1. Determine the length of the listed input vector $\underline{\mathbf{x}}$ in the direction of the listed "reference" vector $\underline{\mathbf{a}}$.

   (a) $\underline{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{\mathbf{a}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   the two vectors point are identical and therefore point in the same direction, so that the length of $\underline{\mathbf{x}}$ in direction of $\underline{\mathbf{a}}$ is length of $\underline{\mathbf{x}}$

$$b = \frac{\underline{\mathbf{a}} \bullet \underline{\mathbf{x}}}{|\underline{\mathbf{a}}|} = \frac{1 \cdot 1 + 1 \cdot 1}{\sqrt{1^2 + 1^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

   (b) $\underline{\mathbf{x}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{\mathbf{a}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$b = \frac{\underline{\mathbf{a}} \bullet \underline{\mathbf{x}}}{|\underline{\mathbf{a}}|} = \frac{1 \cdot 1 + (-1) \cdot 1}{\sqrt{1^2 + 1^2}} = \frac{0}{\sqrt{2}} = 0 \implies \underline{\mathbf{x}} \perp \underline{\mathbf{a}}$$

   (c) $\underline{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\underline{\mathbf{a}} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

$$b = \frac{\underline{\mathbf{a}} \bullet \underline{\mathbf{x}}}{|\underline{\mathbf{a}}|} = \frac{1 \cdot 1 + 1 \cdot 1 + 2 \cdot (-2)}{\sqrt{1^2 + 1^2 + (-2)^2}} = \frac{1 + 1 - 4}{\sqrt{6}} = \frac{-2}{\sqrt{6}} = -\frac{\sqrt{6}}{3}$$

2. Find all of the values of z such that $z^4 = -1$. Express in two ways: as real and imaginary parts and as magnitude and phase angle.

   Solution: *There will be four fourth roots of any complex number. To find them, write both in in magnitude and azimuth ("phase") notation:*

$$\begin{aligned}
z &= |z| \cdot \exp\left[+i \cdot \phi\right] \implies z^4 = \left(|z| \cdot \exp\left[+i \cdot \phi\right]\right)^4 = |z|^4 \cdot \exp\left[+i \cdot 4\phi\right] \\
-1 &= 1 \cdot \exp\left[\pm i\pi\right] = 1 \cdot \exp\left[i \cdot (2n\pi \pm \pi)\right],\ n = 0, \pm 1, \pm 2, \ldots \\
&\quad \textit{equate magnitudes and equate phases} \\
|z|^4 &= 1 \implies |z| = 1 \\
\exp\left[+i \cdot 4\phi\right] &= \exp\left[i \cdot (2n\pi \pm \pi)\right] \implies 4\phi = 2n\pi \pm \pi \\
n &= 0 \implies \phi = \pm\frac{\pi}{4} \implies z_0 = 1 \cdot \exp\left[\pm i\frac{\pi}{4}\right] = \frac{1}{\sqrt{2}} \pm i \cdot \frac{1}{\sqrt{2}} \\
n &= 1 \implies \phi = \pm\frac{\pi}{4} + \frac{\pi}{2} \implies z_1 = 1 \cdot \exp\left[+i\frac{3\pi}{4}\right] = -\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}} \\
\textit{or}\ z_1 &= 1 \cdot \exp\left[-i\frac{\pi}{4}\right] = +\frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}} \\
n &= 2 \implies \phi = \pm\frac{\pi}{4} + \pi = +\frac{5\pi}{4}\ \textit{or}\ +\frac{3\pi}{4} \implies z_2 = -\frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}} \\
&\quad \textit{or}\ -\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}} \\
n &= 3 \implies \phi = \pm\frac{\pi}{4} + \frac{3\pi}{2} = \frac{7\pi}{4}\ \textit{or}\ \frac{5\pi}{4} \implies z_2 = +\frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}} \\
&\quad \textit{or}\ -\frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}}
\end{aligned}$$

3. Write down the matrix that evaluates the difference of adjacent components of the arbitrary 4-element vector $\underline{\mathbf{x}}$, e.g.,

$$\underline{\mathbf{x}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The specification of the difference of adjacent components may be interpreted in different ways, I choose to use this convention:

$$\underline{\mathbf{Ax}} = \underline{\mathbf{A}} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \underline{\mathbf{b}} = \begin{bmatrix} b-a \\ c-b \\ d-c \\ a-d \end{bmatrix}$$

$$\boxed{\underline{\mathbf{A}} = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \\ +1 & 0 & 0 & -1 \end{bmatrix}}$$

4. For the matrix **A:**

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

   (a) Determine if $\underline{\mathbf{A}}$ is invertible, i.e., if $\underline{\mathbf{A}}^{-1}$ exists:

   Solution: *lots of ways to do this, perhaps easiest is to evaluate determinant of* $\underline{\mathbf{A}}$ *and see if it is nonzero (invertible) or zero (not invertible). For general* 2×2 *matrix, determinant is:*

$$\underline{\mathbf{A}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Longrightarrow \det \underline{\mathbf{A}} = ad - bc$$

   In this case,

$$\det \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1 \cdot 1 - (-1) \cdot (-1) = 0 \Longrightarrow \boxed{\textit{NOT INVERTIBLE}}$$

   Another way is to recognize that this matrix evaluates differences of adjacent components of the vector, i.e., it evaluates a discrete derivative. We know from calculus that the integral of the derivative is the original function EXCEPT for the constant part of the original.

   (b) Find the eigenvectors and associated eigenvalues of $\underline{\mathbf{A}}$.

   Solution: *The action of a matrix on an eigenvector is a change of length only; no change of direction. Put another way, the output of the product of a matrix with an eigenvector is a new vector pointed in the same direction as the input vector. For the "nth" eigenvector, the output is:*

$$\underline{\mathbf{A}}\underline{\mathbf{x}}_n = \underline{\mathbf{b}} = \lambda_n \underline{\mathbf{x}}_n \propto \underline{\mathbf{x}}_n$$

   where the scaling factor λ_n *is the eigenvalue corresponding to that eigenvector. The eigenvalues are found from the secular equation:*

$$\begin{aligned}
\det\left(\underline{\mathbf{A}} - \underline{\mathbf{I}}\lambda\right) &= 0 \\
\det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix} \cdot \lambda\right) &= \det\left(\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}\right) \\
&= (1-\lambda)^2 - (-1)^2 = 0 \\
&\Longrightarrow \left(1 - 2\lambda + \lambda^2\right) - 1 = 0 \\
&\Longrightarrow \lambda^2 - 2\lambda = 0 \\
&\Longrightarrow (\lambda - 2) \cdot \lambda = 0 \\
&\Longrightarrow \lambda = +2, 0
\end{aligned}$$

   Find the eigenvectors:

$$\begin{aligned}
\lambda &= 0 \Longrightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
x_1 - x_2 &= 0 \Longrightarrow x_2 = x_1 \\
-x_1 + x_2 &= 0 \Longrightarrow x_2 = x_1
\end{aligned}$$

Eigenvector corresponding to eigenvalue:

$$\boxed{\lambda = 0 \implies \underline{\mathbf{x}} \propto \begin{bmatrix} +1 \\ +1 \end{bmatrix}}$$

Second eigenvector:

$$\begin{aligned} \lambda &= 2 \implies \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ x_1 - x_2 &= 2x_1 \implies x_2 = -x_1 \end{aligned}$$

$$\boxed{\lambda = +2 \implies \underline{\mathbf{x}} \propto \begin{bmatrix} +1 \\ -1 \end{bmatrix}}$$

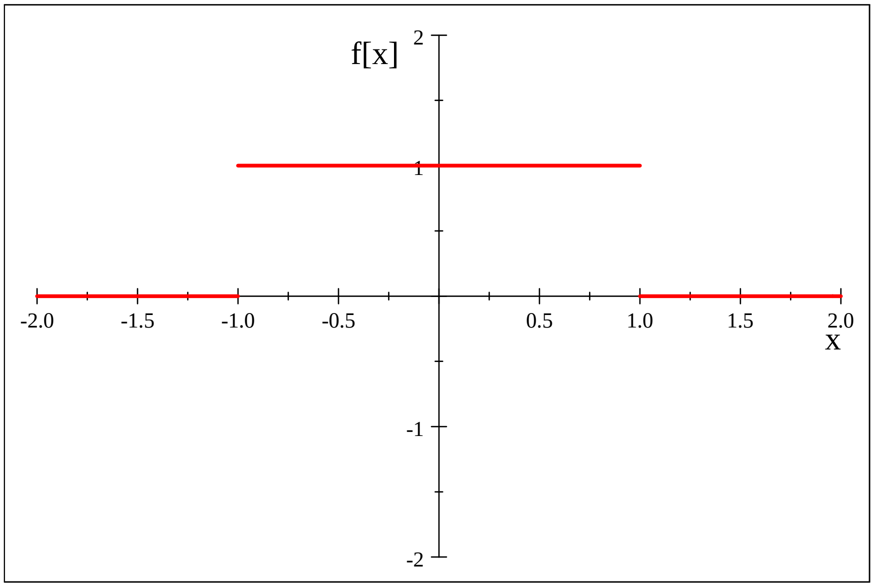
5. A function of one variable is defined by the following expression:

$$f[x] \equiv \begin{cases} 0 & \text{if} \quad |x| > 1 \\ \frac{1}{2} & \text{if} \quad |x| = 1 \\ 1 & \text{if} \quad |x| < 1 \end{cases}$$

(a) Sketch $f[x]$

This is a version of the "rectangle function."

$$f[x] \equiv \begin{cases} 0 & \text{if} \quad x < -1 \\ \frac{1}{2} & \text{if} \quad x = -1 \\ 1 & \text{if} \quad -1 < x < +1 \\ \frac{1}{2} & \text{if} \quad x = +1 \\ 0 & \text{if} \quad x > 1 \end{cases}$$



(where I didn't bother to plot the isolated values at $x = \pm 1$ to save time because of the constraints of the computer software)

(b) For this function $f[x]$, evaluate the integral and sketch as a function of u

$$\int_{x=-\infty}^{x=+\infty} f[x] \cdot \exp[+i \cdot 2\pi \cdot x \cdot u] \; du$$

where $i \equiv \sqrt{-1}$

This integral is actually pretty easy to solve:

$$\begin{aligned}
\int_{x=-\infty}^{x=+\infty} f\left[x\right]\cdot\exp\left[+i\cdot 2\pi\cdot x\cdot u\right]\;dx &= \int_{x=-\infty}^{x=-1} 0\cdot\exp\left[+i\cdot 2\pi\cdot x\cdot u\right]\;dx \\
&\quad + \int_{x=-1}^{x=+1} 1\cdot\exp\left[+i\cdot 2\pi\cdot x\cdot u\right]\;dx \\
&\quad + \int_{x=-\infty}^{x=-1} 0\cdot\exp\left[+i\cdot 2\pi\cdot x\cdot u\right]\;dx \\
&= 0+\int_{x=-1}^{x=+1}\exp\left[+i\cdot 2\pi\cdot x\cdot u\right]\;dx+0 \\
&= \left.\frac{\exp\left[+i\cdot 2\pi\cdot x\cdot u\right]}{+i\cdot 2\pi\cdot u}\right|_{x=-1}^{x=+1} \\
&= \frac{\exp\left[+i\cdot 2\pi\cdot u\right]-\exp\left[-i\cdot 2\pi\cdot u\right]}{+2i\cdot\left(\pi\cdot u\right)} \\
&= \frac{\sin\left[2\pi u\right]}{\pi u} = 2\left(\frac{\sin\left[2\pi u\right]}{2\pi u}\right)
\end{aligned}$$