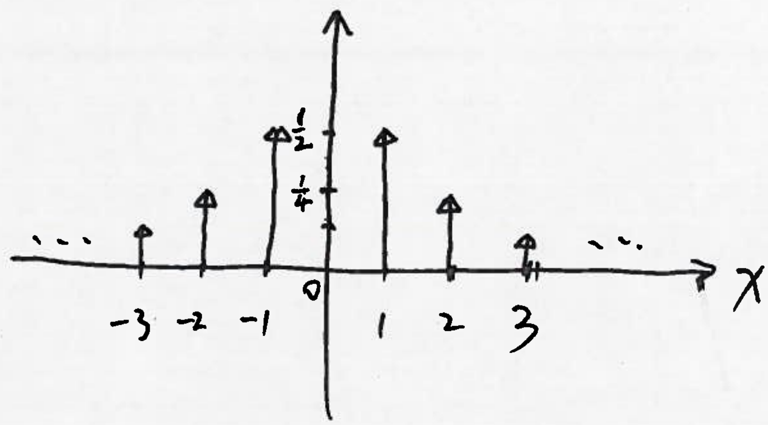
IMGS-616-20141 FOURIER METHODS

HW # 4.

1.

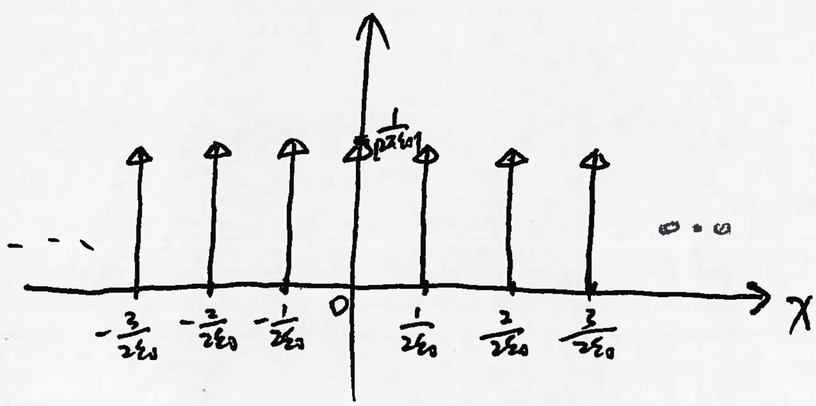
(a) $\sum_{n=1}^{\infty} \delta(x^2-n^2) = \sum_{n=1}^{\infty} \frac{1}{|2n|} \left(\delta(x-n) + \delta(x+n)\right)$



(b).

$\sin(2\pi\xi_0 x) = 0 \quad \Rightarrow \quad x = \frac{n}{2\xi_0}.$

$\delta(\sin(2\pi\xi_0 x)) = \sum_{n=-\infty}^{+\infty} \frac{1}{|2\pi\xi_0|} \cdot \delta(x - \frac{n}{2\xi_0}).$



2.

(a). $S(a_1 f_1(x) + b_1 f_2(x)) = \left[a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right]\cdot(a_1 f_1(x) + b_1 f_2(x))$

$= a_1\left[a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right]\cdot f_1(x)$

$+ b_1\left[a\frac{d}{dx^2} + b\frac{d}{dx} + c\right]\cdot f_2(x)$

$= a_1 S(f_1(x)) + b_1 S(f_2(x))$

S is linear system.

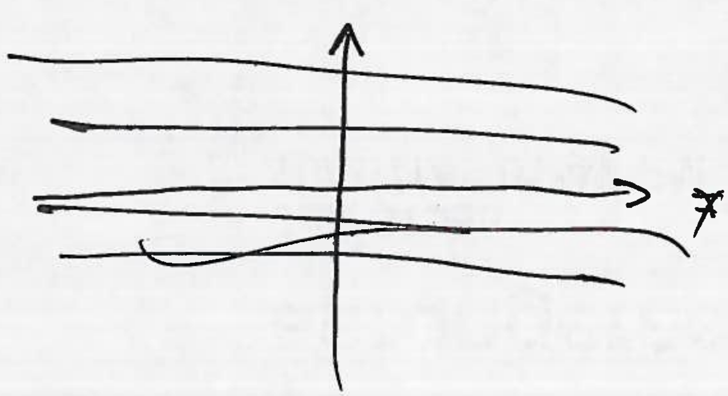
(b). $S(f(x-x_0)) = \left[a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right]\cdot f(x-x_0)$

$= \left[a\frac{d^2}{d(x-x_0)^2} + b\frac{d}{d(x-x_0)} + c\right] f(x-x_0)$

$= g(x-x_0).$

S is shift invariant

~~(c). for a=b=c=1.~~



for $a=b=c=1$

$S(f(x)) = \left[\frac{d^2}{dx^2} + \frac{d}{dx} + 1\right]\cdot\cos(\pi x)$

$= -\pi^2\cdot\cos(\pi x) - \pi\sin(\pi x) + \cos(\pi x)$

3.

a). $S(k_1 f_1(x) + k_2 f_2(x)) = a_0 \cdot (k_1 f_1(x) + k_2 f_2(x))^2 + b_0 (k_1 f_1(x) + k_2 f_2(x))$

$= a_0 \cdot (k_1 f_1(x))^2 + k_2^2 \cdot a_0 f_2^2(x) + 2a_0 k_1 k_2 f_1(x) + f_2(x) + k_1 b_0 f_1(x) + k_2 b_0 f_2(x)$

$\neq k_1 S(f_1(x)) + k_2 S(f_2(x)) = k_1 \cdot a_0 f_1^2(x) + b_0 \cdot k_1 f_1(x) + k_2 a_0 f_2^2(x) + k_2 b_0 f_2(x)$

not linear

$S(f(x-x_0)) = a_0 \cdot (f(x-x_0))^2 + b_0 f(x-x_0) = g(x-x_0)$

shift invariant.

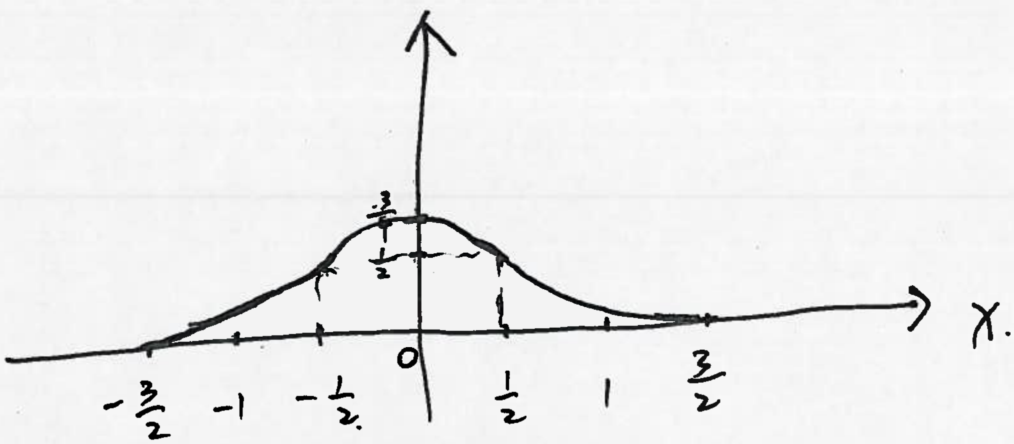
b). for $a_0 = b_0 = 1$,

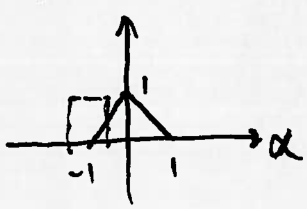
$S(f(x)) = [\exp(-\pi x^2)]^2 + \exp(-\pi x^2) = \exp(-2\pi x^2) + \exp(-\pi x^2).$

5.

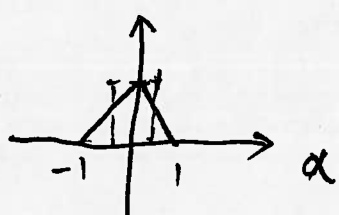
$$TRI(x) * RECT(x) = \int_{-\infty}^{+\infty} TRI(\alpha)\cdot RECT(x-\alpha)\, d\alpha .$$

$$= \begin{cases} 0 & |x| \geq \frac{3}{2} \\ \int_{-1}^{x+\frac{1}{2}} (1-|\alpha|)\, d\alpha & -\frac{3}{2} < x \leq -\frac{1}{2} \\ \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} (1-|\alpha|)\, d\alpha & -\frac{1}{2} < x \leq \frac{1}{2} \\ \int_{x-\frac{1}{2}}^{1} (1-|\alpha|)\, d\alpha & \frac{1}{2} < x \leq \frac{3}{2} \end{cases} = \begin{cases} 0 & |x| \geq \frac{3}{2} \\ \frac{1}{2}(x+\frac{3}{2})^2 & -\frac{3}{2} < x \leq -\frac{1}{2} \\ \frac{3}{4} - x^2 & -\frac{1}{2} < x \leq \frac{1}{2} \\ \frac{1}{2}(x-\frac{3}{2})^2 & \frac{1}{2} < x < \frac{3}{2} \end{cases}$$

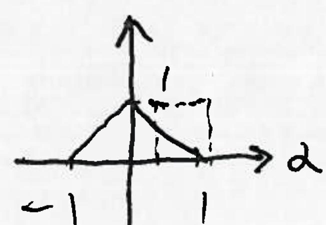




$-\frac{3}{2} < x \leq -\frac{1}{2}$



$-\frac{1}{2} < x \leq \frac{1}{2}$



$\frac{1}{2} < x < \frac{3}{2}$

6.

(a). $f(x) \star f(x) =$

$$f(x) * f^*(-x) = \left[\delta(x) + \delta(x-1) + i(\delta(x-1) + \delta(x-4))\right] *$$
$$\left[\delta(-x) + \delta(-x-1) - i(\delta(-x-1) + \delta(-x-4))\right]$$

using: $\delta(x) * \delta(x-x_0) = \delta(x-x_0)$

$\delta(x-x_0) * \delta(x-x_1) = \delta(x-x_0-x_1)$

$$f(x) * f^*(-x) = 4\delta(x) + \delta(x+1) + \delta(x-1) + \delta(x+3) + \delta(x-3)$$
$$+ i(\delta(x-1) + \delta(x-3) + \delta(x-4) - \delta(x+1) - \delta(x+3) - \delta(x+4))$$

(b) $$f(x) * f(x) = (\delta(x) + \delta(x-1) + i(\delta(x-1) + \delta(x-4)))$$
$$* (\delta(x) + \delta(x-1) + i(\delta(x-1) + \delta(x-4))).$$

$$= \delta(x) + 2\delta(x-1) - \delta(x-8) - 2\delta(x-5)$$
$$+ i \cdot 2(\delta(x-1) + \delta(x-2) + \delta(x-4) + \delta(x-5))$$

7. $\mathcal{f}$ stands for fourier transform.

(a): $F(\xi) = \mathcal{f}(\mathrm{RECT}(\frac{x}{2})) = 2\,\mathrm{sinc}(2\xi)$.

(b) $G(\xi) = \mathcal{f}(\mathrm{RECT}(x-1)) = e^{-i2\pi\xi}\,\mathrm{sinc}(\xi)$.

(c). $H(\xi) = \mathcal{f}(\frac{1}{2}\mathrm{RECT}(\frac{x-1}{2})) = \frac{1}{2}\cdot 2\cdot e^{-i2\pi\xi}\cdot\mathrm{sinc}(2\xi)$

$= e^{-i2\pi\xi}\cdot\mathrm{sinc}(2\xi)$

(d) $P(\xi) = \mathcal{f}(\mathrm{RECT}(x-2)\cdot\exp(+2\pi i x))$

$= \left[e^{-i2\pi\xi\cdot 2}\cdot\mathrm{sinc}(\xi)\right] * \delta(\xi-1)$.

$= e^{-i4\pi(\xi-1)}\cdot\mathrm{sinc}(\xi-1)$.

(e) $R(\xi) = \mathcal{f}(\frac{1}{2}\mathrm{RECT}(\frac{x}{2}) + \mathrm{TRI}(x))$

$= \frac{1}{2}\cdot 2\cdot\mathrm{sinc}(2\xi) + \mathrm{sinc}(\xi)\cdot\mathrm{sinc}(\xi)$

$= \mathrm{sinc}(2\xi) + \mathrm{sinc}^2(\xi)$

8.

(a) $\mathcal{F}(\mathrm{RECT}(x) * \mathrm{RECT}(x))$

$$= \mathrm{sinc}(\xi) \cdot \mathrm{sinc}(\xi) = \mathrm{sinc}^2(\xi)$$

(b) $\mathcal{F}(\mathrm{RECT}(x-1) * \mathrm{RECT}(x))$

$$= e^{-i2\pi\xi} \cdot \mathrm{sinc}(\xi) \cdot \mathrm{sinc}(\xi) = e^{-i2\pi\xi} \cdot \mathrm{sinc}^2(\xi).$$

(c) $\mathcal{F}(\mathrm{RECT}(x-1) * \mathrm{RECT}(x+1)) = \mathcal{F}(\mathrm{RECT}(x) * \delta(x-1) * \mathrm{RECT}(x) * \delta(x+1))$

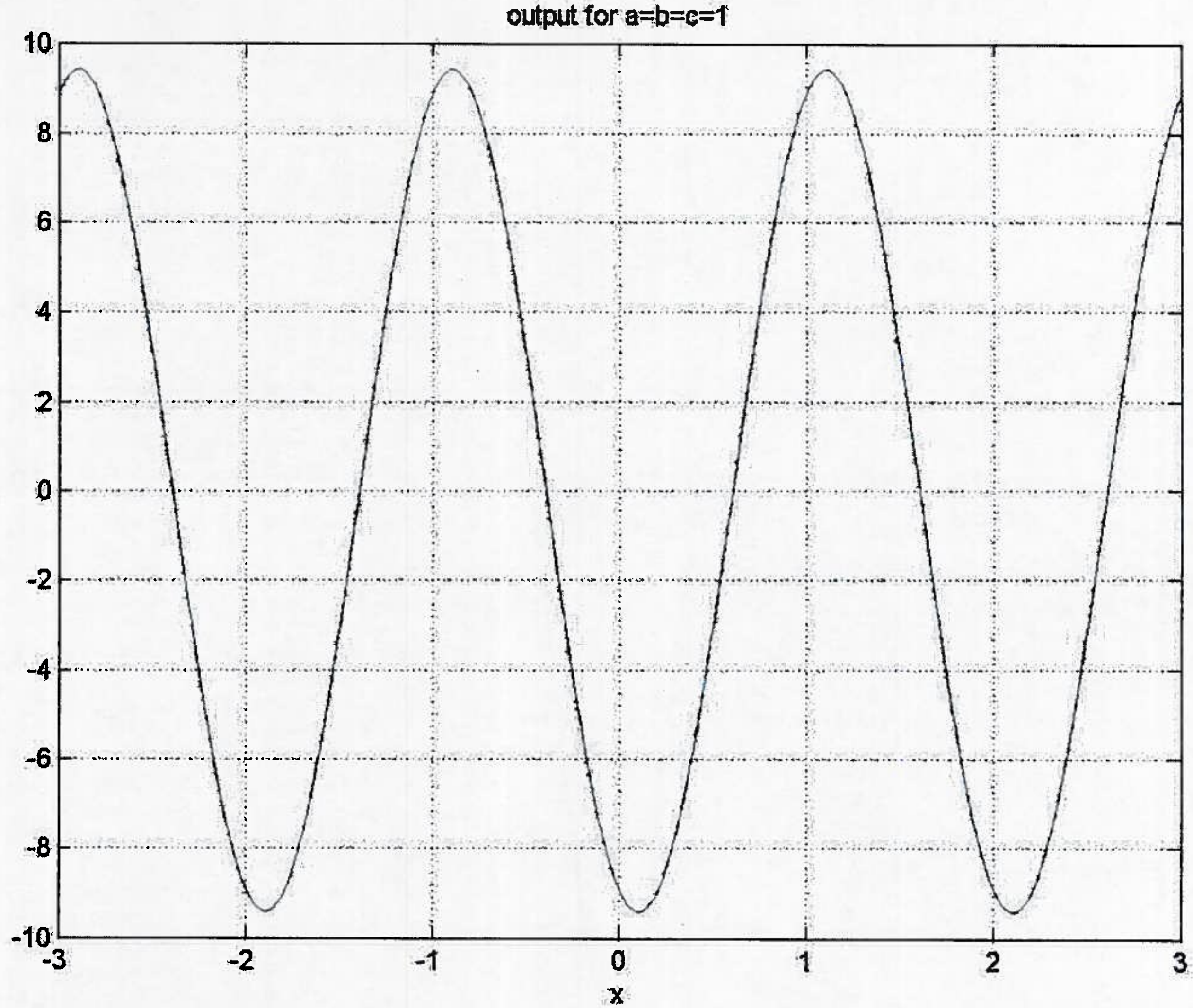
$$= \mathcal{F}(\mathrm{RECT}(x) * \mathrm{RECT}(x)) = \mathrm{sinc}^2(\xi)$$

(d) $\mathcal{F}(\mathrm{RECT}(x-1) \star \mathrm{RECT}(x+1))$

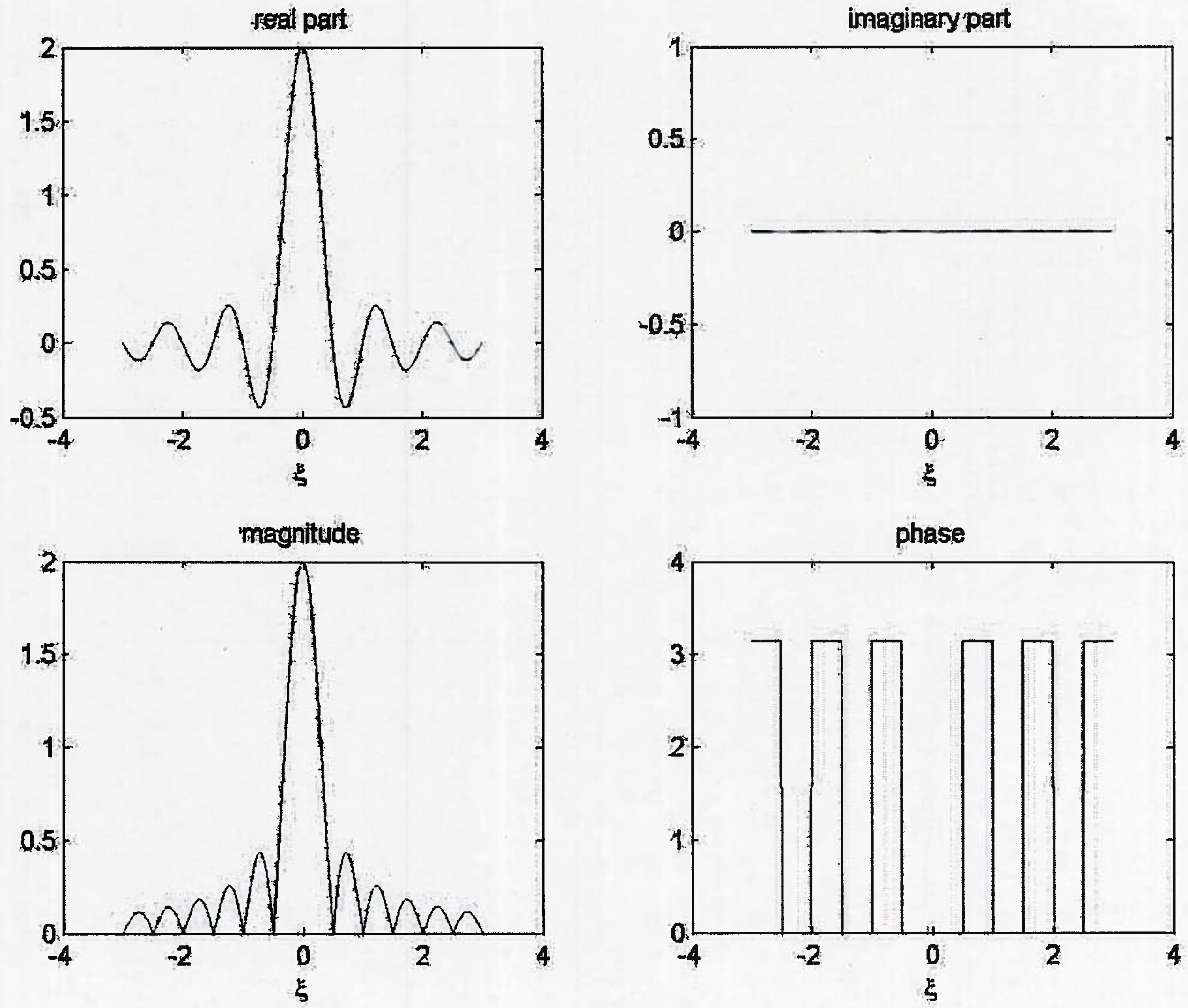
$$= e^{-i2\pi\xi} \cdot \mathrm{sinc}(\xi) \cdot \left(\mathrm{sinc}(\xi) \cdot e^{i2\pi\xi}\right)^*$$

$$= e^{-i4\pi\xi} \cdot \mathrm{sinc}^2(\xi)$$

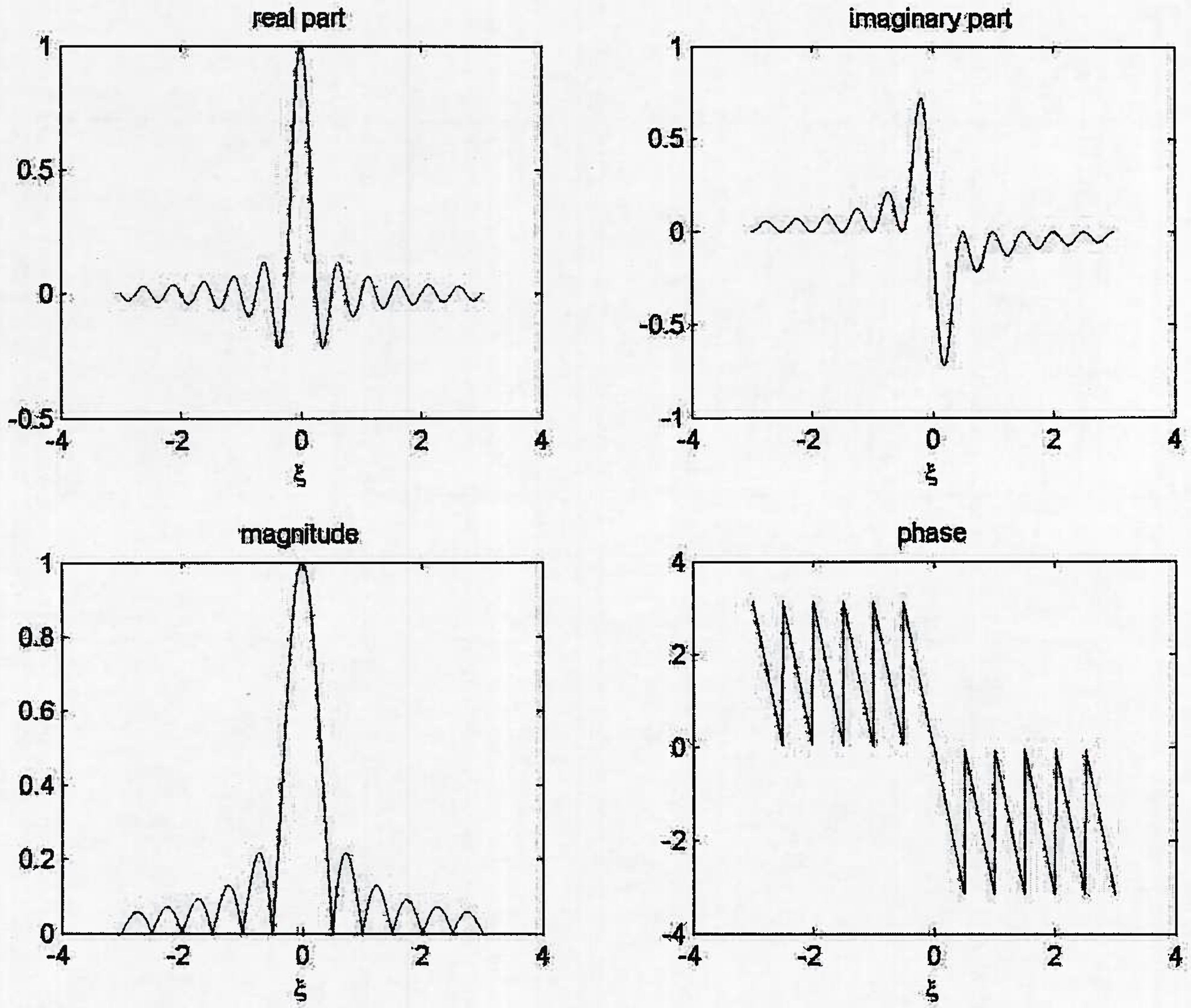
2 c)



7 a)



7 c)


real part
imaginary part
magnitude
phase
1
0.5
0
-0.5
-1
0.8
0.6
0.4
0.2
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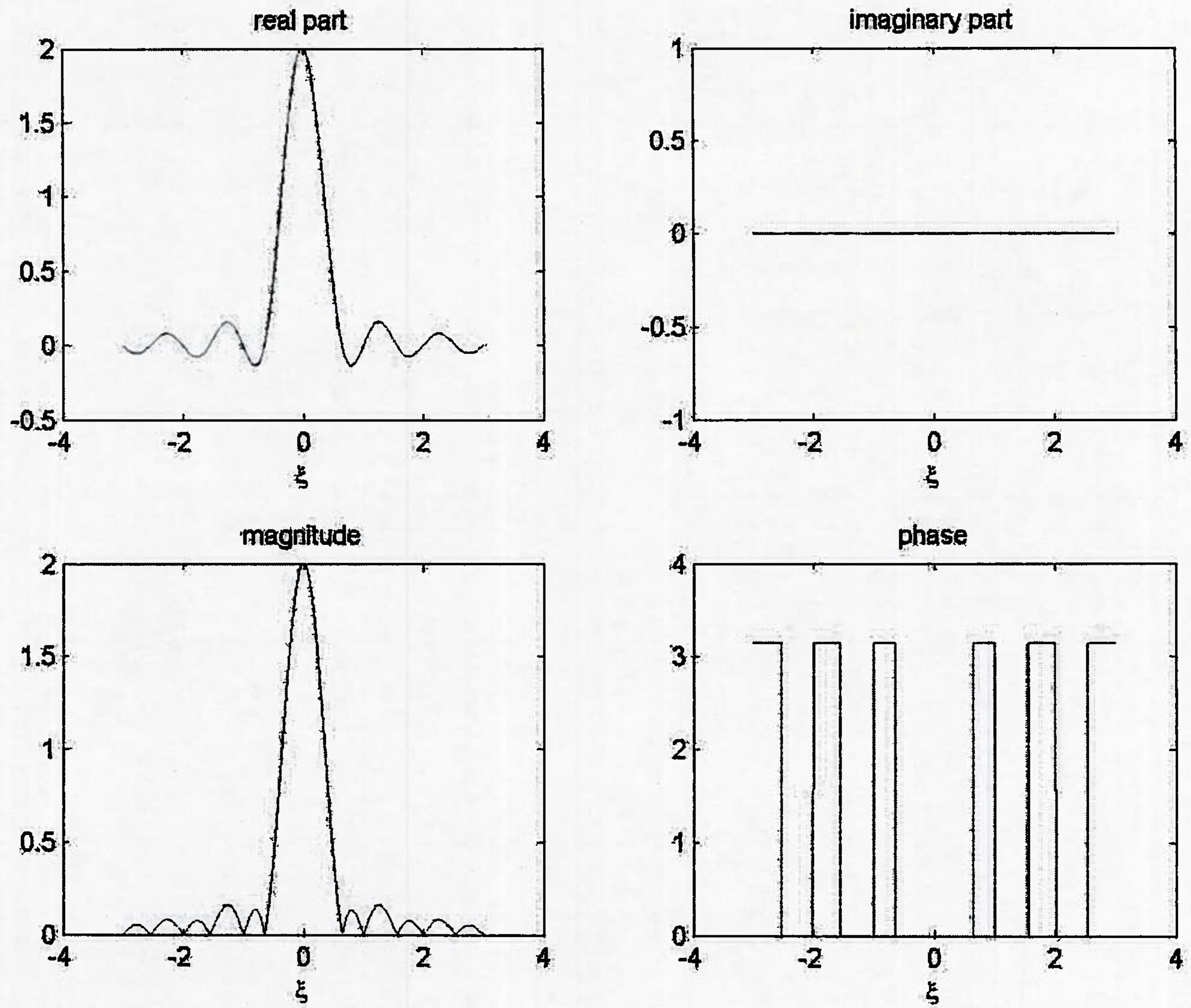
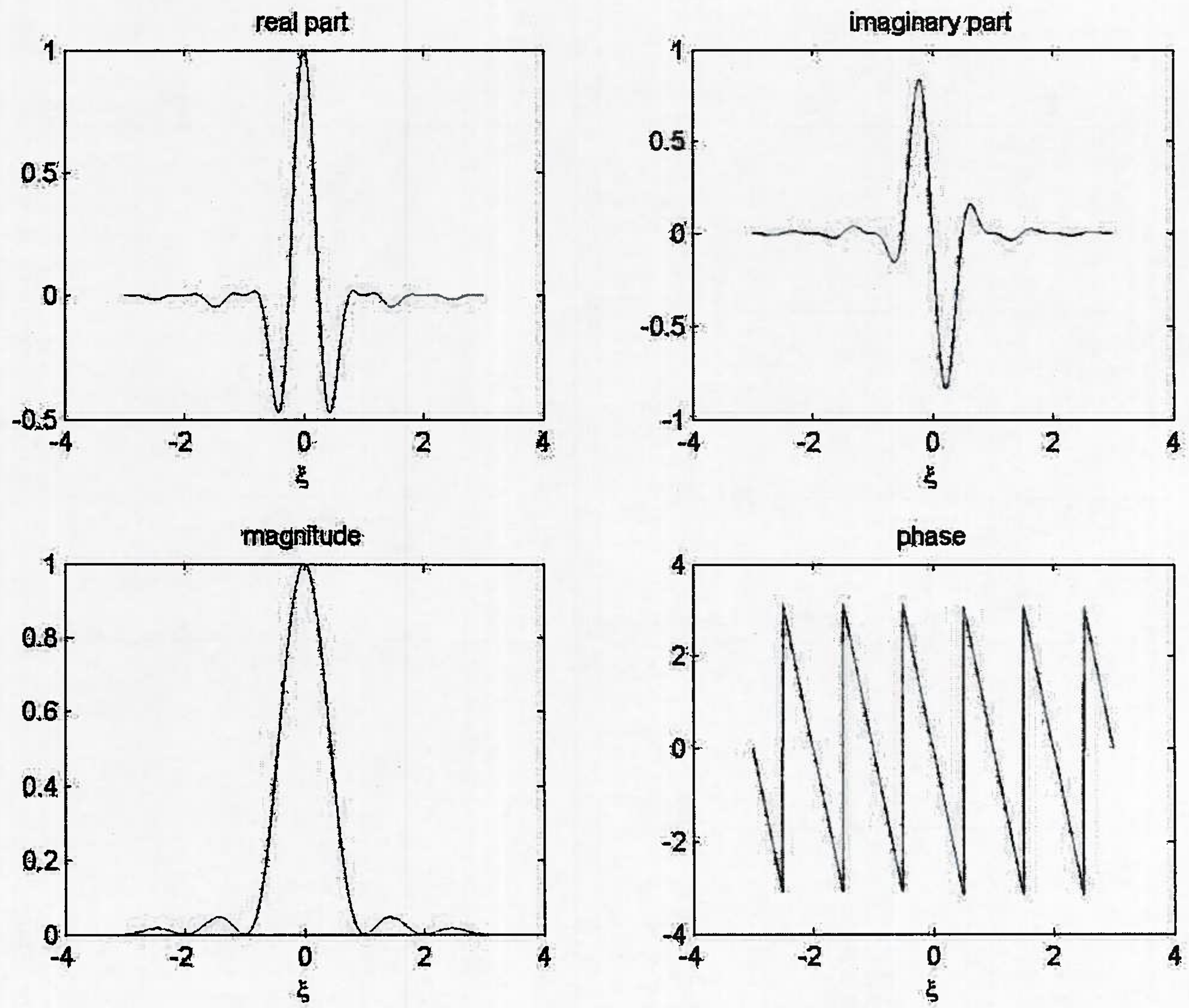
-2
-4
ξ

7e)



8 b)



8 d)