SIMG-616-20142 Solutions to Exam #2

First comment: *lots of people disobeyed the first law of problem solving: "draw the picture!" (though of course I often use a more colorful modifying adjective). Perhaps the problem is deciding which graph to draw ... I'll try to illustrate for the different problems where relevant. Many didn't seem to see that convolution with Dirac delta functions usually does not require transformation to the frequency domain, just use:*

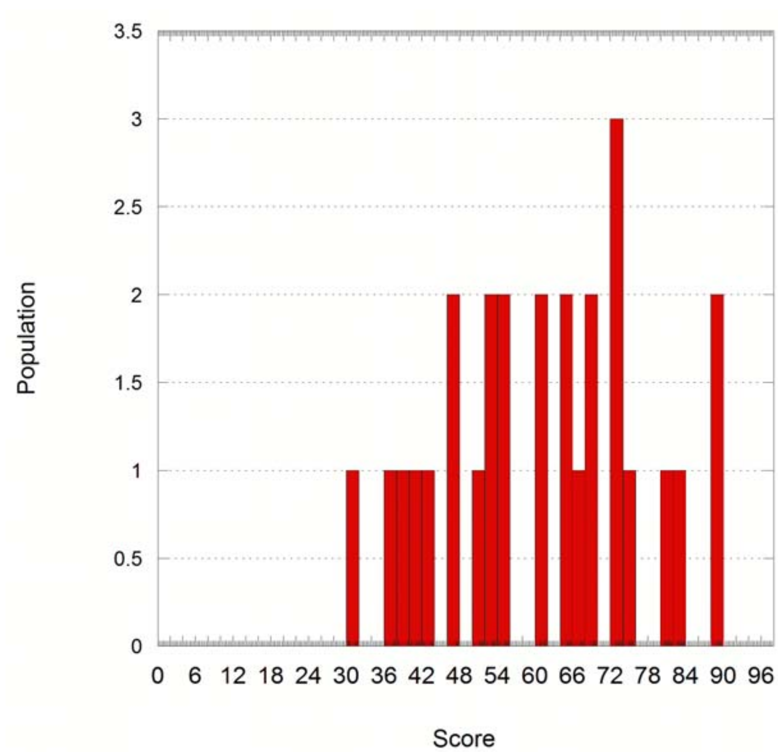
$$f[x] * \delta[x - x_0] = f[x - x_0]$$

Note the difference if the two functions are multiplied instead of convolved; the result is a scaled Dirac delta function

$$f[x] \cdot \delta[x - x_0] = f[x_0] \cdot \delta[x - x_0]$$

Statistics:

Metric	Score
μ	**60.4**
σ	**15.7**
Max	**89**
Min	**31**
Median	**61**



1. (10%) For a function $f[x]$ that is imaginary and "odd" (antisymmetric), DERIVE an expression that describes the character of its spectrum (i.e., is the spectrum real/imaginary/both and/or even/odd/both?).

Solution: *This question is asking for statements of how even and odd functions are affected by Fourier analysis. Start with:*

$$F[\xi] = \int_{-\infty}^{+\infty} f[x] \ \exp[-i \cdot 2\pi\xi x] \ dx$$

Substitute a general expression for the input function decomposed into real and imaginary parts and even and odd parts:

$$f[x] = \mathrm{Re}\{f_{\mathrm{even}}[x]\} + i \cdot \mathrm{Im}\{f_{\mathrm{even}}[x]\} + \mathrm{Re}\{f_{\mathrm{odd}}[x]\} + i \cdot \mathrm{Im}\{f_{\mathrm{odd}}[x]\}$$

If $f[x]$ is imaginary and odd, then only one of these is nonzero:

$$f[x] = 0 + i \cdot 0 + 0 + i \cdot \mathrm{Im}\{f_{\mathrm{odd}}[x]\} = i \cdot \mathrm{Im}\{f_{\mathrm{odd}}[x]\}$$

Substitute this into the Fourier integral:

$$\begin{aligned}
F[\xi] &= \int_{-\infty}^{+\infty} (i \cdot \mathrm{Im}\{f_{\mathrm{odd}}[x]\}) \cdot (\cos[2\pi\xi x] - i \cdot \sin[2\pi\xi x]) \ dx \\
&= i \cdot \int_{-\infty}^{+\infty} \mathrm{Im}\{f_{\mathrm{odd}}[x]\} \cdot (\cos[2\pi\xi x] - i \cdot \sin[2\pi\xi x]) \ dx \\
&= i \cdot \int_{-\infty}^{+\infty} \mathrm{Im}\{f_{\mathrm{odd}}[x]\} \cdot \cos[2\pi\xi x] \ dx + i \cdot \int_{-\infty}^{+\infty} \mathrm{Im}\{f_{\mathrm{odd}}[x]\} \cdot (-i \cdot \sin[2\pi\xi x]) \ dx
\end{aligned}$$

The product of odd part $f_{\mathrm{odd}}[x]$ and the even cosine in the first integral yields an odd function, whose area MUST be zero, so the first integral evaluates to zero ... poof! The product of the odd part of f and the odd sine is even, regardless of the value or sign of ξ, so the area may be nonzero. The product of the two factors of $\pm i$ is $+1$.

$$\begin{aligned}
F[\xi] &= i \cdot 0 + i \cdot (-i) \int_{-\infty}^{+\infty} \mathrm{Im}\{f_{\mathrm{odd}}[x]\} \cdot (\sin[2\pi\xi x]) \ dx \\
&= +1 \cdot \int_{-\infty}^{+\infty} \mathrm{Im}\{f_{\mathrm{odd}}[x]\} \cdot (\sin[2\pi\xi x]) \ dx
\end{aligned}$$

The sine function changes sign for positive or negative ξ, so the sign of the integral also changes. This ensures that the spectrum is an odd function of ξ. In short, the spectrum of the odd imaginary space-domain function is $\boxed{\textit{REAL and ODD}}$.

One simple confirmation: we know the spectrum of the real and odd sine is imaginary and odd

$$\mathcal{F}\{\sin[2\pi\xi_0 x]\} = i \cdot \left(\frac{1}{2}\delta[\xi + \xi_0] - \frac{1}{2}\delta[\xi - \xi_0]\right)$$

Now use the linearity of the Fourier transform by multiplying both sides by "i":

$$\begin{aligned}
\mathcal{F}\{i \cdot \sin[2\pi\xi_0 x]\} &= i \cdot \mathcal{F}\{\sin[2\pi\xi_0 x]\} = i \cdot i \cdot \left(\frac{1}{2}\delta[\xi + \xi_0] - \frac{1}{2}\delta[\xi - \xi_0]\right) \\
&= \frac{1}{2}\delta[\xi - \xi_0] - \frac{1}{2}\delta[\xi + \xi_0]
\end{aligned}$$

which is real and odd.

2. (10%) Describe the conditions on the real-valued parameters b_0, d_0, and A_0 such that this expression is valid:

$$SINC\left[\frac{x}{b_0}\right] * SINC^3\left[\frac{x}{d_0}\right] = A_0 \cdot SINC^3\left[\frac{x}{d_0}\right]$$

(you need not derive the conditions, but you need to show why they must be valid; sketches will be helpful).

Solution: *This is a case where sketches would be helpful. We did not derive an expression for the spectrum of $SINC^3[x]$, but clearly we have*

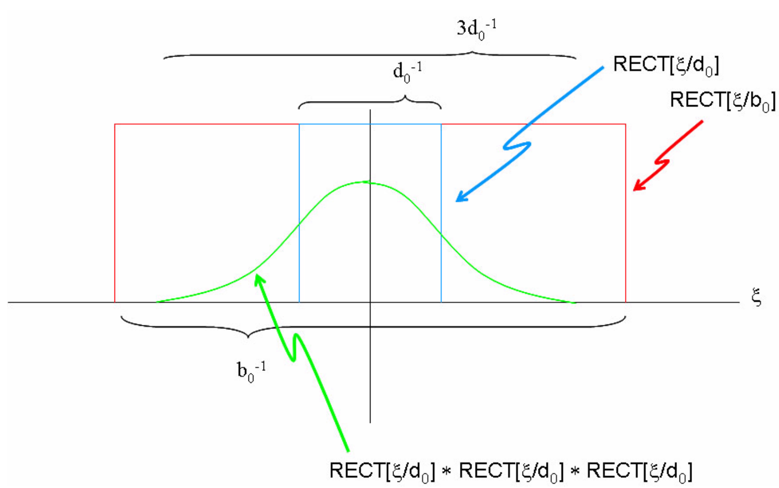
$$SINC^3\left[\frac{x}{d_0}\right] = SINC\left[\frac{x}{d_0}\right] \cdot SINC\left[\frac{x}{d_0}\right] \cdot SINC\left[\frac{x}{d_0}\right]$$

and we do know the modulation theorem:

$$\mathcal{F}_1\{SINC[x] \cdot SINC[x] \cdot SINC[x]\} = RECT[\xi] * RECT[\xi] * RECT[\xi]$$

The support of the spectrum of $SINC^3[x]$ therefore is three "units." The scaling and modulation theorems tell us that:

$$\begin{aligned}
\mathcal{F}_1\left\{SINC^3\left[\frac{x}{d_0}\right]\right\} &= (|d_0| \cdot RECT[d_0\xi]) * (|d_0| \cdot RECT[d_0\xi]) * (|d_0| \cdot RECT[d_0\xi]) \\
&= |d_0|^3 \cdot \left(RECT\left[\frac{\xi}{d_0^{-1}}\right] * RECT\left[\frac{\xi}{d_0^{-1}}\right] * RECT\left[\frac{\xi}{d_0^{-1}}\right]\right) \\
&\Longrightarrow \text{support of spectrum of } SINC^3\left[\frac{x}{d_0}\right] \text{ is } \frac{3}{d_0}
\end{aligned}$$



Comparison of spectra showing widths of spectrum of $SINC\left[\frac{x}{b_0}\right]$ and of $SINC^3\left[\frac{x}{d_0}\right]$

$$\mathcal{F}\left\{SINC\left[\frac{x}{b_0}\right]\right\} = |b_0| \cdot RECT[b_0\xi] = |b_0| \cdot RECT\left[\frac{\xi}{b_0^{-1}}\right]$$

To ensure that the spatial variation of the convolution $SINC\left[\frac{x}{b_0}\right] * SINC^3\left[\frac{x}{d_0}\right]$ *is identical to that of* $SINC^3\left[\frac{x}{d_0}\right]$, *then the width of the spectrum of the* $SINC$ *must be wider than that of the* $SINC^3$*:*

$$\text{support of } RECT\left[\frac{\xi}{b_0^{-1}}\right] \quad > \quad \text{support of } RECT\left[\frac{\xi}{d_0^{-1}}\right] * RECT\left[\frac{\xi}{d_0^{-1}}\right] * RECT\left[\frac{\xi}{d_0^{-1}}\right]$$

$$\Longrightarrow \quad \frac{1}{|b_0|} > \frac{3}{d_0} \Longrightarrow |d_0| > 3 \cdot |b_0|$$

Now look at the amplitude A_0 *of the output:*

$$\begin{aligned}
g[x] &= A_0 \cdot SINC^3\left[\frac{x}{d_0}\right] = \mathcal{F}_1^{-1}\{G[\xi]\} \\
G[\xi] &= |b_0| \cdot |d_0|^3 \cdot \left(RECT\left[\frac{\xi}{d_0^{-1}}\right] * RECT\left[\frac{\xi}{d_0^{-1}}\right] * RECT\left[\frac{\xi}{d_0^{-1}}\right]\right) \\
g[x] &= |b_0| \cdot \mathcal{F}_1^{-1}\left\{|d_0|^3 \cdot \left(RECT\left[\frac{\xi}{d_0^{-1}}\right] * RECT\left[\frac{\xi}{d_0^{-1}}\right] * RECT\left[\frac{\xi}{d_0^{-1}}\right]\right)\right\} \\
&= |b_0| \cdot SINC^3\left[\frac{x}{d_0}\right]
\end{aligned}$$

which we set equal to the presented expression:

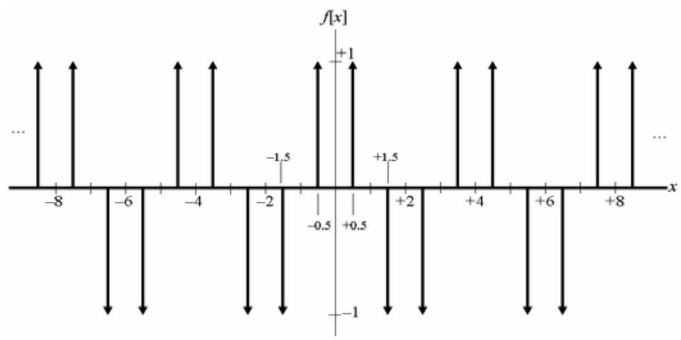
$$\begin{aligned}
g[x] \quad &= \quad |b_0| \cdot SINC^3\left[\frac{x}{d_0}\right] = A_0 \cdot SINC^3\left[\frac{x}{d_0}\right] \\
\Longrightarrow \quad & |b_0| = A_0
\end{aligned}$$

so the two conditions to be satisfied are:

$$\boxed{|b_0| < \frac{|d_0|}{3} \Longrightarrow |d_0| > 3 \cdot |b_0|}$$

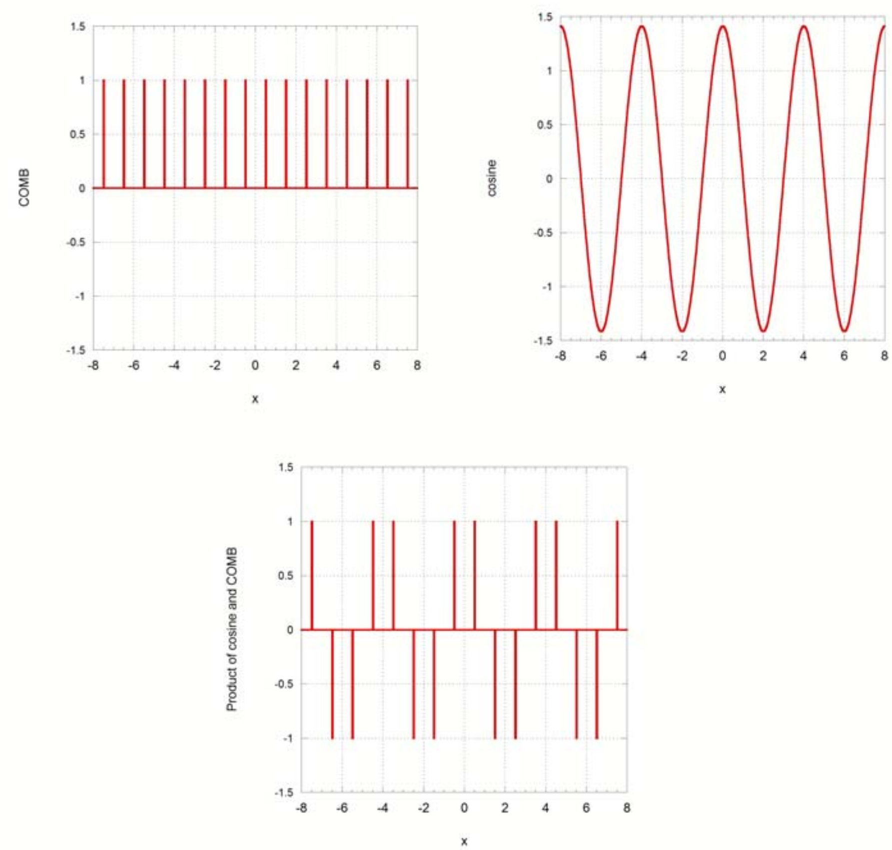
$$\boxed{A_0 = |b_0|}$$

3. (20%) Evaluate AND SKETCH the Fourier transform of the function shown in the graph; note that the function continues in same manner over the entire infinite domain.



Solution: *We can see right away that $f[x]$ is real, even, discrete, and periodic, so we'd expect the spectrum to be real, even, periodic, and discrete. If $F[\xi]$ does not fulfill these conditions, something is wrong.*

The "trick" here is to write down a convenient expression for $f[x]$ that leads to a convenient expression for $F[\xi]$. Lots of choices for $f[x]$ are possible, but I think the easiest results from recognizing that $f[x]$ is the product of a displaced COMB and a cosine. The period of the function clearly is 4 units, so that is the period of the cosine. The cosine would have unit amplitude at the origin, so to get unit amplitude at $x = \pm\frac{1}{2}$, which is $\frac{1}{8}$ of the period ($\frac{\pi}{4}$ radians), the cosine must be scaled by $\sqrt{2}$. The component functions are plotted



The corresponding expression for $f[x]$ *is:*

$$f[x] = \sqrt{2} \cdot \cos\left[2\pi \frac{x}{4}\right] \cdot COMB\left[x - \frac{1}{2}\right]$$

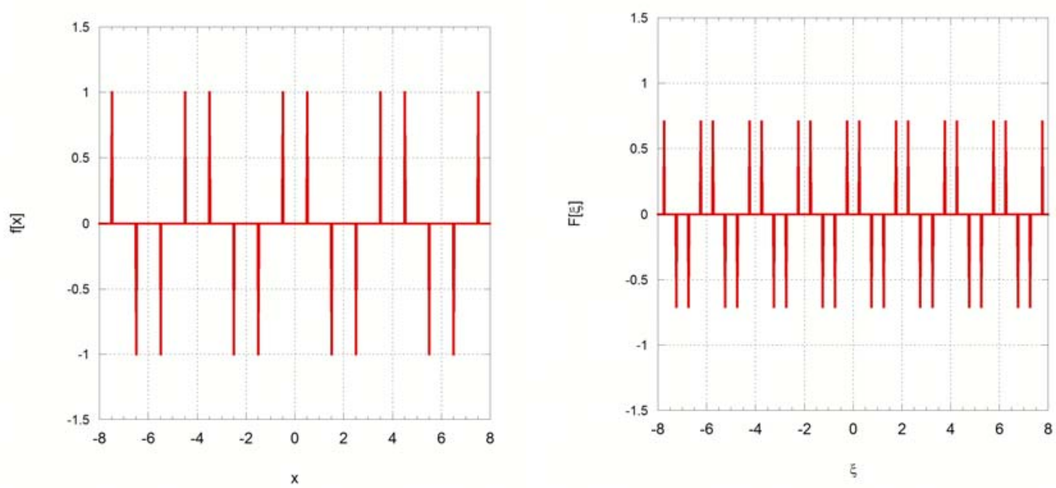
So the spectrum is:

$$\begin{aligned} F[\xi] &= \sqrt{2} \cdot \left(\frac{1}{2} \cdot \delta\left[\xi + \frac{1}{4}\right] + \frac{1}{2} \cdot \delta\left[\xi - \frac{1}{4}\right]\right) * \left(COMB[\xi] \cdot \exp\left[-i \cdot 2\pi \cdot \xi \cdot \frac{1}{2}\right]\right) \\ &= \frac{1}{\sqrt{2}} \cdot \left(\left(\delta\left[\xi + \frac{1}{4}\right] + \delta\left[\xi - \frac{1}{4}\right]\right)\right) * (COMB[\xi] \cdot \exp[-i\pi\xi]) \end{aligned}$$

Note that:

$$\begin{aligned} COMB[\xi] \cdot \exp[-i\pi\xi] &= \sum_{k=-\infty}^{+\infty} \delta[\xi - k] \cdot \exp[-i\pi\xi] \\ &= \sum_{k=-\infty}^{+\infty} \delta[\xi - k] \cdot \exp[-i\pi k] \\ &= \sum_{k=-\infty}^{+\infty} (-1)^k \cdot \delta[\xi - k] \end{aligned}$$

$$
\begin{aligned}
F[\xi] &= \frac{1}{\sqrt{2}} \cdot \left(\left(\delta\left[\xi + \frac{1}{4}\right] + \delta\left[\xi - \frac{1}{4}\right] \right) \right) * \sum_{k=-\infty}^{+\infty} (-1)^k \cdot \delta[\xi - k] \\
&= \frac{1}{\sqrt{2}} \cdot \sum_{k=-\infty}^{+\infty} (-1)^k \cdot \delta[\xi - k] * \left(\delta\left[\xi + \frac{1}{4}\right] + \delta\left[\xi - \frac{1}{4}\right] \right) \\
&= \frac{1}{\sqrt{2}} \cdot \sum_{k=-\infty}^{+\infty} (-1)^k \cdot \delta\left[\xi - \left(k + \frac{1}{4}\right)\right] + \delta\left[\xi - \left(k - \frac{1}{4}\right)\right]
\end{aligned}
$$



(a) $f[x]$; (b) $F[\xi]$ on "same scale" showing that the spectrum has the same form but different amplitude and scale factor.

4. (20%) For the 2-D input function $f[x,y]$ and impulse response $h[x,y]$:

$$\begin{aligned} f[x,y] &= \exp\left[-\pi\left(\left(\frac{x}{4}\right)^2+(2y)^2\right)\right] \\ h[x,y] &= \delta[x]\cdot 1[y]+1[x]\cdot\delta[y] \end{aligned}$$

n.b., the function is a separable 2-D Gaussian and the impulse response is the sum of two separable parts

(a) Evaluate the transfer function $H[\xi,\eta]$ for this system

Solution:

$$\begin{aligned} H[\xi,\eta] &= \mathcal{F}_2\{\delta[x]\cdot 1[y]+1[x]\cdot\delta[y]\} \\ &= \mathcal{F}_2\{\delta[x]\cdot 1[y]\}+\mathcal{F}_2\{1[x]\cdot\delta[y]\} \text{ (by linearity)} \\ &= \mathcal{F}_1\{\delta[x]\}\cdot\mathcal{F}_1\{1[y]\}+\mathcal{F}_1\{1[x]\}\cdot\mathcal{F}_1\{\delta[y]\} \text{ (by separability)} \\ &= 1[\xi]\cdot\delta[\eta]+\delta[\xi]\cdot 1[\eta] \\ &= \delta[\xi]\cdot 1[\eta]+1[\xi]\cdot\delta[\eta] \end{aligned}$$

(which you probably knew already since the Fourier transform of the CROSS function is a CROSS)

A small number seemed to forget that the transform of a separable function is the PRODUCT of the transforms, NOT the CONVOLUTION of the component transforms .... ARRGH!

(b) Evaluate the output $g[x,y]=f[x,y]*h[x,y]$ and sketch profiles $g[x,0]$ and $g[0,y]$

Solution: *Note that $f[x,y]$ is separable:*

$$\begin{aligned} f[x,y] &= \exp\left[-\pi\left(\left(\frac{x}{4}\right)^2+(2y)^2\right)\right] \\ &= \exp\left[-\pi\left(\frac{x}{4}\right)^2\right]\cdot\exp\left[-\pi(2y)^2\right] \end{aligned}$$

Also, a few read these functions as "chirps" instead of as "Gaussians." (another ARRGH!). You may evaluate the convolution in either domain, but the space-domain solution is perhaps easier, though a diversion to the frequency domain simplifies the result:

$$\begin{aligned} g[x,y] = \exp\left[-\pi\left(\frac{x}{4}\right)^2\right]\cdot\exp\left[-\pi(2y)^2\right]*(\delta[x]\cdot 1[y]+1[x]\cdot\delta[y]) \\ = \left(\exp\left[-\pi\left(\frac{x}{4}\right)^2\right]\cdot\exp\left[-\pi(2y)^2\right]\right)*(\delta[x]\cdot 1[y]) \\ + \left(\exp\left[-\pi\left(\frac{x}{4}\right)^2\right]\cdot\exp\left[-\pi(2y)^2\right]\right)*(1[x]\cdot\delta[y]) \\ = \left(\exp\left[-\pi\left(\frac{x}{4}\right)^2\right]*\delta[x]\right)\cdot\left(\exp\left[-\pi(2y)^2\right]*1[y]\right) \\ + \left(\exp\left[-\pi\left(\frac{x}{4}\right)^2\right]*1[x]\right)\cdot\left(\exp\left[-\pi(2y)^2\right]*\delta[y]\right) \end{aligned}$$

The component convolutions are:

$$\exp\left[-\pi\left(\frac{x}{4}\right)^2\right] * \delta[x] = \exp\left[-\pi\left(\frac{x}{4}\right)^2\right]$$

$$\begin{aligned}
\exp\left[-\pi(2y)^2\right] * 1[y] &= \mathcal{F}_1^{-1}\left\{\frac{1}{2}\exp\left[-\pi\left(\frac{\eta}{2}\right)^2\right]\cdot\delta[\eta]\right\} \\
&= \mathcal{F}_1^{-1}\left\{\frac{1}{2}\exp\left[-\pi\left(\frac{0}{2}\right)^2\right]\cdot\delta[\eta]\right\} \\
&= \frac{1}{2}\cdot 1[y]
\end{aligned}$$

$$\begin{aligned}
\exp\left[-\pi\left(\frac{x}{4}\right)^2\right] * 1[x] &= \mathcal{F}_1^{-1}\left\{4\cdot\exp\left[-\pi(4\xi)^2\right]\cdot\delta[\xi]\right\} \\
&= \mathcal{F}_1^{-1}\left\{4\cdot\exp\left[-\pi(4\cdot 0)^2\right]\cdot\delta[\xi]\right\} \\
&= \mathcal{F}_1^{-1}\left\{4\cdot\delta[\xi]\right\} \\
&= 4\cdot 1[x]
\end{aligned}$$

$$\exp\left[-\pi(2y)^2\right] * \delta[y] = \exp\left[-\pi(2y)^2\right]$$

Insert these terms to get the final result:

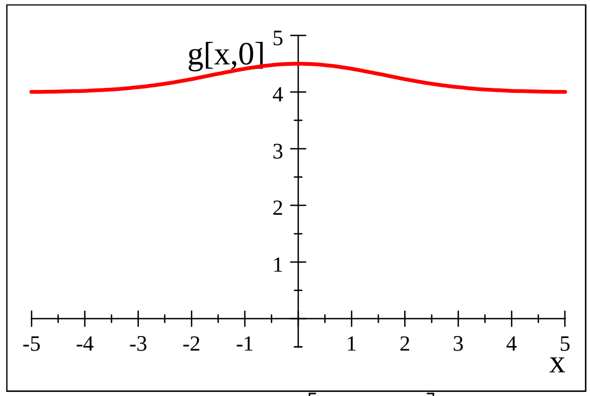
$$g[x,y] = \exp\left[-\pi\left(\frac{x}{4}\right)^2\right]\cdot\frac{1}{2}\cdot 1[y] + 4\cdot 1[x]\cdot\exp\left[-\pi(2y)^2\right]$$

$$\boxed{g[x,y] = \frac{1}{2}\exp\left[-\pi\left(\frac{x}{4}\right)^2\right] + 4\cdot\exp\left[-\pi(2y)^2\right]}$$

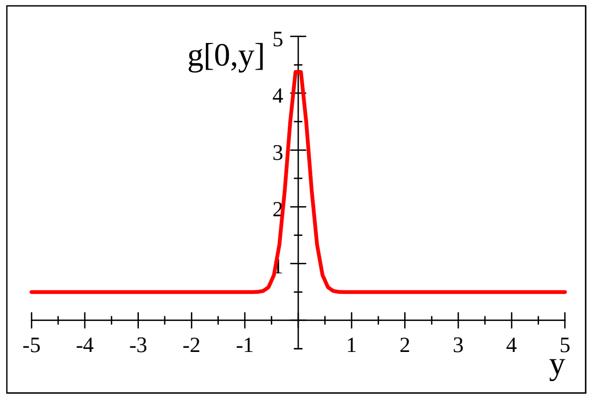
$$\boxed{g[x,0] = \frac{1}{2}\exp\left[-\pi\left(\frac{x}{4}\right)^2\right] + 4}$$

$$\boxed{g[0,y] = \frac{1}{2} + 4\cdot\exp\left[-\pi(2y)^2\right]}$$

Note that the amplitudes of both expressions at the origin are identically 4.5*; they have to match since this is a (the only) common point to both profiles. Both profiles extend to* $\pm\infty$*; the x-profile asymptotically approaches* 4 *and the y-profile asymptotically approaches* $\frac{1}{2}$.



$$g\left[x,0\right]=\frac{1}{2}\exp\left[-\pi\left(\frac{x}{4}\right)^2\right]+4$$



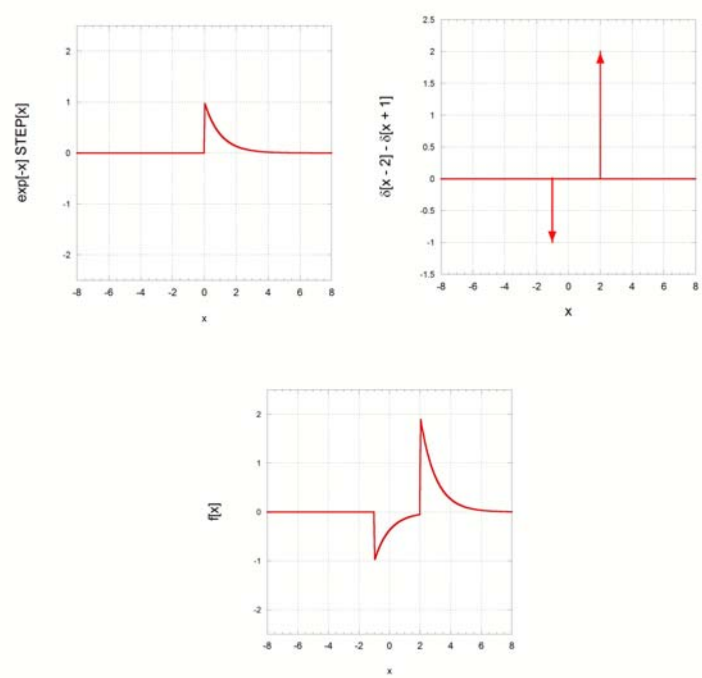
$$g\left[0,y\right]=\frac{1}{2}+4\cdot\exp\left[-\pi\left(2y\right)^2\right]$$

5. (40%) Consider the 1-D signal:

$$f[x] = (\exp[-x] \cdot STEP[x]) * (2 \cdot \delta[x-2] - 1 \cdot \delta[x+1])$$

(a) Sketch $f[x]$

Solution: *This is two replicas of the decaying exponential function: one each with positive and negative amplitude. The graphs of the component functions are also shown:*



(b) This signal is applied to a system that evaluates the crosscorrelation with the function:

$$m[x] = \exp[-x] \cdot STEP[x]$$

so that:

$$g[x] = f[x] \bigstar m[x]$$

Find an expression for and sketch $g[x]$.

Solution: *The expression for $g[x]$ is:*

$$\begin{aligned} g[x] &= f[x] \bigstar m[x] = f[x] * m^*[-x] \\ &= (\exp[-x] \cdot STEP[x]) * (2 \cdot \delta[x-2] - 1 \cdot \delta[x+1]) * (\exp[-(-x)] \cdot STEP[-x])^* \end{aligned}$$

Since $m[x]$ is real-valued, the complex conjugate has no effect:

$$g[x] = (\exp[-x] \cdot STEP[x]) * (2 \cdot \delta[x-2] - 1 \cdot \delta[x+1]) * (\exp[x] \cdot STEP[-x])$$

Convolution is associative, so we can evaluate the autocorrelation of the decaying exponential and then convolve with the pair of Dirac delta functions:

$$\begin{aligned}&(\exp[-x]\cdot STEP[x]) * (2\cdot\delta[x-2] - 1\cdot\delta[x+1]) * (\exp[x]\cdot STEP[-x])\\ &= \{(\exp[-x]\cdot STEP[x]) * (\exp[x]\cdot STEP[-x])\} * (2\cdot\delta[x-2] - 1\cdot\delta[x+1])\end{aligned}$$

which indicates that the output consists of two scaled replicas of $(\exp[-x]\cdot STEP[x]) * (\exp[x]\cdot STEP[-x])$*, so evaluate that first and then convolve with the two Dirac delta functions (which just locates and scales the autocorrelation). The autocorrelation is easy to do IN THE SPACE DOMAIN!!!*

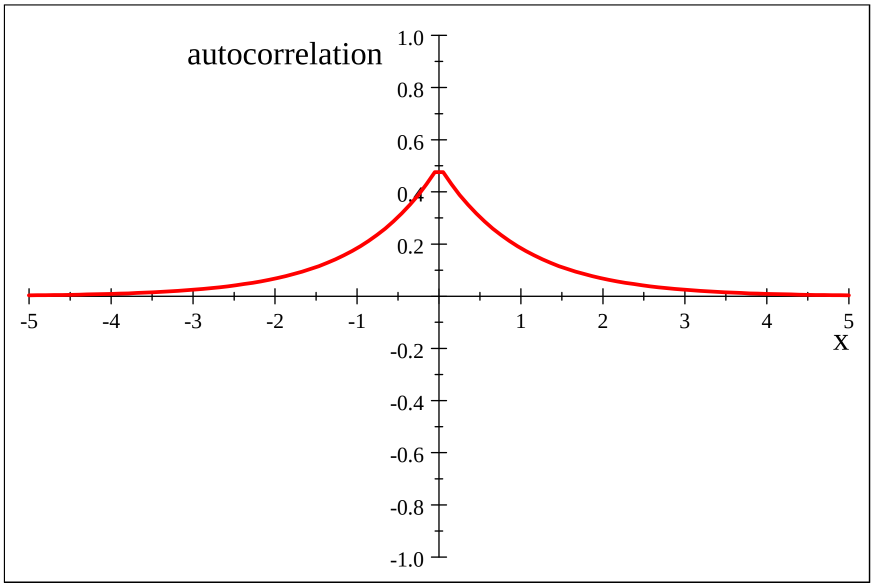
$$\begin{aligned}&(\exp[-x]\cdot STEP[x]) * (\exp[x]\cdot STEP[-x])\\ =& \int_{-\infty}^{+\infty} (\exp[-\alpha]\cdot STEP[\alpha])\cdot(\exp[x-\alpha]\cdot STEP[-(x-\alpha)])\ d\alpha\\ =& \int_{-\infty}^{+\infty} (\exp[-\alpha]\cdot STEP[\alpha])\cdot(\exp[x]\cdot\exp[-\alpha]\cdot STEP[-(x-\alpha)])\ d\alpha\\ =& \exp[+x]\int_{-\infty}^{+\infty} (\exp[-2\alpha]\cdot STEP[\alpha])\cdot STEP[\alpha-x]\ d\alpha\end{aligned}$$

$$\begin{aligned}x &> 0 \Longrightarrow \exp[+x]\cdot\int_{\alpha=0}^{\alpha=+|x|}\exp[-2\alpha]\ d\alpha\\ &= \exp[+x]\cdot\left.\frac{\exp[-2\alpha]}{-2}\right|_{\alpha=x}^{\alpha=+\infty}\ \text{\textit{for} } x>0\ = -\frac{1}{2}\exp[+x]\cdot(0-\exp[-2x])\\ &= +\frac{1}{2}\exp[-x]\ \text{ if } x>0\\ x &< 0 \Longrightarrow \exp[+x]\cdot\int_{\alpha=0}^{\alpha=+\infty}\exp[-2\alpha]\ d\alpha\\ &= \exp[+x]\cdot\left.\frac{\exp[-2\alpha]}{-2}\right|_{\alpha=0}^{\alpha=\infty} = -\frac{1}{2}\exp[+x]\cdot(0-1) = \frac{1}{2}\exp[+x]\end{aligned}$$

Combine the two limiting expressions:

$$\left.\begin{array}{rl} +\frac{1}{2}\exp[-x] & \text{if}\quad x>0\\ +\frac{1}{2} & \text{if}\quad x=0\\ +\frac{1}{2}\exp[+x] & \text{if}\quad x<0\end{array}\right| = \frac{1}{2}\exp[-|x|]$$

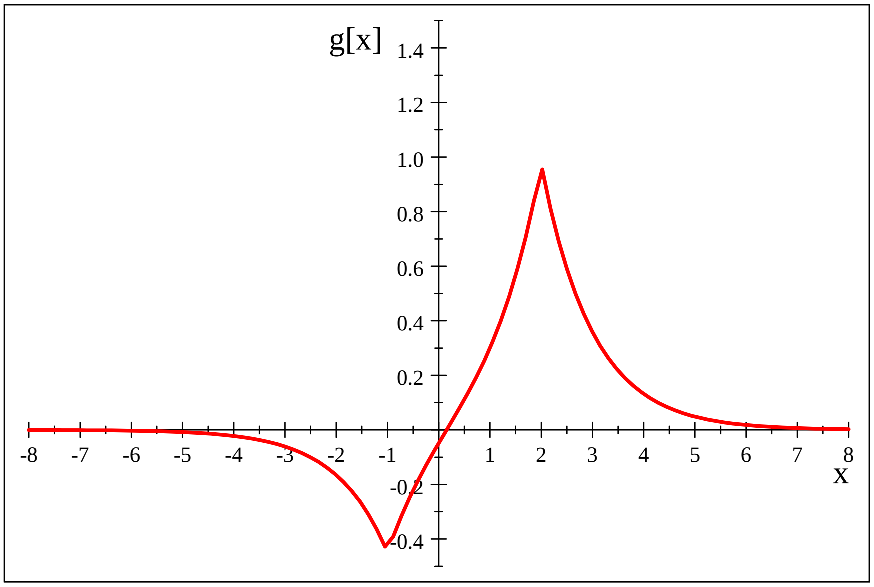
The graph of the autocorrelation of the decaying exponential is:



$$(\exp[-x] \cdot STEP[x]) \bigstar (\exp[-x] \cdot STEP[x]) = \frac{1}{2}\exp[-|x|] = \frac{1}{2}\exp[-|x|]$$

The only remaining step is to convolve the autocorrelation with the pair of Dirac delta functions; the result is the sum of two replicas of the autocorrelation of the decaying exponential:

$$\begin{aligned} g[x] &= \frac{1}{2} \cdot \exp[-|x|] * (2 \cdot \delta[x-2] - 1 \cdot \delta[x+1]) \\ &= 2 \cdot \frac{1}{2} \cdot \exp[-|x-2|] - \frac{1}{2} \cdot \exp[-|x+1|] \\ &= \exp[-|x-2|] - \frac{1}{2} \cdot \exp[-|x+1|] \end{aligned}$$



$$\{(\exp[-x] \cdot STEP[x]) * (\exp[x] \cdot STEP[-x])\} * (2 \cdot \delta[x-2] - 1 \cdot \delta[x+1])$$

6. (40%) The impulse response of a 2-D system is a constant-magnitude circularly symmetric quadratic-phase factor:

$$h[x,y]=\exp\left[+i\pi\frac{(x^2+y^2)}{\alpha_0^2}\right]\Longrightarrow h_r(r)=\exp\left[+i\pi\left(\frac{r}{\alpha_0}\right)^2\right]$$

where α_0 has dimensions of length. The input to the system is:

$$f[x,y]=\delta[x]\cdot(\delta[y+y_0]+\delta[y-y_0])$$

(a) Find an expression for the transfer function $H[\xi,\eta]$ of the system; alternatively, you may evaluate in the form $H(\rho)$.

Solution: *the impulse response is separable:*

$$h[x,y]=\exp\left[+i\pi\frac{(x^2+y^2)}{\alpha_0^2}\right]=\exp\left[+i\pi\frac{x^2}{\alpha_0^2}\right]\cdot\exp\left[+i\pi\frac{y^2}{\alpha_0^2}\right]$$

The 1-D transforms of the component parts are identical but for variable:

$$\begin{aligned}\mathcal{F}_1\left\{\exp\left[+i\pi\frac{x^2}{\alpha_0^2}\right]\right\}&=|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\exp\left[-i\pi(\alpha_0\xi)^2\right]\\ \mathcal{F}_1\left\{\exp\left[+i\pi\frac{y^2}{\alpha_0^2}\right]\right\}&=|\alpha_0|\cdot\exp\left[+i\frac{\pi}{4}\right]\exp\left[-i\pi(\alpha_0\eta)^2\right]\\ H[\xi,\eta]&=|\alpha_0|^2\cdot\exp\left[+i\frac{\pi}{2}\right]\exp\left[-i\pi\alpha_0^2\left(\xi^2+\eta^2\right)\right]\end{aligned}$$

$$\boxed{H[\xi,\eta]=i\cdot|\alpha_0|^2\cdot\exp\left[-i\pi\frac{\left(\xi^2+\eta^2\right)}{\left(\alpha_0^2\right)^{-1}}\right]}$$

(b) Evaluate the output amplitude $g[x,y]=f[x,y]*h[x,y]$

Solution: *the separability of the terms and the fact that $f[x,y]$ is a pair of Dirac delta functions makes this solution easy in the space domain; the output is just the sum of two displaced replicas of the impulse response:*

$$f[x]*\delta[x-x_0]=f[x-x_0]$$

In this case, the ouput is the convolution of separable 2-D functions:

$$\begin{aligned}g[x,y]&=\{\delta[x]\cdot(\delta[y+y_0]+\delta[y-y_0])\}*\left\{\exp\left[+i\pi\frac{x^2}{\alpha_0^2}\right]\cdot\exp\left[+i\pi\frac{y^2}{\alpha_0^2}\right]\right\}\\ &=\left\{\delta[x]*\exp\left[+i\pi\frac{x^2}{\alpha_0^2}\right]\right\}\cdot(\delta[y+y_0]+\delta[y-y_0])*\exp\left[+i\pi\frac{y^2}{\alpha_0^2}\right]\end{aligned}$$

$$\boxed{g[x,y]=\exp\left[+i\pi\frac{x^2}{\alpha_0^2}\right]\cdot\left(\exp\left[+i\pi\frac{(y+y_0)^2}{\alpha_0^2}\right]+\exp\left[+i\pi\frac{(y-y_0)^2}{\alpha_0^2}\right]\right)}$$

This answer is sufficient, but note that it is possible to simplify this expression by expanding the exponents:

$$\begin{aligned}
&\exp\left[+i\pi\frac{(y+y_0)^2}{\alpha_0^2}\right]+\exp\left[+i\pi\frac{(y-y_0)^2}{\alpha_0^2}\right]\\
=\ &\exp\left[+i\pi\frac{(y^2+y_0^2+2yy_0)}{\alpha_0^2}\right]+\exp\left[+i\pi\frac{(y^2+y_0^2-2yy_0)}{\alpha_0^2}\right]\\
=\ &\exp\left[+i\pi\frac{(y^2+y_0^2)}{\alpha_0^2}\right]\cdot\left(\exp\left[+i\pi\frac{2yy_0}{\alpha_0^2}\right]+\exp\left[-i\pi\frac{2yy_0}{\alpha_0^2}\right]\right)\\
=\ &\exp\left[+i\pi\frac{(y^2+y_0^2)}{\alpha_0^2}\right]\cdot 2\cdot\cos\left[2\pi\frac{yy_0}{\alpha_0^2}\right]
\end{aligned}$$

$$g[x,y] = \exp\left[+i\pi\frac{x^2}{\alpha_0^2}\right]\cdot\exp\left[+i\pi\frac{(y^2+y_0^2)}{\alpha_0^2}\right]\cdot 2\cdot\cos\left[2\pi\frac{yy_0}{\alpha_0^2}\right]$$

$$\boxed{g[x,y]=\exp\left[+i\pi\frac{x^2+y^2}{\alpha_0^2}\right]\cdot\exp\left[+i\pi\frac{y_0^2}{\alpha_0^2}\right]\cdot 2\cdot\cos\left[2\pi\frac{yy_0}{\alpha_0^2}\right]}$$

(c) Evaluate and simplify the expression for the squared magnitude $|g[x,y]|^2$

Easiest to use the second expression in part (b):

$$\begin{aligned}
|g[x,y]|^2 &= \left|\exp\left[+i\pi\frac{x^2+y^2}{\alpha_0^2}\right]\cdot\exp\left[+i\pi\frac{y_0^2}{\alpha_0^2}\right]\cdot 2\cdot\cos\left[2\pi\frac{yy_0}{\alpha_0^2}\right]\right|^2\\
&= \left|\exp\left[+i\pi\frac{x^2+y^2}{\alpha_0^2}\right]\right|^2\cdot\left|\exp\left[+i\pi\frac{y_0^2}{\alpha_0^2}\right]\right|^2\cdot\left|2\cdot\cos\left[2\pi\frac{yy_0}{\alpha_0^2}\right]\right|^2\\
&= 1\cdot 1\cdot 4\cdot\cos^2\left[2\pi\frac{yy_0}{\alpha_0^2}\right]\\
&= 4\cdot\cos^2\left[2\pi\frac{yy_0}{\alpha_0^2}\right]
\end{aligned}$$

$$\boxed{|g[x,y]|^2=4\cdot 1[x]\cdot\cos^2\left[2\pi\frac{y}{\left(\frac{\alpha_0^2}{y_0}\right)}\right]}$$

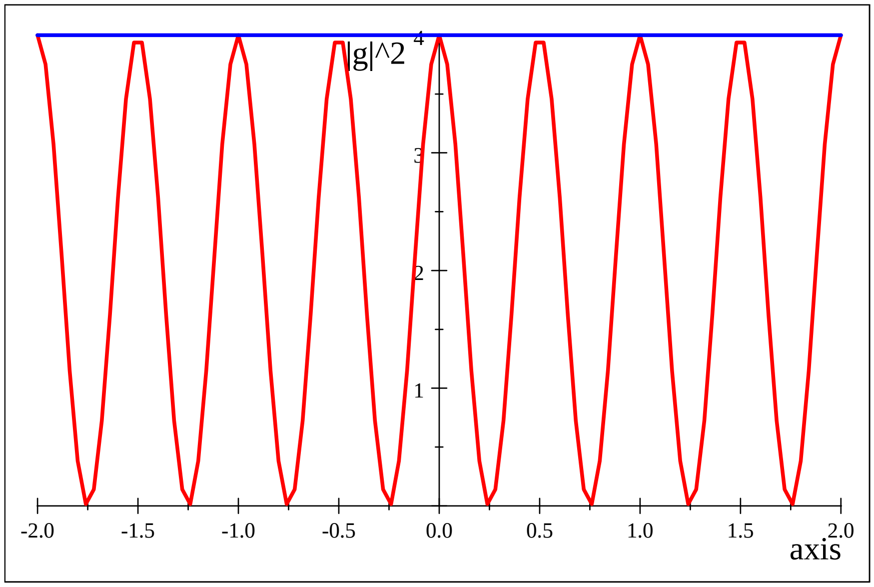
which is the square of a cosine with period $\frac{\alpha_0^2}{y_0}$.

ARRGH: note that the squared magnitude of a sum is NOT the sum of the squared magnitudes! (as some tried to use!)

(d) Sketch $|g[x,0]|^2$ and $|g[0,y]|^2$

$$|g[x,0]|^2 = 4\cdot\cos^2\left[2\pi\frac{0}{\left(\frac{\alpha_0^2}{y_0}\right)}\right]=4\cdot 1[x]$$

$$|g[0,y]|^2 = 4\cdot\cos^2\left[2\pi\frac{y}{\left(\frac{\alpha_0^2}{y_0}\right)}\right]$$



(blue) profile of $|g[x,0]|^2$ *along x axis; (red) profile of* $|g[0,y]|^2$ *along y axis.*
This is an example of diffraction in the Fresnel diffraction region.