

HW # 6.

1.

$$(a): \text{ volume} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) dx dy = 2.$$

(use central theorem will be easier).

$$f(x,0) = \text{sinc}\left(\frac{x}{2}\right).$$

$$f(0,y) = \text{sinc}(y)$$

even,

$$\text{sinc}\left(\frac{x}{2}\right)$$

$$\text{sinc}(y)$$

odd:

$$0$$

$$0$$

$$(b): \text{ volume} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\text{RECT}\left(\frac{x}{2}, \frac{3}{4}\right) - \text{RECT}\left(x, \frac{3}{2}\right)) dx dy = 8 - 2 = 6.$$

$$f(x,0) = \text{RECT}\left[\frac{x}{2}\right] - \text{RECT}(x).$$

$$f(0,y) = \text{RECT}\left[\frac{3}{4}\right] - \text{RECT}\left(\frac{3}{2}\right).$$

even



odd:

$$0$$

$$0$$

$$(c): \text{ volume} = \int_0^{+2\pi} \int_0^{+\infty} p(r, \theta) dr d\theta = \frac{3}{4}\pi$$

$$p(x,0) = \text{RECT}\left[\frac{x}{2}\right]$$

$$p(0,y) = \text{RECT}\left(\frac{3}{2}\right) - \text{RECT}(y)$$

even



odd

$$0$$

$$0$$

(d): volume, n . $\frac{1}{2} V(\text{CYL}(\frac{r}{2})) - \frac{1}{2} V(\text{CYL}(r))$
 $= \frac{1}{2} \cdot \frac{\pi}{4} \cdot 2^2 - \frac{1}{2} \cdot \frac{\pi}{4} = \frac{3}{8} \pi$

$f(x,0) = \frac{1}{2} (\text{RECT}(\frac{x}{2}) - \text{RECT}(x))$ $f(0,y) = [\text{RECT}(\frac{y}{2}) + \text{RECT}(y)]$

↓

even

• STEP (y).
 $\frac{1}{2} (\text{RECT}(y - \frac{3}{2}) + \text{RECT}(-y - \frac{3}{2}))$

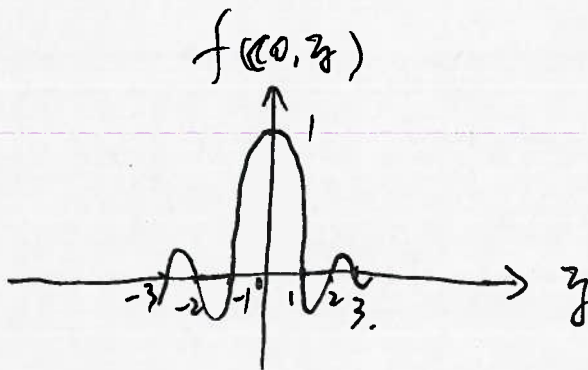
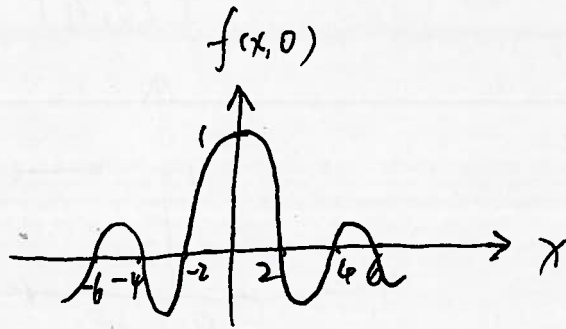
odd

0

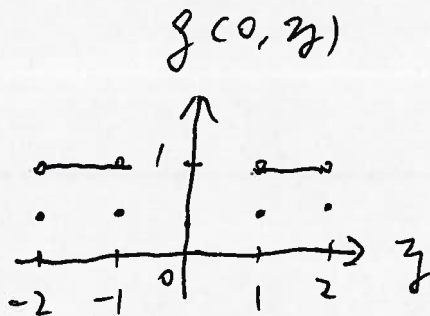
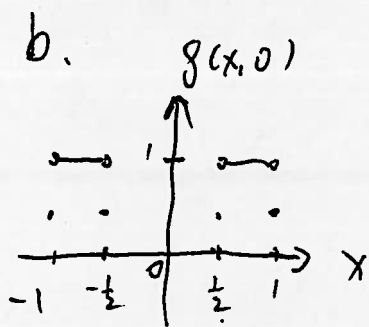
$\frac{1}{2} (\text{RECT}(y - \frac{3}{2}) \pm \text{RECT}(-y - \frac{3}{2}))$

1.

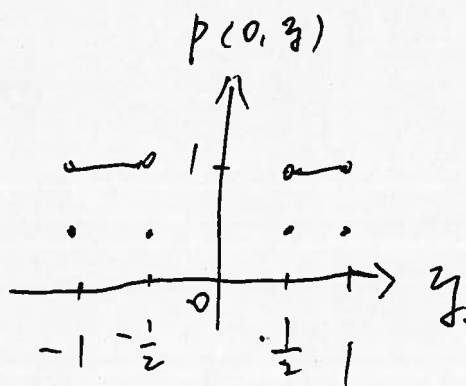
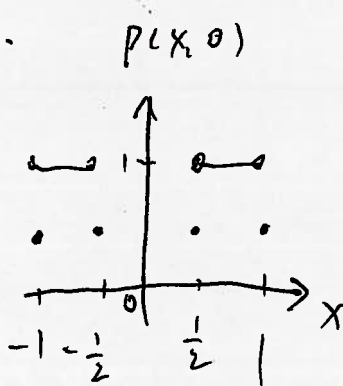
a.



b.



c.



1051-616 PARTIAL Solution Set #6

2. (Book 10.6) Find the Fourier transforms of the following 2-D separable functions and sketch them as profiles or as "images":

- (a) $COR\left[\frac{x}{2}, 2y\right]$, where "COR" is the "corral" function defined in §7.3.6
 (b) $RECT[x, y] * (\delta[x] \cdot 1[y])$
 (c) $RECT[x, y] * (\delta[x-1] \cdot 1[y-1])$
 (d) $RECT[x, y] * CROSS[x, y]$, where $CROSS[x, y] \equiv \delta[x] \cdot 1[y] + 1[x] \cdot \delta[y]$
 (e) $COR[x, y] * COR[x, y]$ where "COR" is the "corral" function defined in §7.3.6

$$COR[x, y] \equiv \left(\delta\left[x + \frac{1}{2}\right] + \delta\left[x - \frac{1}{2}\right] \right) \cdot RECT[y] + RECT[x] \cdot \left(\delta\left[y + \frac{1}{2}\right] + \delta\left[y - \frac{1}{2}\right] \right)$$

$$\mathcal{F}_2\{COR[x, y]\} = 2 \cdot \cos\left[2\pi\xi \cdot \frac{1}{2}\right] \cdot SINC[\eta] + SINC[\xi] \cdot 2 \cdot \cos\left[2\pi\eta \cdot \frac{1}{2}\right]$$

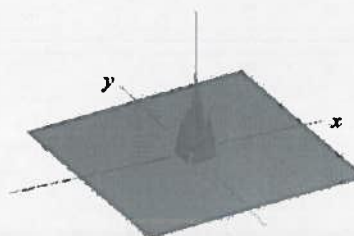
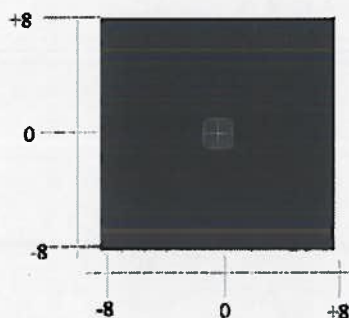
$$= 2 \cdot (\cos[\pi\xi] \cdot SINC[\eta] + SINC[\xi] \cdot \cos[\pi\eta])$$

$$COR[x, y] * COR[x, y] = \mathcal{F}_2^{-1} \left\{ \left(2 \cdot (\cos[\pi\xi] \cdot SINC[\eta] + SINC[\xi] \cdot \cos[\pi\eta]) \right)^2 \right\}$$

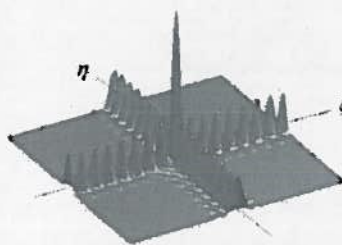
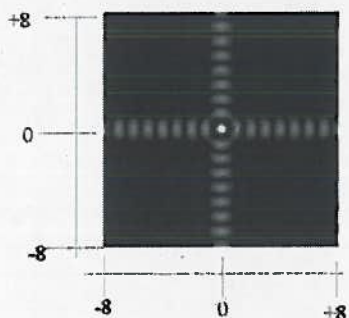
$$= 4 \cdot \mathcal{F}_2^{-1} \left\{ \left((\cos^2[\pi\xi] \cdot SINC^2[\eta] + SINC^2[\xi] \cdot \cos^2[\pi\eta]) + 2 \cos[\pi\xi] \cos[\pi\eta] \cdot SINC[\xi] \cdot SINC[\eta] \right) \right\}$$

$$= 4 \cdot (\delta[x+1] + 2 \cdot \delta[x] + \delta[x-1]) \cdot TRI[y] + TRI[x] \cdot (\delta[y+1] + 2 \cdot \delta[y] + \delta[y-1]) + RECT[x+1]$$

$$f[x, y] = COR[x, y] * COR[x, y]$$



$$F[\xi, \eta]$$



3. Use the Fourier transforms of $\exp[\pm i\pi x^2]$ to derive the 2-D transform

$$\mathcal{F}_2 \{ \exp[\pm i\pi(x^2)] \cdot \exp[\pm i\pi(y^2)] \} = \mathcal{F}_2 \{ \exp[\pm i\pi(x^2 + y^2)] \} = \mathcal{F}_2 \{ \exp[\pm i\pi r^2] \}$$

$$\mathcal{F}_1 \{ \exp[+i\pi x^2] \} = \exp\left[+i\frac{\pi}{4}\right] \cdot \exp[-i\pi\xi^2]$$

$$\mathcal{F}_1 \{ \exp[+i\pi y^2] \} = \exp\left[+i\frac{\pi}{4}\right] \cdot \exp[-i\pi\eta^2]$$

$$\exp[+i\pi x^2] \cdot \exp[+i\pi y^2] = \exp[+i\pi(x^2 + y^2)] = \exp[+i\pi r^2]$$

Note that this is the product of two *ORTHOGONAL* functions, not of two functions along the same axis. You can evaluate the transforms individually and then multiply them. By separability:

$$\begin{aligned} \mathcal{F}_2 \{ \exp[+i\pi r^2] \} &= \mathcal{F}_1 \{ \exp[+i\pi x^2] \} \cdot \mathcal{F}_1 \{ \exp[+i\pi y^2] \} \\ &= \left(\exp\left[+i\frac{\pi}{4}\right] \cdot \exp[-i\pi\xi^2] \right) \cdot \left(\exp\left[+i\frac{\pi}{4}\right] \cdot \exp[-i\pi\eta^2] \right) \\ &= \left(\exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[+i\frac{\pi}{4}\right] \right) \cdot \left(\exp[-i\pi\xi^2] \cdot \exp[-i\pi\eta^2] \right) \\ &= \exp\left[+i\frac{\pi}{2}\right] \cdot \left(\exp[-i\pi(\xi^2 + \eta^2)] \right) \\ &= i \cdot \left(\exp[-i\pi\rho^2] \right) \text{ where } \rho^2 \equiv \xi^2 + \eta^2 \end{aligned}$$

If you want to use the modulation theorem, you must create two 2-D functions that, when convolved, yield the 2-D function::

$$\begin{aligned} (\exp[+i\pi x^2] \cdot \delta[y]) * (\delta[x] \cdot \exp[+i\pi y^2]) &= \exp[+i\pi(x^2 + y^2)] \\ \mathcal{F}_2 \{ (\exp[+i\pi x^2] \cdot \delta[y]) * (\delta[x] \cdot \exp[+i\pi y^2]) \} &= \mathcal{F}_2 \{ (\exp[+i\pi x^2] \cdot \delta[y]) \} \cdot \mathcal{F}_2 \{ (\delta[x] \cdot \exp[+i\pi y^2]) \} \\ &= \left(\exp\left[+i\frac{\pi}{4}\right] \exp[-i\pi\xi^2] \cdot 1[\eta] \right) \cdot \left(1[\xi] \exp\left[+i\frac{\pi}{4}\right] \exp[-i\pi\eta^2] \right) \\ &= \exp\left[+i\frac{\pi}{2}\right] \cdot \exp[-i\pi(\xi^2 + \eta^2)] \end{aligned}$$

$$\boxed{\mathcal{F}_2 \{ (\exp[+i\pi x^2] \cdot \delta[y]) * (\delta[x] \cdot \exp[+i\pi y^2]) \} = i \cdot \exp[-i\pi(\xi^2 + \eta^2)]}$$

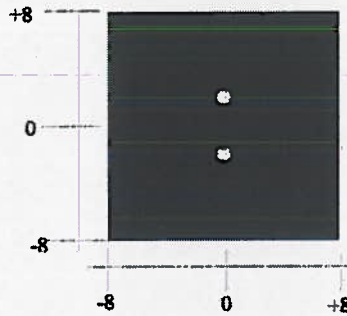
4. Find the results of the convolution and sketch it:

(a) $CYL(r) * (\delta[x, y + 2] + \delta[x, y - 2])$

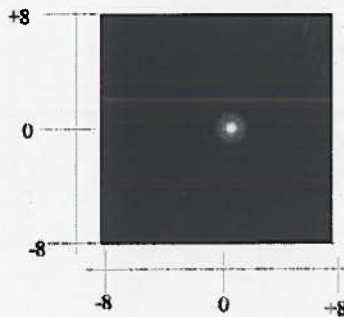
$$\begin{aligned} CYL(r) * (\delta[x, y + 2] + \delta[x, y - 2]) &= CYL(\sqrt{x^2 + y^2}) * (\delta[x, y + 2] + \delta[x, y - 2]) \\ &= (CYL(\sqrt{x^2 + y^2}) * \delta[x, y + 2]) + (CYL(\sqrt{x^2 + y^2}) * \delta[x, y - 2]) \\ &= CYL(\sqrt{x^2 + (y + 2)^2}) + CYL(\sqrt{x^2 + (y - 2)^2}) \end{aligned}$$

This is the sum of two cylinders centered at $[x, y] = [0, \pm 2]$

$$f[x, y] = CYL(r) * \delta[x] \cdot (\delta[y + 2] + \delta[y - 2])$$



$$f[x, y] = GAUS(r) * \delta[x - 1, y]$$

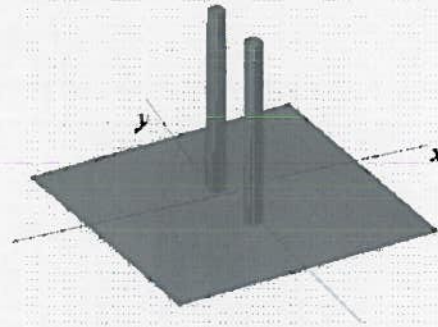
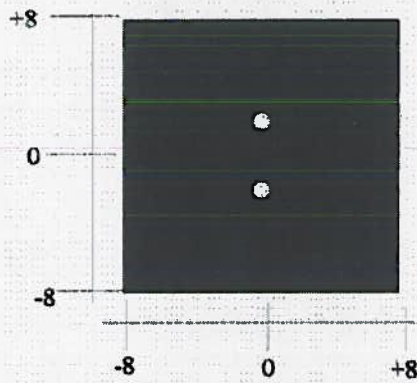


1. (Book §11.2) Evaluate the Fourier transforms of the following functions and sketch them:

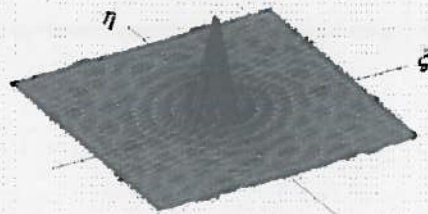
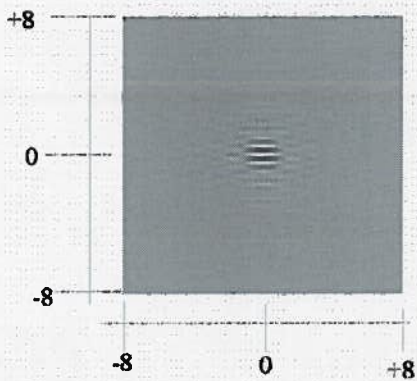
(a) $CYL(r) * (\delta[x] (\delta[y+2] + \delta[y-2]))$

$$\begin{aligned} \mathcal{F}_2 \{CYL(r) * (\delta[x] (\delta[y+2] + \delta[y-2]))\} &= \mathcal{F}_2 \{CYL(r) * (\delta[x] (\delta[y+2] + \delta[y-2]))\} \\ &= \mathcal{F}_2 \{CYL(r)\} \cdot \mathcal{F}_2 \{\delta[x] (\delta[y+2] + \delta[y-2])\} \\ &= \frac{\pi}{4} SOMB(\rho) \cdot (1[\xi] \cdot 2 \cos(2\pi \cdot 2 \cdot \eta)) \end{aligned}$$

$$f[x,y] = CYL(r) * \delta[x] \cdot (\delta[y+2] + \delta[y-2])$$



$$F[\xi, \eta]$$

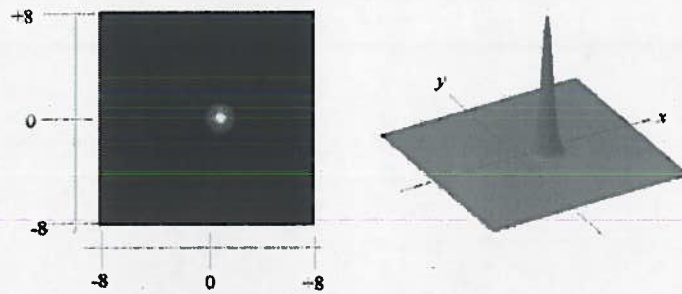


(b) $GAUS(r) * \delta[x-1, y]$

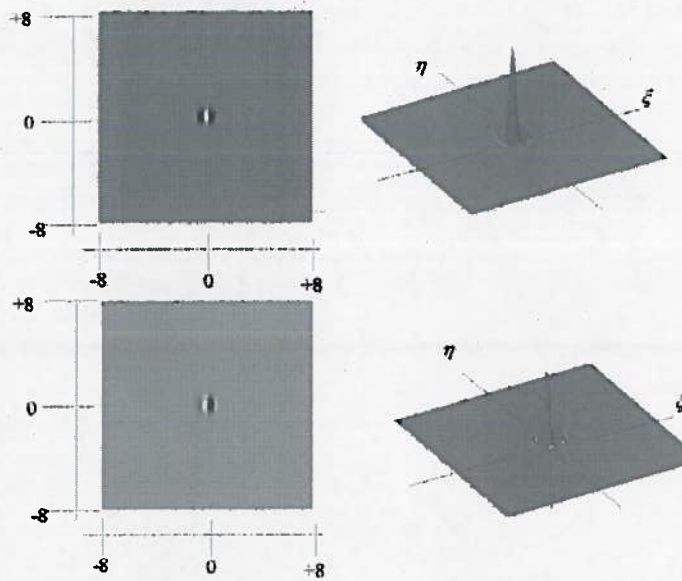
$$\begin{aligned} \mathcal{F}_2\{GAUS(r) * \delta[x-1, y]\} &= GAUS(\rho) \cdot (\exp[-2\pi i \xi \cdot 1]) \cdot 1[\eta] \\ &= GAUS(\rho) \cdot (\cos[2\pi \xi] - i \sin[2\pi \xi]) \\ &= (GAUS(\rho) \cdot \cos[2\pi \xi]) - i \cdot (GAUS(\rho) \cdot \sin[2\pi \xi]) \end{aligned}$$

This is a Gaussian modulated by a complex exponential.

$$f[x, y] = RECT[x, y] * (\delta[x-1] \cdot I[y-1]) = RECT[x-1] \cdot I[y]$$



$$F[\xi, \eta] = SINC[\xi] \exp[+2\pi i \xi] \cdot \delta[\eta]$$



(c) $J_0(2\pi r) + J_0(4\pi r)$

$$J_0(2\pi \cdot r) = J_0(2\pi \cdot 1 \cdot r) \Rightarrow \rho_0 = 1$$

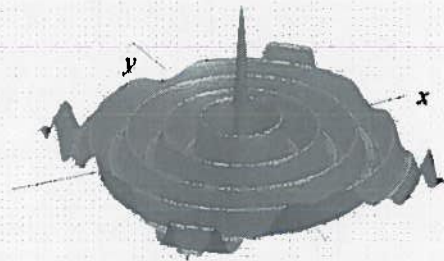
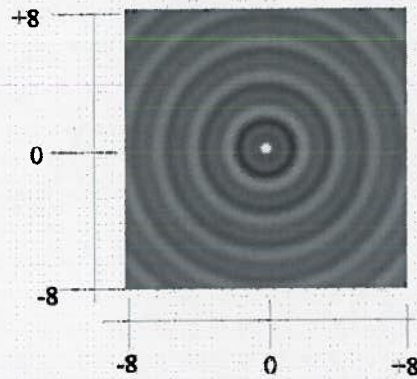
$$\mathcal{F}_2\{J_0(2\pi \cdot 1 \cdot r)\} = \mathcal{H}_0\{J_0(2\pi \cdot 1 \cdot r)\} = \frac{1}{2\pi \cdot 1} \delta(\rho - 1)$$

$$J_0(4\pi \cdot r) = J_0(2\pi \cdot 2 \cdot r) \Rightarrow \rho_0 = 2$$

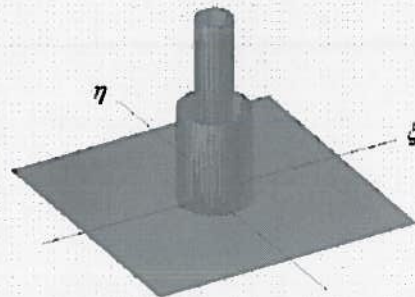
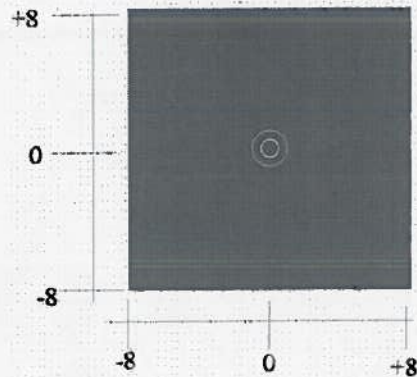
$$\mathcal{F}_2\{J_0(2\pi \cdot 2 \cdot r)\} = \mathcal{H}_0\{J_0(2\pi \cdot 2 \cdot r)\} = \frac{1}{2\pi \cdot 2} \delta(\rho - 2) = \frac{1}{4\pi} \delta(\rho - 2)$$

$$\mathcal{F}_2\{J_0(2\pi r) + J_0(4\pi r)\} = \frac{1}{2\pi} \delta(\rho - 1) + \frac{1}{4\pi} \delta(\rho - 2)$$

$$f[x,y] = \text{CYL}(r) * \delta[x] \cdot (\delta[y+2] + \delta[y-2])$$



$$F[\xi, \eta]$$



$$(d) \exp \left[-i\pi \frac{r^2}{4} \right] + \exp \left[+i\pi \frac{r^2}{4} \right]$$

$$\begin{aligned} & \exp \left[-i\pi \frac{r^2}{4} \right] + \exp \left[+i\pi \frac{r^2}{4} \right] \\ &= \exp \left[-i\pi \frac{x^2 + y^2}{2^2} \right] + \exp \left[+i\pi \frac{x^2 + y^2}{2^2} \right] = \exp \left[-i\pi \frac{x^2}{2^2} \right] \cdot \exp \left[-i\pi \frac{y^2}{2^2} \right] + \exp \left[+i\pi \frac{x^2}{2^2} \right] \cdot \exp \left[+i\pi \frac{y^2}{2^2} \right] \\ &= 2 \cdot \cos \left[\pi \frac{x^2 + y^2}{2^2} \right] = 2 \cdot \cos \left[\pi \left(\frac{r}{2} \right)^2 \right] \end{aligned}$$

$$\begin{aligned} & \mathcal{F}_2 \left\{ \exp \left[-i\pi \frac{r^2}{4} \right] + \exp \left[+i\pi \frac{r^2}{4} \right] \right\} \\ &= \left(|2| \cdot \exp \left[+i\frac{\pi}{4} \right] \exp \left[-i\pi (2\xi)^2 \right] \right) \cdot \left(|2| \cdot \exp \left[+i\frac{\pi}{4} \right] \exp \left[-i\pi (2\eta)^2 \right] \right) \\ &\quad + \left(|2| \cdot \exp \left[-i\frac{\pi}{4} \right] \exp \left[+i\pi (2\xi)^2 \right] \right) \cdot \left(|2| \cdot \exp \left[-i\frac{\pi}{4} \right] \exp \left[+i\pi (2\eta)^2 \right] \right) \\ &= 4 \cdot i \cdot \exp \left[-i\pi (2\xi)^2 \right] \cdot \exp \left[-i\pi (2\eta)^2 \right] + 4 \cdot (-i) \cdot \exp \left[+i\pi (2\xi)^2 \right] \cdot \exp \left[+i\pi (2\eta)^2 \right] \\ &= 4 \cdot i \cdot \left(\exp \left[-i\pi \cdot 4 \cdot \rho^2 \right] - \exp \left[+i\pi \cdot 4 \cdot \rho^2 \right] \right) \\ &= -4 \cdot i \cdot \left(\exp \left[+i\pi \cdot 4 \cdot \rho^2 \right] - \exp \left[-i\pi \cdot 4 \cdot \rho^2 \right] \right) \\ &= -4 \cdot i \cdot (2i \cdot \sin [\pi \cdot 4 \cdot \rho^2]) = +8 \cdot \sin [\pi \cdot 4 \cdot \rho^2] \end{aligned}$$

$$\boxed{\mathcal{F}_2 \left\{ \exp \left[-i\pi \frac{r^2}{4} \right] + \exp \left[+i\pi \frac{r^2}{4} \right] \right\} = +8 \cdot \sin \left[\pi \cdot \left(\frac{\rho}{\frac{1}{2}} \right)^2 \right]}$$

A circularly symmetric sine chirp function with chirp rate $\frac{1}{2}$

6. Find the transfer function of the imaging systems with the following impulse responses:

(a) $h_a(r) = J_0(2\pi r) + J_0(\pi r)$

$$J_0(2\pi \cdot r) = J_0(2\pi \cdot 1 \cdot r) \Rightarrow \rho_0 = 1$$

$$\mathcal{F}_2\{J_0(2\pi \cdot 1 \cdot r)\} = \mathcal{H}_0\{J_0(2\pi \cdot 1 \cdot r)\} = \frac{1}{2\pi \cdot 1} \delta(\rho - 1)$$

$$J_0(4\pi \cdot r) = J_0(2\pi \cdot 2 \cdot r) \Rightarrow \rho_0 = 2$$

$$\mathcal{F}_2\{J_0(2\pi \cdot 2 \cdot r)\} = \mathcal{H}_0\{J_0(2\pi \cdot 2 \cdot r)\} = \frac{1}{2\pi \cdot 2} \delta(\rho - 2) = \frac{1}{4\pi} \delta(\rho - 2)$$

$$\mathcal{F}_2\{J_0(2\pi r) + J_0(4\pi r)\} = \boxed{H_a(\rho) = \frac{1}{2\pi} \delta(\rho - 1) + \frac{1}{4\pi} \delta(\rho - 2)}$$

(b) $h_b(r) = \text{SOMB}\left(\frac{r}{10}\right)$

$$H_b(\rho) = H_0\left\{\text{SOMB}\left(\frac{r}{10}\right)\right\} = \frac{4}{\pi \cdot 10^2} \cdot \text{CYL}(10\rho) = \frac{1}{25\pi} \cdot \text{CYL}\left(\frac{\rho}{\frac{1}{10}}\right)$$

which is very "narrow" because the impulse response is very "wide"; this is an ideal lowpass filter with cutoff frequency at $\rho_{\text{cutoff}} = \sqrt{\xi^2 + \eta^2} = \frac{1}{20}$ cycle per unit length.

(c) $h_c(r) = -r^2 \text{GAUS}(r)$

$$H_c(\rho) = \mathcal{F}_2\{-r^2 \text{GAUS}(r)\} = \mathcal{H}_0\{-r^2 \text{GAUS}(r)\}$$

$$\mathcal{F}_2\{r^2 \text{GAUS}(r)\} = \mathcal{F}_2\{x^2 + y^2\} * \mathcal{F}_2\{\text{GAUS}(x^2 + y^2)\}$$

$$= -\frac{1}{4\pi^2} (\delta''[\xi] + \delta''[\eta]) * e^{-\pi(\xi^2 + \eta^2)}$$

$$= -\frac{1}{4\pi^2} \left[\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) e^{-\pi(\xi^2 + \eta^2)} \right]$$

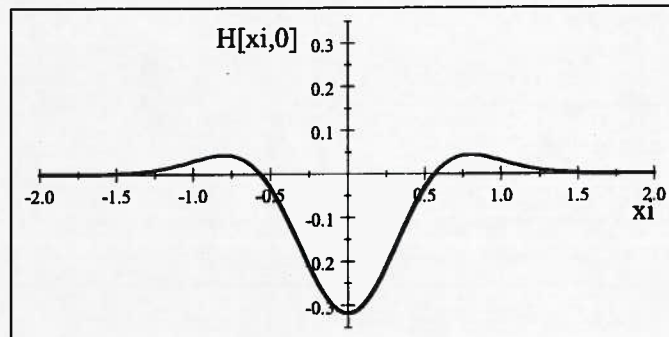
$$= \left(\frac{1}{\pi} - (\xi^2 + \eta^2) \right) e^{-\pi(\xi^2 + \eta^2)}$$

$$= \left(\frac{1}{\pi} - \rho^2 \right) \cdot \text{GAUS}(\rho)$$

$$\Rightarrow H_c(\rho) = \left(\rho^2 - \frac{1}{\pi} \right) \cdot \text{GAUS}(\rho)$$

$$H_c[\xi, \eta = 0] = \left((\xi^2 + 0^2) - \frac{1}{\pi} \right) \cdot \exp[-\pi(\xi^2 + 0^2)]$$

$$= \left(\xi^2 - \frac{1}{\pi} \right) \cdot \exp[-\pi\xi^2]$$



7. (Book §12.3) Evaluate AND SKETCH the results of the following 2-D operations, where the symbols “*” and “★” denote 2-D convolution and correlation, respectively:

$$(a) (\cos[\pi\xi] \cdot \text{SINC}[\eta] + \text{SINC}[\xi] \cdot \cos[\pi\eta]) \star (\cos[2\pi\xi] \cdot \text{SINC}[\frac{\eta}{2}] + \text{SINC}[\frac{\xi}{2}] \cdot \cos[2\pi\eta])$$

$$F[\xi, \eta] \star M[\xi, \eta] = F[\xi, \eta] * M^*[-\xi, -\eta]$$

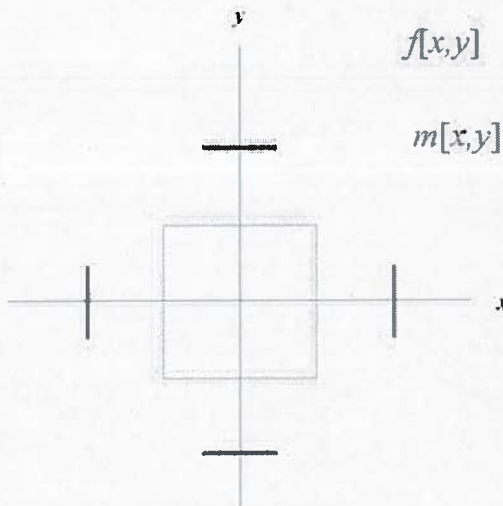
but $M[\xi, \eta]$ is real and even, so this is equivalent to a convolution. We can transform back to the space domain and multiply:

$$f[x, y] = \mathcal{F}_2^{-1}\{F[\xi, \eta]\} = \frac{1}{2} \left(\delta\left[x + \frac{1}{2}\right] + \delta\left[x - \frac{1}{2}\right] \right) \cdot \text{RECT}[y] + \text{RECT}[x] \cdot \frac{1}{2} \left(\delta\left[y + \frac{1}{2}\right] + \delta\left[y - \frac{1}{2}\right] \right)$$

which is the “corral” function. The space-domain representation of the second function is:

$$\begin{aligned} m[x, y] &= \mathcal{F}_2^{-1}\left\{ \cos[2\pi\xi] \cdot \text{SINC}\left[\frac{\eta}{2}\right] + \text{SINC}\left[\frac{\xi}{2}\right] \cdot \cos[2\pi\eta] \right\} \\ &= \frac{1}{2} (\delta[x+1] + \delta[x-1]) \cdot 2 \cdot \text{RECT}\left[\frac{y}{\frac{1}{2}}\right] + 2 \cdot \text{RECT}\left[\frac{x}{\frac{1}{2}}\right] \cdot \frac{1}{2} (\delta[y+1] + \delta[y-1]) \\ &= (\delta[x+1] + \delta[x-1]) \cdot \text{RECT}\left[\frac{y}{\frac{1}{2}}\right] + \text{RECT}\left[\frac{x}{\frac{1}{2}}\right] \cdot (\delta[y+1] + \delta[y-1]) \end{aligned}$$

which is composed of four “line-delta” functions of length $\frac{1}{2}$ along y centered at $x = \pm 1$ and along x centered at $y = \pm 1$. We must now multiply these two functions together.



As shown from the sketch, the line deltas do not overlap so the product in the space domain is zero:

$$f[x, y] \cdot m[x, y] = 0[x, y] \implies F[\xi, \eta] * M[\xi, \eta] = 0[\xi, \eta]$$

(b) $J_0(2\pi\rho_0 r) * J_0(2\pi\rho_1 r)$, where $\rho_0 \neq \rho_1$

This is really the same problem as (a) but in the circular case. If we evaluate the spectra of the two components, we see that they are two ring delta functions:

$$\mathcal{H}_0 \{J_0(2\pi\rho_0 r)\} = \frac{1}{2\pi\rho_0} \delta(\rho - \rho_0)$$

$$\mathcal{H}_0 \{J_0(2\pi\rho_1 r)\} = \frac{1}{2\pi\rho_1} \delta(\rho - \rho_1)$$

If $\rho_0 \neq \rho_1$, then the transforms do not overlap and the product is zero:

$$J_0(2\pi\rho_0 r) * J_0(2\pi\rho_1 r) = \mathcal{H}_0^{-1} \left\{ \frac{1}{2\pi\rho_0} \delta(\rho - \rho_0) \cdot \frac{1}{2\pi\rho_1} \delta(\rho - \rho_1) \right\} = \mathcal{H}_0 \{0(\rho)\} = 0(r)$$

(c) $CYL(r) * (\delta[x] \cdot 1[y])$

This is the Radon transform of the cylinder function, which was evaluated in the book:

$$\begin{aligned} CYL(r) * (\delta[x] \cdot 1[y]) &= RECT[x] \cdot \int_{y=-\sqrt{\frac{1}{4}-x^2}}^{+\sqrt{\frac{1}{4}-x^2}} 1 \cdot dy \\ &= RECT[x] \cdot 2 \cdot \sqrt{\frac{1}{4}-x^2} \\ &= \sqrt{1-4x^2} \cdot RECT[x] \end{aligned}$$

