac{1}{2\pi} \cdot \left(\pi \cdot \frac{1}{b_0^n} \cdot n \cdot x^{n-1}\right) = \frac{n \cdot x^{n-1}}{2\,b_0^n}$$

Then: $\frac{1}{2\Delta x} = \xi$. is the ideal. nyquist sampling limit.

$$\frac{1}{2\Delta x} = \frac{n \cdot x^{n-1}}{2\,b_0^n} \qquad x^{n-1} = \frac{b_0^n}{n \cdot \Delta x}.$$

for this x, the chirp. function is just aliased.

7.

(b).

$$RECT\left[\frac{x-2.5}{5}\right]\cdot \exp[+i\alpha x^2] \xrightarrow[\approx]{\mathcal{F}} RECT\left[\frac{\xi-2.5}{5}\right]\cdot e^{i\frac{\pi}{4}}\cdot \exp(-i\alpha\xi^2)$$

(c).

$$\hat{g}(x) = f(x) * f(x) \approx \mathcal{F}^{-1}\left[\hat{F}(\xi)\cdot\hat{F}(\xi)\right] = \mathcal{F}^{-1}\left[e^{i\frac{\pi}{2}} RECT\left(\frac{\xi-2.5}{5}\right]\cdot \exp(-i2\alpha\xi^2)\right]$$

(d).

$$f(x) \star f(x) \approx \mathcal{F}^{-1}\left[\hat{F}(\xi)\cdot\hat{F}^*(\xi)\right] = \mathcal{F}^{-1}\left[RECT\left[\frac{\xi-2.5}{5}\right]\right]$$

$$= 5\,SINC(5x)\cdot e^{i\cdot 5\pi x}.$$

(e).

$$R(f(x)) = \frac{1}{2}[\exp(i\alpha x^2)+\exp(-i\alpha x^2))\cdot RECT\left(\frac{x-2.5}{5}\right).$$

$$\Downarrow$$

$$\approx \frac{1}{2} RECT\left(\frac{\xi-2.5}{5}\right) e^{i\frac{\pi}{4}} \exp(-i\alpha\xi^2) + \frac{1}{2} RECT\left(\frac{\xi+2.5}{5}\right) e^{-i\frac{\pi}{4}} \exp(i\alpha\xi^2).$$

8.

(a) $\cos\left[\pi\left(\frac{x}{2}\right)^2\right]\cdot RECT\left[\frac{x-4}{2}\right] = \frac{1}{2}\left(\exp\left(i\pi\left(\frac{x}{2}\right)^2\right) + \exp\left(-i\pi\left(\frac{x}{2}\right)^2\right)\right)\cdot RECT\left(\ldots\right.$

$\mathcal{F}\Downarrow$

$\approx \frac{1}{2} RECT\left[\frac{\xi-4/4}{2/4}\right]\cdot e^{i\frac{\pi}{4}} \exp\left(-i\pi(2\xi)^2\right)\cdot 2 + \frac{1}{2} RECT\left[\frac{\xi+4/4}{2/4}\right]\cdot e^{-i\frac{\pi}{4}} \exp\left(+i\pi(2\xi)^2\right)\cdot \ldots$

$= RECT\left[\frac{\xi-1}{1/2}\right]\cdot \exp\left(-i\left(\pi\cdot 4\xi^2-\frac{\pi}{4}\right)\right) + RECT\left[\frac{\xi+1}{1/2}\right]\cdot \exp\left(i\left(\pi\cdot 4\xi^2-\frac{\pi}{4}\right)\right).$

(b) $\cos\left(\pi\cdot\left(\frac{x}{2}\right)^2\right)\cdot SINC\left(\frac{x-4}{2}\right) = \frac{1}{2}\left(\exp\left(i\pi\left(\frac{x}{2}\right)^2\right) + \exp\left(-i\pi\left(\frac{x}{2}\right)^2\right)\right)\cdot SINC\left(\frac{x-4}{2}\right.$

$\mathcal{F}\Downarrow$

$\approx SINC\left(\frac{\xi-1}{1/2}\right)\cdot \exp\left(-i\left(\pi\cdot 4\xi^2-\frac{\pi}{4}\right)\right) + SINC\left(\frac{\xi+1}{1/2}\right)\cdot \exp\left(i\left(\pi\cdot 4\xi^2-\frac{\pi}{4}\right)\right).$

(c) $\sin\left(\pi\left(\frac{x}{4}\right)^2\right)\cdot SINC\left(\frac{x-4}{4}\right) = \frac{1}{2i}\left(\exp\left(i\pi\left(\frac{x}{4}\right)^2\right) - \exp\left(-i\pi\left(\frac{x}{4}\right)^2\right)\right)\cdot SINC\left(\frac{x-4}{4}\right).$

$\mathcal{F}\Downarrow$

$\approx \frac{1}{2i}\cdot e^{i\frac{\pi}{4}}\cdot \exp\left(-i\pi(4\xi)^2\right)\cdot 4\cdot SINC\left(\frac{\xi-4/16}{4/16}\right) - \frac{1}{2i}\cdot e^{-i\frac{\pi}{4}}\cdot \exp\left(i\pi(4\xi)^2\right)\cdot 4\cdot SINC\left(\frac{\xi+4/16}{4/16}\right)$

$= -2i\cdot SINC(4\xi-1)\cdot \exp\left(-i\left(\pi(4\xi)^2-\frac{\pi}{4}\right)\right) + 2i\cdot SINC(4\xi+1)\cdot \exp\left(i\left(\pi(4\xi)^2-\frac{\pi}{4}\right)\right).$