

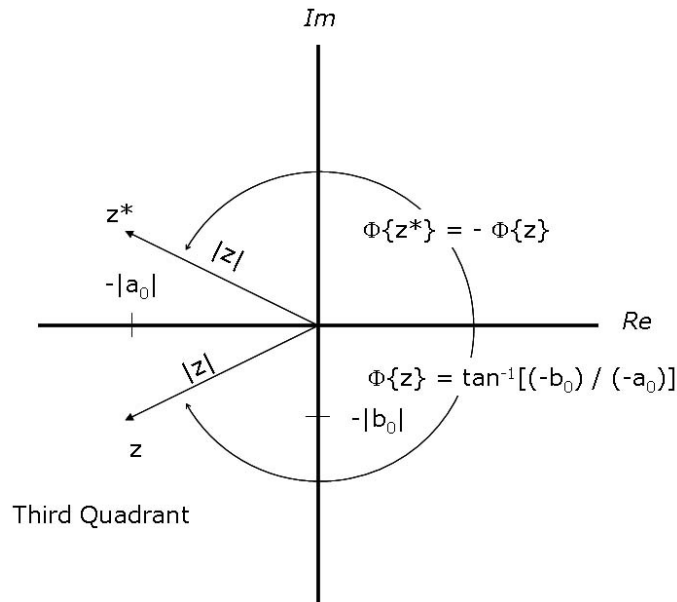
General Comments: *The goal of the exam was for students to demonstrate that they could use the material FROM THE COURSE to solve the problems, e.g., use the known eigenvectors of circulant matrices to solve for the eigenvalues. Some students used other methods to solve some problems, often incorrectly, but even if the answers were correct, some points were deducted because of the goal of the exam.*

Other Big Comment: **READ AND FOLLOW DIRECTIONS!**

1. (10%) On an Argand diagram for an arbitrary complex number $z_0 = a_0 + i \cdot b_0$, where $a_0 < 0$ and $b_0 < 0$, show the locations of the five complex numbers z , z^* , z^2 , $z^{\frac{1}{2}}$, and $z^{-1} \equiv \frac{1}{z}$; the superscript “*” denotes the “complex conjugate” of the associated complex number.

Solution: First note that the specified values for a_0 and b_0 place the complex number in the third quadrant so that the condition on the phase angle is $-\frac{\pi}{2} < \phi < -\pi$. If the “naive” calculation of the phase angle is used as only the inverse tangent of the ratio of the imaginary part to real part, then the angle will be placed in the first quadrant.

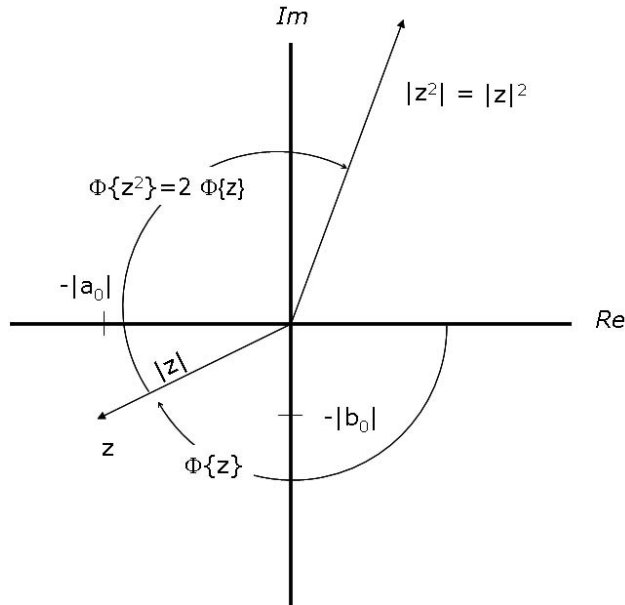
$$\begin{aligned}
 a_0 < 0 &\implies a_0 = -|a_0| \\
 z &= -|a_0| + i \cdot (-|b_0|) \\
 |z| &= \sqrt{|a_0|^2 + |b_0|^2} \\
 \Phi\{z\} &= \tan^{-1} \left[\frac{-|b_0|}{-|a_0|} \right] = \tan^{-1} \left[\frac{|b_0|}{|a_0|} \right] \pm \pi \\
 \\ \\
 z^* &= -|a_0| - i \cdot (-|b_0|) = -|a_0| + i \cdot |b_0| \\
 |z^*| &= |z| = \sqrt{|a_0|^2 + |b_0|^2} \\
 \Phi\{z^*\} &= \tan^{-1} \left[\frac{+|b_0|}{-|a_0|} \right] \\
 \Phi\{z^2\} &= \tan^{-1} \left[\frac{2 \cdot |a_0| \cdot |b_0|}{|a_0|^2 - |b_0|^2} \right] = 2 \cdot \Phi\{z\}
 \end{aligned}$$



$$z = (-|a_0|) + i \cdot (-|b_0|) \text{ and } z^* = (-|a_0|) - i \cdot (-|b_0|) = -|a_0| + i \cdot |b_0|$$

$$\begin{aligned}
z^2 &= (-|a_0| + i \cdot (-|b_0|))^2 = (-|a_0|)^2 + (i \cdot -|b_0|)^2 + i \cdot 2 \cdot (-|a_0| \cdot -|b_0|) \\
&= +|a_0|^2 - |b_0|^2 + i \cdot (2 \cdot |a_0| \cdot |b_0|) \\
|z^2| &= \left(\sqrt{|a_0|^2 + |b_0|^2} \right)^2 = |a_0|^2 + |b_0|^2 \\
\Phi\{z^2\} &= \tan^{-1} \left[\frac{2 \cdot |a_0| \cdot |b_0|}{|a_0|^2 - |b_0|^2} \right] = 2 \cdot \Phi\{z\}
\end{aligned}$$

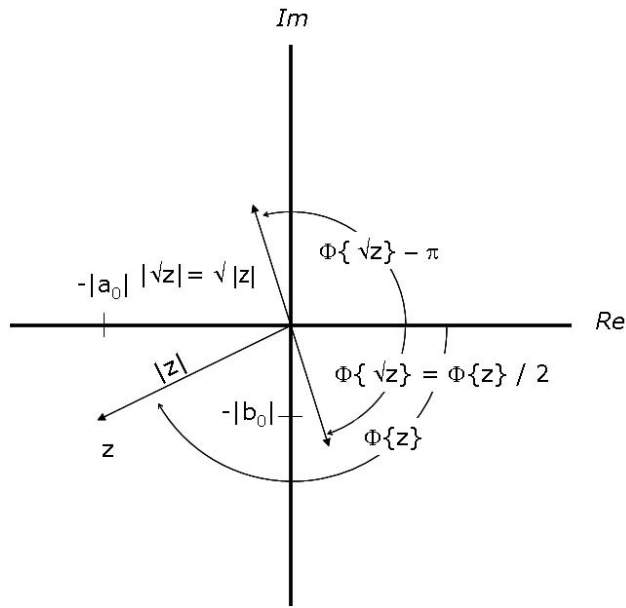
Since z is in the third quadrant, then z^2 must be in either the second quadrant (if $-\frac{\pi}{2} < \Phi\{z\} < -\frac{3\pi}{4}$) or the first quadrant (if $-\frac{3\pi}{4} < \Phi\{z\} < -\pi$)



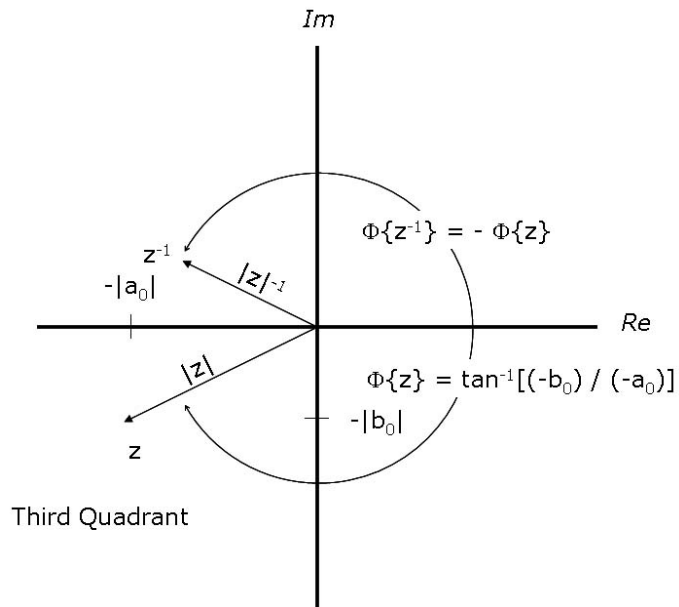
z^2 compared to z

DeMoivre's theorem is used to evaluate the square root, but remember that there are TWO square roots; the second is the negative of the first. The phase angle of the first root is half that of z and so will be in the fourth quadrant; the second root is in the second quadrant.

$$\begin{aligned}
z^{\frac{1}{2}} &= u \implies u^2 = z = -|a_0| + i \cdot (-|b_0|) \\
z^{\frac{1}{2}} &= |z|^{\frac{1}{2}} \exp \left\{ i \cdot \frac{\Phi\{z\}}{2} \right\} \text{ and } z^{\frac{1}{2}} = |z|^{\frac{1}{2}} \exp \left\{ i \cdot \left(\frac{\Phi\{z\}}{2} \pm \pi \right) \right\}
\end{aligned}$$



$$z^{-1} = \frac{1}{z} = \frac{1}{-|a_0| + i \cdot (-|b_0|)}$$



Note that the azimuth (phase) angles of z^{-1} and z^* are identically $-\Phi\{z\}$.

3. (20%) Evaluate the discrete Fourier transform of each of these 4-element vectors with elements indexed from $n = 0$ to $n = N - 1 = 3$.

$$(a) \underline{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \underline{\mathbf{D}}^{-1} \underline{\mathbf{x}}_1 &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}^* \underline{\mathbf{x}}_1 \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

which means that $\underline{\mathbf{x}}_1$ is composed of only the “0th” eigenvector of a 4×4 circulant matrix and is twice the length of that vector. In other words, $\underline{\mathbf{x}}_1$ is composed only of a constant vector.

$$(b) \underline{\mathbf{x}}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This is an impulse; its DFT is:

$$\underline{\mathbf{D}}^{-1} \underline{\mathbf{x}}_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

which means that $\underline{\mathbf{x}}_2$ is composed of equal amounts of each of the four eigenvectors of a 4×4 circulant matrix.

- (c) Compare the inputs and outputs of (a) and (b) to comment on the similarities.

The DFT of a constant is an impulse and that of an impulse is a constant vector. This makes sense because of the similarity of the matrices for the forward DFT (analysis) and inverse DFT (synthesis), which only differ by the sign of the imaginary part, which has no impact on even and real-valued input vectors.

4. (20%) The eigenvalues of a particular 6×6 circulant matrix:

$$\underline{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

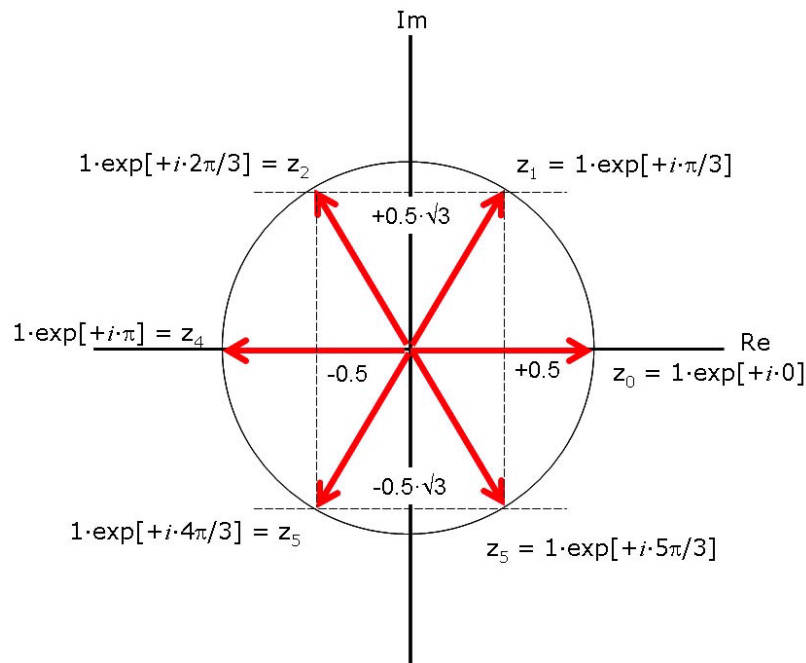
satisfy the equation:

$$\lambda^6 - 1 = 0$$

where the λ_n are generally complex valued. Find expressions for the eigenvalues and associated normalized eigenvectors of $\underline{\mathbf{A}}$ as BOTH real and imaginary parts AND as magnitude and phase.

Solution: Must solve for the six values of λ , which are the six roots of unity $\lambda^6 = 1$. These are the solutions of:

$$\begin{aligned} \lambda_n &= |1|^{\frac{1}{6}} \cdot \exp\left[+i \cdot 2\pi \cdot \frac{n}{6}\right] \\ &= 1 \cdot \exp\left[+i \cdot \frac{\pi}{3} \cdot n\right], \quad n = 0, 1, 2, 3, 4, 5 \end{aligned}$$



Graphical solution to the six roots of unity; the magnitude of each is one and the phase angles are separated by $\Delta\phi = \frac{2\pi}{6} = \frac{\pi}{3}$ radians.

$$\begin{aligned}
\lambda_0 &= \exp \left[+i \cdot \frac{\pi}{3} \cdot 0 \right] = 1 \cdot \exp [i \cdot 0] = 1 + 0i \\
\lambda_1 &= \exp \left[+i \cdot \frac{\pi}{3} \cdot 1 \right] = \cos \left[\frac{\pi}{3} \right] + i \cdot \sin \left[\frac{\pi}{3} \right] = \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \\
\lambda_2 &= \exp \left[+i \cdot \frac{\pi}{3} \cdot 2 \right] = \cos \left[\frac{2\pi}{3} \right] + i \cdot \sin \left[\frac{2\pi}{3} \right] = -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \\
\lambda_3 &= \exp \left[+i \cdot \frac{\pi}{3} \cdot 3 \right] = \exp [i \cdot \pi] = \cos [\pi] + i \cdot \sin [\pi] = -1 + i \cdot 0 \\
\lambda_4 &= \exp \left[+i \cdot \frac{\pi}{3} \cdot 4 \right] = \cos \left[\frac{4\pi}{3} \right] + i \cdot \sin \left[\frac{4\pi}{3} \right] = -\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} \\
\lambda_5 &= \exp \left[+i \cdot \frac{\pi}{3} \cdot 5 \right] = \cos \left[\frac{5\pi}{3} \right] + i \cdot \sin \left[\frac{5\pi}{3} \right] = +\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}
\end{aligned}$$

Associated eigenvectors are the six vectors that oscillate at frequencies determined by the value of k where $0 \leq k \leq N - 1$:

k	$\xi = \frac{k \text{ cycles}}{N \text{ sample}}$
0	0
1	$\frac{1}{6} \implies \Delta\phi = \frac{2\pi}{6} = \frac{\pi \text{ radians}}{3 \text{ sample}}$
2	$\frac{1}{3} \implies \Delta\phi = \frac{4\pi}{6} = \frac{2\pi \text{ radians}}{3 \text{ sample}}$
3	$\frac{1}{2} \implies \Delta\phi = \frac{6\pi}{6} = \pi \frac{\text{radians}}{\text{sample}}$
4	$\frac{2}{3} \implies \Delta\phi = \frac{8\pi}{6} = \frac{4\pi \text{ radians}}{3 \text{ sample}}$
5	$\frac{5}{6} \implies \Delta\phi = \frac{10\pi}{6} = \frac{5\pi \text{ radians}}{3 \text{ sample}}$

The eigenvalues and associated eigenvectors are:

$$\begin{aligned}
\lambda_0 &= 1 \implies \hat{\mathbf{x}}_0 = \frac{1}{\sqrt{6}} \begin{bmatrix} \exp \left[+i \cdot \frac{0 \cdot 2\pi}{6} \right] \\ \exp \left[+i \cdot \frac{0 \cdot 1 \cdot 2\pi}{6} \right] \\ \exp \left[+i \cdot \frac{0 \cdot 2 \cdot 2\pi}{6} \right] \\ \exp \left[+i \cdot \frac{0 \cdot 3 \cdot 2\pi}{6} \right] \\ \exp \left[+i \cdot \frac{0 \cdot 4 \cdot 2\pi}{6} \right] \\ \exp \left[+i \cdot \frac{0 \cdot 5 \cdot 2\pi}{6} \right] \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
\xi_0 &= 0 \frac{\text{cycles}}{\text{sample}}
\end{aligned}$$

$$\lambda_1 = \exp\left[i \cdot \frac{\pi}{3}\right] \Rightarrow \hat{\mathbf{x}}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} \exp\left[+i \cdot \frac{1 \cdot 0 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{1 \cdot 1 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{1 \cdot 2 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{1 \cdot 3 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{1 \cdot 4 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{1 \cdot 5 \cdot 2\pi}{6}\right] \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \\ -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \\ -1 \\ -\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} \\ \frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\xi_0 = \frac{1 \text{ cycles}}{6 \text{ sample}}$$

$$\lambda_2 = \exp\left[i \cdot \frac{2\pi}{3}\right] \Rightarrow \hat{\mathbf{x}}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} \exp\left[+i \cdot \frac{2 \cdot 0 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{2 \cdot 1 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{2 \cdot 2 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{2 \cdot 3 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{2 \cdot 4 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{2 \cdot 5 \cdot 2\pi}{6}\right] \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} +1 \\ -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \\ -\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} \\ +1 \\ -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \\ -\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\xi_1 = \frac{1 \text{ cycles}}{3 \text{ sample}}$$

$$\lambda_3 = \exp\left[i \cdot \frac{3\pi}{3}\right] = -1 \Rightarrow \hat{\mathbf{x}}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} \exp\left[+i \cdot \frac{3 \cdot 0 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{3 \cdot 1 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{3 \cdot 2 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{3 \cdot 3 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{3 \cdot 4 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{3 \cdot 5 \cdot 2\pi}{6}\right] \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \\ +1 \\ -1 \end{bmatrix}$$

$$\xi_2 = \frac{1 \text{ cycles}}{2 \text{ sample}}$$

$$\lambda_4 = \exp\left[i \cdot \frac{4\pi}{3}\right] \Rightarrow \hat{\mathbf{x}}_4 = \frac{1}{\sqrt{6}} \begin{bmatrix} \exp\left[+i \cdot \frac{4 \cdot 0 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{4 \cdot 1 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{4 \cdot 2 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{4 \cdot 3 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{4 \cdot 4 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{4 \cdot 5 \cdot 2\pi}{6}\right] \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} +1 \\ -\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} \\ -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \\ +1 \\ -\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} \\ -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\xi_4 = \frac{2 \text{ cycles}}{3 \text{ sample}}$$

$$\lambda_5 = \exp\left[i \cdot \frac{5\pi}{3}\right] \Rightarrow \hat{\mathbf{x}}_5 = \frac{1}{\sqrt{6}} \begin{bmatrix} \exp\left[+i \cdot \frac{5 \cdot 0 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{5 \cdot 1 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{5 \cdot 2 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{5 \cdot 3 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{5 \cdot 4 \cdot 2\pi}{6}\right] \\ \exp\left[+i \cdot \frac{5 \cdot 5 \cdot 2\pi}{6}\right] \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} +1 \\ +\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} \\ -\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2} \\ -1 \\ -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \\ +\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\xi_5 = \frac{5 \text{ cycles}}{6 \text{ sample}}$$

5. (40%) For the 4×4 circulant matrix listed:

$$\underline{\mathbf{A}} = \begin{bmatrix} +1 & 0 & -1 & 0 \\ 0 & +1 & 0 & -1 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 \end{bmatrix}$$

- (a) Find expressions for the vectors in the null subspace that are “blocked” by the system AND for the vectors that are “passed” by the system so that the output is nonzero.

The matrix is circulant, so we **KNOW** the eigenvectors. The point of the course is to use the eigenvectors and eigenvalues, so that is what was expected. Some people used other methods that gave results that are not as easily interpreted and were penalized since you should be using the tools of the course to solve these problems; we want to apply the methods to imaging where the other techniques are not as useful.

Some people tried to evaluate $\mathbf{D}^{-1}\mathbf{A}\mathbf{D}$, but the intensity of the math should have been a clue that there HAS to be an easier way. **All that is necessary is to determine which of the 4 known eigenvectors are blocked by the system, i.e., for which vectors $\underline{\mathbf{x}}_n$ is $\underline{\mathbf{A}}\underline{\mathbf{x}}_n = \underline{\mathbf{0}}$:**

$$\underline{\mathbf{A}}\underline{\hat{\mathbf{x}}}_0 = \begin{bmatrix} +1 & 0 & -1 & 0 \\ 0 & +1 & 0 & -1 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{\mathbf{0}} \implies \boxed{\lambda_0 = 0}$$

$$\underline{\mathbf{A}}\underline{\hat{\mathbf{x}}}_1 = \begin{bmatrix} +1 & 0 & -1 & 0 \\ 0 & +1 & 0 & -1 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ +\frac{i}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = 2 \cdot \begin{bmatrix} \frac{1}{2} \\ +\frac{i}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \end{bmatrix} \implies \boxed{\lambda_1 = +2}$$

$$\underline{\mathbf{A}}\underline{\hat{\mathbf{x}}}_2 = \begin{bmatrix} +1 & 0 & -1 & 0 \\ 0 & +1 & 0 & -1 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{\mathbf{0}} \implies \boxed{\lambda_2 = 0}$$

$$\underline{\mathbf{A}}\underline{\hat{\mathbf{x}}}_3 = \begin{bmatrix} +1 & 0 & -1 & 0 \\ 0 & +1 & 0 & -1 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{i}{2} \\ -\frac{i}{2} \\ +\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} = 2 \cdot \begin{bmatrix} \frac{1}{2} \\ -\frac{i}{2} \\ -\frac{i}{2} \\ +\frac{1}{2} \end{bmatrix} \implies \boxed{\lambda_3 = +2}$$

so the vectors in the null subspace have the form of a weighted sum of the two eigenvectors that are blocked by the system.

$$\boxed{\{\mathbf{x}_n\} = \alpha \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \beta \cdot \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \alpha + \beta \\ \alpha - \beta \\ \alpha + \beta \\ \alpha - \beta \end{bmatrix}}$$

where α, β are any complex weights. The diagonal form of this matrix is obtained by placing the eigenvalues in their proper locations:

$$\underline{\Lambda} = \begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & +2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +2 \end{bmatrix}$$

- (b) Evaluate the projection of any vector in the null subspace onto any vector “passed” by the system:

Duh: any vector in the null subspace is orthogonal to any vector that is not in the null subspace, so the projection HAS TO BE 0!

- (c) Determine if the matrix is invertible and give reasons.

Clearly not invertible because two eigenvalues are zero

6. (40%) A shift-invariant operation acts on 4 samples of a function $f[n]$ that may be represented as a 4-element vector $\underline{\mathbf{x}}$. For the “first” output pixel, the operator evaluates a weighted sum of the four “input” pixels:

$$g[0] = 0.1 \cdot f[0] + 0.2 \cdot f[1] + 0.3 \cdot f[2] + 0.4 \cdot f[3]$$

- (a) Write down the 4×4 matrix $\underline{\mathbf{A}}$ that implements this operation.

$$\underline{\mathbf{A}} = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \end{bmatrix}$$

- (b) Evaluate the corresponding 4×4 diagonal matrix $\underline{\mathbf{\Lambda}}$

Again, many tried to evaluate $\mathbf{D}^{-1}\mathbf{A}\mathbf{D}$, but found that the math gets intense quickly; the EASY way (least prone to arithmetic error) is to find the four eigenvalues individually and construct the diagonal matrix with the eigenvalues in the proper order. This was discussed in class, but apparently many missed it...

$$\begin{aligned} \underline{\mathbf{A}}\hat{\mathbf{x}}_0 &= \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = 1 \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\ \Rightarrow & \boxed{\lambda_0 = +1} \end{aligned}$$

$$\begin{aligned} \underline{\mathbf{A}}\hat{\mathbf{x}}_1 &= \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{bmatrix} = \begin{bmatrix} -0.1 - 0.1i \\ 0.1 - 0.1i \\ 0.1 + 0.1i \\ -0.1 + 0.1i \end{bmatrix} = \frac{2}{10}(-1 - i) \cdot \begin{bmatrix} \frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{bmatrix} \\ \Rightarrow & \boxed{\lambda_1 = -\frac{1}{5}(1 + i)} \end{aligned}$$

$$\begin{aligned} \underline{\mathbf{A}}\hat{\mathbf{x}}_2 &= \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -0.1 \\ 0.1 \\ -0.1 \\ 0.1 \end{bmatrix} = -\frac{2}{10} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \\ \Rightarrow & \boxed{\lambda_2 = -\frac{1}{5}} \end{aligned}$$

$$\begin{aligned} \underline{\mathbf{A}}\hat{\mathbf{x}}_3 &= \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ +\frac{i}{2} \end{bmatrix} = \begin{bmatrix} -0.1 + 0.1i \\ 0.1 + 0.1i \\ 0.1 - 0.1i \\ -0.1 - 0.1i \end{bmatrix} = \frac{2}{10} \cdot (-1 + i) \cdot \begin{bmatrix} \frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ +\frac{i}{2} \end{bmatrix} \\ \Rightarrow & \boxed{\lambda_3 = -\frac{1}{5}(1 - i)} \end{aligned}$$

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{5}(1+i) & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & -\frac{1}{5}(1-i) \end{bmatrix}$$

(c) Use the result of part (b) to determine if the action of the system is invertible

None of the four eigenvalues is zero so the matrix is invertible.

(d) If the answer to (c) is “yes,” then find the inverse matrix $\underline{\mathbf{A}}^{-1}$; if the answer to (c) is “no,” then find the pseudoinverse matrix $\underline{\mathbf{A}}^\dagger$

Since we know the eigenvalues, we just find the form in the original basis of the diagonal matrix formed from the reciprocals of the eigenvalues. In this case the math is “doable” since the matrix is diagonal.

$$\begin{aligned} \underline{\mathbf{A}}^{-1} &= \begin{bmatrix} 1^{-1} & 0 & 0 & 0 \\ 0 & (-\frac{1}{5}(1+i))^{-1} & 0 & 0 \\ 0 & 0 & (-\frac{1}{5})^{-1} & 0 \\ 0 & 0 & 0 & (-\frac{1}{5}(1-i))^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{5}{2}(1-i) & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -\frac{5}{2}(1+i) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \underline{\mathbf{A}}^{-1} &= \underline{\mathbf{D}}\underline{\mathbf{A}}\underline{\mathbf{D}}^{-1} \\ &= \left(\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{5}{2}(1-i) & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -\frac{5}{2}(1+i) \end{bmatrix} \\ &\quad \cdot \left(\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{bmatrix} \right) \\ &= \left(\frac{1}{2} \begin{bmatrix} 1 & -\frac{5}{2} + \frac{5}{2}i & -5 & -\frac{5}{2} - \frac{5}{2}i \\ 1 & -\frac{5}{2} - \frac{5}{2}i & 5 & -\frac{5}{2} + \frac{5}{2}i \\ 1 & \frac{5}{2} - \frac{5}{2}i & -5 & \frac{5}{2} + \frac{5}{2}i \\ 1 & \frac{5}{2} + \frac{5}{2}i & 5 & \frac{5}{2} - \frac{5}{2}i \end{bmatrix} \right) \cdot \left(\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{bmatrix} \right) \\ &= \frac{1}{4} \cdot \begin{bmatrix} -9 & 11 & 1 & 1 \\ 1 & -9 & 11 & 1 \\ 1 & 1 & -9 & 11 \\ 11 & 1 & 1 & -9 \end{bmatrix} \end{aligned}$$