

Mathematical Series (You Should Know)

Mathematical series representations are very useful tools for describing images or for solving/approximating the solutions to imaging problems. They may be used to “expand” a function into terms that are individual monomial expressions (i.e., “powers”) of the coordinate

Geometric Series

Adjacent terms in a geometric series exhibit a constant ratio, e.g., if the scale factor for adjacent terms in the series is t , the series has the form:

$$1 + t + t^2 + t^3 + \dots = \sum_{n=0}^{\infty} t^n$$

If $|t| < 1$, this solution converges to a simple (and EASILY remembered) expression:

$$\boxed{\sum_{n=0}^{\infty} t^n = \frac{1}{1-t} \text{ if } |t| < 1}$$

This series pops up frequently in science and it is useful to remember the solution. We may “turn the problem around” by using a truncated series as an approximation for the ratio:

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \cong \sum_{n=0}^N t^n$$

where N is some maximum power in the series

Examples:

1.

$$\begin{aligned} (0.9)^{-1} &= \frac{1}{0.9} = \frac{1}{1-0.1} = (0.1)^0 + (0.1)^1 + (0.1)^2 + (0.1)^3 + \dots \\ &= 1 + 0.1 + 0.01 + 0.001 + \dots = 1.11111\dots \equiv (0.9)^{-1} \end{aligned}$$

This particular series converges fairly quickly and because of the small value of $t = 0.1$; this series may be truncated after a few terms and still obtain a fairly accurate value.

2.

$$\begin{aligned} \frac{1}{0.25} &= 4 \\ &= \frac{1}{1-0.75} = (0.75)^0 + (0.75)^1 + (0.75)^2 + (0.75)^3 + (0.75)^4 + (0.75)^5 + \dots \\ &= (1 + 0.75 + 0.5625 + 0.421875 + 0.31640635) + \dots \\ &= 3.05078125 + \dots < 4 \end{aligned}$$

Note that this series converges slowly because $t = 0.25$ is relatively “large;” the sum of a few terms is a poor approximation to the end result.

Finite Geometric Series

The truncated geometric series also may be rewritten into a simple expression. Consider the finite series that includes $N + 1$ terms:

$$\begin{aligned}\sum_{n=0}^N t^n &= 1 + t + t^2 + t^3 + \dots + t^N \\ &= \sum_{n=0}^{\infty} t^n - \sum_{n=N+1}^{\infty} t^n\end{aligned}$$

We may write this as the difference of two infinite geometric series:

$$\begin{aligned}\sum_{n=0}^N t^n &= (1 + t + t^2 + t^3 + \dots + t^N + t^{N+1} + \dots) - (t^{N+1} + t^{N+2} + \dots) \\ &= \sum_{n=0}^{\infty} t^n - \sum_{n=N+1}^{\infty} t^n\end{aligned}$$

Now change the summation variable for the second infinite series from n to $u \equiv n - (N + 1) \implies n = u + N + 1$:

$$\sum_{n=N+1}^{\infty} t^n = \sum_{u=0}^{\infty} t^{u+(N+1)} = \sum_{u=0}^{\infty} t^u \cdot t^{N+1} = t^{N+1} \cdot \sum_{u=0}^{\infty} t^u = t^{N+1} \cdot \frac{1}{1-t}$$

The expressions for the two series may now be combined:

$$\begin{aligned}\sum_{n=0}^N t^n &= \sum_{n=0}^{\infty} t^n - \sum_{n=N+1}^{\infty} t^n \\ &= \left(\frac{1}{1-t}\right) - t^{N+1} \cdot \frac{1}{1-t}\end{aligned}$$

$$\sum_{n=0}^N t^n = \frac{1 - t^{N+1}}{1 - t} \text{ if } |t| < 1$$

This is also often shown in the form where the maximum power in the series is $N - 1$ so that there are N terms:

$$\sum_{n=0}^{N-1} t^n = \frac{1 - t^N}{1 - t} \text{ if } |t| < 1$$

Examples:

1.

$$\begin{aligned}\sum_{n=0}^4 (0.1)^n &= 1 + 0.1 + 0.01 + 0.001 + 0.0001 = 1.1111 \\ t = 0.1, N = 4 &\implies \frac{1 - t^{N+1}}{1 - t} = \frac{1 - (0.1)^5}{1 - (0.1)} = 1.1111\end{aligned}$$

2.

$$\begin{aligned}\sum_{n=0}^3 (0.75)^n &= 2.7344 \\ \frac{1 - t^{N+1}}{1 - t} &= \frac{1 - (0.75)^4}{1 - (0.75)} = 2.7344\end{aligned}$$

Binomial Expansion:

$$\begin{aligned}(1+x)^n &= \frac{1}{0!} + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n!}{(n-r)!r!}x^r + \dots \\ &= 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + \dots\end{aligned}$$

The definition of the “binomial coefficient” is often substituted to make the expression more concise:

$$\binom{n}{r} \equiv \frac{n!}{(n-r)! \cdot r!}$$

$$\begin{aligned}(1+x)^n &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{r}x^{n-r}x^r + \dots \\ &= \boxed{(1+x)^n = \sum_{r=0}^{\infty} \binom{n}{r} \cdot x^{n-r} \cdot x^r}\end{aligned}$$

If n is a positive integer, the series includes $n+1$ terms. If n is NOT a positive integer, the series converges if $|x| < 1$. If $n > 0$, the series also converges if $|x| = 1$.

$$\begin{aligned}\implies (1-x)^n &= 1 + n(-x) + \frac{n(n-1)}{2}(-x)^2 + \frac{n(n-1)(n-2)}{6}(-x)^3 + \dots \\ &= 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots\end{aligned}$$

Again, the series may be truncated to approximate the result. In many areas of science, the series is often truncated after the second term, so the summation only includes the constant and the linear term (e.g., Fresnel approximation to optical diffraction) so that:

$$(1 \pm x)^n \cong 1 \pm nx \text{ if } |x| \gtrsim 0$$

Examples:

1.

$$\begin{aligned}\frac{1}{1-x} &= (1-x)^{-1} = 1 - (-1)x + \frac{(-1)(-2)}{2!}x^2 - \frac{(-1)(-2)(-3)}{3!}x^3 + \dots \\ &= 1 + x + \frac{2!}{2!}x^2 - \frac{(3!)}{3!}x^3 + \dots \\ &= 1 + x + x^2 + x^3 + \dots\end{aligned}$$

which demonstrates that the geometric series may be written as a binomial expansion.

2. Square root

$$\begin{aligned}(1+x)^{\frac{1}{2}} &= \sqrt{1+x} = 1 + \frac{1}{2} \cdot x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2} \cdot x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6} \cdot x^3 + \dots \\ &= 1 + \frac{1}{2} \cdot x - \frac{1}{8} \cdot x^2 + \frac{1}{16} \cdot x^3 + \dots\end{aligned}$$

In words, this expresses the square root of $1+x$ as a series of the powers of x with decreasing weights. Truncation to two terms yields an approximation for the square root that is most accurate for $|x| \gtrsim 0$.

$$(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2} \cdot x - \frac{1}{8} \cdot x^2 \cong 1 + \frac{1}{2} \cdot x$$

$$\frac{17}{16} = 1.0625 = 1 + \frac{1}{16}$$

$$\sqrt{\frac{17}{16}} = 1.030776406\dots$$

$$\text{first-order approximation: } \sqrt{\left(1 + \frac{1}{16}\right)} \cong 1 + \frac{1}{2} \cdot \frac{1}{16} = \frac{33}{32} = 1.03125$$

$$\text{second-order approximation: } \sqrt{\left(1 + \frac{1}{16}\right)} \cong 1 + \frac{1}{2} \cdot \frac{1}{16} - \frac{1}{8} \left(\frac{1}{16}\right)^2 = \frac{2111}{2048} = 1.030761719\dots$$

3. Cube root

$$\begin{aligned} (1-x)^{\frac{1}{3}} &= \sqrt[3]{(1-x)} = 1 + \frac{1}{3} \cdot (-x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2} \cdot (-x)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6} \cdot (-x)^3 + \dots \\ &= 1 - \frac{1}{3} \cdot x - \frac{1}{9} \cdot x^2 + \frac{5}{81} \cdot x^3 + \dots \\ &\cong 1 + \frac{1}{3} \cdot x - \frac{1}{9} \cdot x^2 \cong 1 + \frac{1}{3} \cdot x \end{aligned}$$

$$\frac{15}{16} = 1 - \frac{1}{16}$$

$$\left(\frac{15}{16}\right)^{\frac{1}{3}} = 0.97871691\dots$$

$$\left(1 - \frac{1}{16}\right)^{\frac{1}{3}} \cong 1 + \frac{1}{3} \cdot \left(-\frac{1}{16}\right) = \frac{47}{48} = 0.97916666\dots$$

$$\left(1 - \frac{1}{16}\right)^{\frac{1}{3}} \cong 1 + \frac{1}{3} \cdot \left(-\frac{1}{16}\right) + \frac{1}{9} \cdot \left(-\frac{1}{16}\right)^2 = \frac{2257}{2304} = 0.979600694\dots$$

$$\left(1 - \frac{1}{16}\right)^{\frac{1}{3}} \cong 1 + \frac{1}{3} \cdot \left(-\frac{1}{16}\right) + \frac{1}{9} \cdot \left(-\frac{1}{16}\right)^2 + \frac{5}{81} \cdot \left(-\frac{1}{16}\right)^3 = \frac{325\,003}{331\,776} = 0.979585624\dots$$

Exponential Series

Without proof, we state that we write e raised to a numerical power u as a series in the powers of u :

$$\begin{aligned}\exp [u] &= e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \\ &= \frac{1}{0!} + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots \\ &= 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + \dots\end{aligned}$$

which may be easily generalized to any base:

$$a^u = \exp [u \cdot \log [a]] = \sum_{n=0}^{\infty} \frac{[u \cdot \log [a]]^n}{n!}$$

For example, we can write any power of 10 in this form:

$$\begin{aligned}10^u &= \sum_{n=0}^{\infty} \frac{[u \cdot \log [10]]^n}{n!} \\ &\cong \sum_{n=0}^{\infty} \frac{(\log [10])^n}{n!} \cdot u^n \\ \text{where } \log [10] &\cong 2.302585\end{aligned}$$

Complex Exponential Series

The generalization of the exponential series for complex-valued powers:

$$\begin{aligned}\exp [\pm i\theta] &= \exp [\pm i\theta] = \sum_{n=0}^{\infty} \frac{(\pm i\theta)^n}{n!} \\ &= \frac{1}{0!} \pm \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} \mp \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} \pm \dots \\ &= 1 \pm i\theta + \frac{1}{2}i^2\theta^2 \mp \frac{1}{6}i^3\theta^3 + \frac{1}{24}i^4\theta^4 \pm \frac{1}{120}i^5\theta^5 + \dots \\ &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \dots\right) \pm i \cdot \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots\right)\end{aligned}$$

From Euler relation:

$$\exp [+i\theta] = \cos [\theta] + i \sin [\theta]$$

Equate real and imaginary parts:

$$\begin{aligned}\cos [\theta] &= \frac{1}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \dots \\ \lim_{\theta \rightarrow 0} \cos [\theta] &= 1 \\ \sin [\theta] &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots \\ \lim_{\theta \rightarrow 0} \sin [\theta] &= \theta\end{aligned}$$

The linear approximation of the complex exponential is often used in diffraction theory:

$$\exp [+i\theta] \cong 1 + i \cdot \theta$$

Maclaurin Series:

“Predict” the value of the function $f[x]$ based on its value and its derivatives evaluated at the origin of coordinates ($x = 0$):

$$\begin{aligned} f[x] &= \frac{x^0}{0!} \cdot f[0] + \frac{x^1}{1!} \cdot \left. \frac{df}{dx} \right|_{x=0} + \frac{x^2}{2!} \cdot \left. \frac{d^2f}{dx^2} \right|_{x=0} + \frac{x^3}{3!} \cdot \left. \frac{d^3f}{dx^3} \right|_{x=0} + \dots \\ &= \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} \cdot \left. \frac{d^n f}{dx^n} \right|_{x=0} \right) \\ &= f[0] + x \cdot f'[0] + \frac{x^2}{2} \cdot f''[0] + \frac{x^3}{6} \cdot f'''[0] + \dots = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} \cdot f^{(n)}[0] \right) \end{aligned}$$

where the “exponential” notation for differentiation has been used:

$$\left. \frac{d^n}{dx^n} \{f[x]\} \right|_{x=0} \equiv f^{(n)}[0]$$

Taylor Series:

Generalization of the Maclaurin series to “predict” value of $f[x + x_0]$ based on the value of the function and its derivatives evaluated at x_0 :

$$\begin{aligned} f[x + x_0] &= \frac{x_0^0}{0!} \cdot f[x]_{x=x_0} + \frac{x_0^1}{1!} \cdot \left. \frac{df}{dx} \right|_{x=x_0} + \frac{x_0^2}{2!} \cdot \left. \frac{d^2f}{dx^2} \right|_{x=x_0} + \frac{x_0^3}{3!} \cdot \left. \frac{d^3f}{dx^3} \right|_{x=x_0} + \dots \\ &= \frac{x_0^0}{0!} \cdot f[x_0] + \frac{x_0}{1!} \cdot \left. \frac{df}{dx} \right|_{x=x_0} + \frac{x_0^2}{2!} \cdot \left. \frac{d^2f}{dx^2} \right|_{x=x_0} + \frac{x_0^3}{3!} \cdot \left. \frac{d^3f}{dx^3} \right|_{x=x_0} + \dots \\ &= \sum_{n=0}^{\infty} \left(\frac{x_0^n}{n!} \cdot \left. \frac{d^n f}{dx^n} \right|_{x=x_0} \right) \\ &= f[x_0] + x_0 \cdot f'[x_0] + \frac{x_0^2}{2} \cdot f''[x_0] + \frac{x_0^3}{6} \cdot f'''[x_0] + \dots = \sum_{n=0}^{\infty} \left(\frac{x_0^n}{n!} \cdot f^{(n)}[x_0] \right) \end{aligned}$$

[Arfken, §1, Schey, Feynman V1 §2]

Could also think of it as evaluating the value of $f[x]$ based on the function and its derivatives evaluated at $x - x_0$

$$\begin{aligned} f[x] &= \frac{x_0^0}{0!} \cdot f[x_0] + \frac{x_0^1}{1!} \cdot \left. \frac{df}{dx} \right|_{x=x-x_0} + \frac{x_0^2}{2!} \cdot \left. \frac{d^2f}{dx^2} \right|_{x=x-x_0} + \frac{x_0^3}{3!} \cdot \left. \frac{d^3f}{dx^3} \right|_{x=x-x_0} + \dots \\ &= \sum_{n=0}^{\infty} \left(\frac{x_0^n}{n!} \cdot \left. \frac{d^n f}{dx^n} \right|_{x=x-x_0} \right) \\ &= f[x - x_0] + x_0 \cdot f'[x - x_0] + \frac{x_0^2}{2} \cdot f''[x - x_0] + \frac{x_0^3}{6} \cdot f'''[x - x_0] + \dots \\ &= \sum_{n=0}^{\infty} \left(\frac{x_0^n}{n!} \cdot f^{(n)}[x - x_0] \right) \end{aligned}$$

Again, it often is useful to truncate the Taylor series to approximate the value of the function.