

0. Read Chapter 5 and Chapter 6.1 in the book (you may skip §6.1.10 and §6.1.13)

1. A 2×2 sampled “object” with real-valued gray values has the form $f[n, m] = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

It also may be represented as a 4-D vector by “stacking” the columns:

$$f[n] \rightarrow \underline{\mathbf{x}} \equiv \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

(this is the “lexicographically” ordered 1-D vector that represents the 2-D sampled image). Find the matrix operators $\underline{\mathbf{A}}_n$ that produce the following output vectors $\underline{\mathbf{b}}_n$ (“output images”) when applied to $\underline{\mathbf{x}}$:

- (a) $\underline{\mathbf{b}}_A$ is vector for the original image after exchanging the rows and columns, so that

$$\text{the } 2 \times 2 \text{ output image is } g[n, m] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- (b) $\underline{\mathbf{b}}_B$ is vector corresponding to the original image rotated by $+\frac{\pi}{2}$ about the “center” of the array (the intersection of the “four corners” of the array)

- (c) $\underline{\mathbf{b}}_C$ is the original image rotated by $+\pi$

- (d) The elements of the output image are the sums of the rows and of the columns

$$\text{of } \underline{\mathbf{x}} \text{ so that } \underline{\mathbf{b}}_D = \begin{bmatrix} a + c \\ b + d \\ a + b \\ c + d \end{bmatrix}. \text{ (this is the crudest analogy to the measurements$$

made in computed tomography and magnetic resonance imaging, where sums are measured of the unknown data along parallel lines)

- (e) Find the inverses $\underline{\mathbf{A}}_n^{-1}$ of the matrix operators *if they exist*. If $\underline{\mathbf{A}}_n^{-1}$ does not exist, then explain why.

- (f) (OPTIONAL BONUS) Extend the case of part (d) by adding measurements of the sums of the diagonal elements, so that $(\underline{\mathbf{b}}_F)_5 = a + d$ and $(\underline{\mathbf{b}}_F)_6 = b + c$. Find the inverse matrix that evaluates the four unknowns $[a, b, c, d]$ in $\underline{\mathbf{x}}$ from the six measurements in the output vector $\underline{\mathbf{b}}_F$.

2. In each case, calculate the complex-valued “projection” of the vector $\underline{\mathbf{x}}$ onto the direction of the vector $\underline{\mathbf{a}}$, i.e., evaluate $\hat{\underline{\mathbf{a}}} \bullet \underline{\mathbf{x}}$ in each case.

$$(a) \underline{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \underline{\mathbf{a}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$(b) \underline{\mathbf{x}} = \begin{bmatrix} i \\ i \\ 0 \end{bmatrix}, \underline{\mathbf{a}} = \begin{bmatrix} 0 \\ i \\ i \end{bmatrix}$$

$$(c) \underline{\mathbf{x}} = \begin{bmatrix} 1+i \\ 1 \\ 1-i \end{bmatrix}, \underline{\mathbf{a}} = \begin{bmatrix} 1 \\ 1+i \\ 1-i \end{bmatrix}$$

3. Find the normalized eigenvectors (unit length) and corresponding eigenvalues of each matrix

$$(a) \underline{\mathbf{A}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(b) \underline{\mathbf{A}}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(c) \underline{\mathbf{A}}_3 = \begin{bmatrix} +1 & -1 & 0 \\ 0 & +1 & -1 \\ -1 & 0 & +1 \end{bmatrix}$$

- (d) For each case, write down the matrix $\underline{\mathbf{A}}$.
- (e) Write down the 4 normalized eigenvectors of $\underline{\mathbf{A}}$ (in words, the length of each vector is unity).
- (f) Write down the diagonalizing matrix $\underline{\mathbf{D}}$ (the matrix that generates a diagonal matrix $\underline{\mathbf{\Lambda}}$ via:

$$\underline{\mathbf{D}}^{-1} \underline{\mathbf{A}} \underline{\mathbf{D}} = \underline{\mathbf{\Lambda}}$$

- (g) Evaluate the product just given to find the diagonal form $\underline{\mathbf{\Lambda}}_n$ and thus the four eigenvalues λ_n .
- (h) For each matrix, make graphs of the magnitude and of the phase of the eigenvalues on a plot where the indices k of the four eigenvalues λ_k form the horizontal axis and the complex-valued amplitude is plotted along the vertical axis. In other words, the axis coordinates are labeled by $k = 1, k = 2, k = 3, k = 4$.
- (i) Explain why each matrix is or is not invertible.

4. The following definitions of 8-element row vectors represent the first rows of the corresponding 8×8 circulant matrix \mathbf{A} ; in all cases, assume that the samples of the vector “wrap around” when you get to the edge (which is equivalent to assuming that the input function from which the vector was derived is periodic); all matrices are *circulant*.

$\underline{\mathbf{v}}_1$ calculates the numerical average of 2 adjacent pixels in the vector

$\underline{\mathbf{v}}_2$ calculates the numerical average of 3 adjacent pixels in the vector

$\underline{\mathbf{v}}_3$ calculates the numerical average of the 4 pixels in the vector

$\underline{\mathbf{v}}_4$ calculates the numerical average of pairs of pixels in the vector separated by one pixel

$\underline{\mathbf{v}}_5$ calculates the numerical difference of adjacent pixels in the vector

$\underline{\mathbf{v}}_6$ calculates the numerical difference of the adjacent pixels that were calculated for $\underline{\mathbf{v}}_5$

$\underline{\mathbf{v}}_7$ calculates the difference of the pixel and the average of the two adjacent neighbors

- (a) For each of the seven cases, write down the matrix $\underline{\mathbf{A}}$.
- (b) Write down the 8 normalized eigenvectors of $\underline{\mathbf{A}}$ (in words, the length of each vector is unity) ordered by oscillation frequency, so that the sequence of components of $\underline{\hat{\mathbf{x}}}_0$ oscillate most slowly, etc.
- (c) Write down the diagonalizing matrix $\underline{\mathbf{D}}$ (the matrix that generates a diagonal matrix $\underline{\mathbf{A}}$ via:

$$\underline{\mathbf{D}}^{-1} \underline{\mathbf{A}} \underline{\mathbf{D}} = \underline{\mathbf{A}}$$

- (d) Evaluate the product just given to find the diagonal form $\underline{\mathbf{A}}_n$ and thus the eight eigenvalues λ_n for each of the seven cases. You may do this by computer, but they are easy (though perhaps tedious) to calculate by hand.
- (e) For each matrix, make graphs of the magnitude and of the phase of the eigenvalues on a plot where the indices k of the four eigenvalues λ_k form the horizontal axis and the complex-valued amplitude is plotted along the vertical axis. In other words, the axis coordinates are labeled by $k = 0, k = 1, \dots, k = 7$
- (f) Explain why each of the seven matrices is or is not invertible.