SIMG-716
Linear Imaging Mathematics I, Handout 04

# 1 1-D Real-Valued "Special" Functions

- 1-D special functions defined over real-valued domain with independent spatial variable $x$
- Many functions (e.g., $f[x] = RECT[x]$) defined by specifying amplitude $f$ at different coordinates by "rule" or mathematical relation.

## 1.1 Scaling "width"

- "width" varied by scaling independent variable $x$ by real-valued factor $b_0$ to evaluate $f\left[\frac{x}{b_0}\right]$
- consider $f[x]$ with compact support such that $f[x] = 0$ for $|x| > 1$

1. scaled function $f\left[\frac{x}{b_0}\right] = 0$ for $\left|\frac{x}{b_0}\right| > 1 \implies |x| > |b_0|$
2. scaled function is "wider" if $b_0 > 1$
3. scaled function is "narrower" if $b_0 < 1$
4. function is scaled and "reversed" if $b_0 < 0$
5. scaled function is not defined if $b_0 = 0$
6. If $f[x]$ has finite support, then $b_0$ is the scale factor for both that support interval and of the area of the function.
7. If $f[x]$ has infinite support, then $b_0$ is the scale factor of separation distance between "features" of the function (such as the locations of zeros).

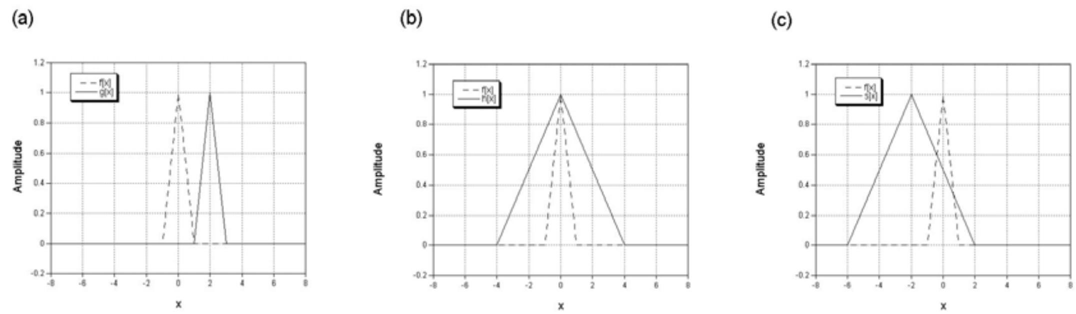
- A few functions are defined by a single criterion over entire infinite domain
  - Examples include sinusoids and Gaussian function $e^{-\pi x^2}$.
  - These functions may be scaled by complex-valued width parameters to produce functions with complex-valued amplitudes.

## 1.2 Translating the "central ordinate" of the function

- recast independent variable $x \to x - x_0$
  - amplitude originally at origin $x = 0$ is now located where $x - x_0 = 0 \implies x = +x_0$
- Both scaling and translation may be applied to a function at one time by applying the general argument $\frac{x-x_0}{b_0}$; if $f[x]$ has compact support limited by $x = a_0$ and $x = d_0$ (so width is $d_0 - a_0$), then $f\left[\frac{x-x_0}{b_0}\right]$ has support limited by:

$$\frac{x-x_0}{b_0} = a_0 \implies x = x_0 + a_0 b_0$$
$$\frac{x-x_0}{b_0} = d_0 \implies x = x_0 + d_0 b_0$$

so it is centered at $x = x_0$ with width $(d_0 - a_0) \cdot b_0$




Effects of parameters of argument of function $f\left[x\right]$: (a) shifting $f\left[x-2\right]$; (b) scaling $f\left[\frac{x}{4}\right]$; (c) combination $f\left[\frac{x+2}{4}\right]$.

## 1.3 1-D Constant Function

- "local" amplitude $\propto x^0$
- Two examples are given own symbols:

### 1.3.1 unit constant $f[x] = 1[x]$:

- infinite support
- infinite area.

$$1[x] \equiv 1 \text{ for all } x$$

- NOTE: $f[x] = 2$ for all $x$ is NOT $f[x] = 2[x]$ (not defined), but rather $f[x] = 2 \cdot 1[x]$

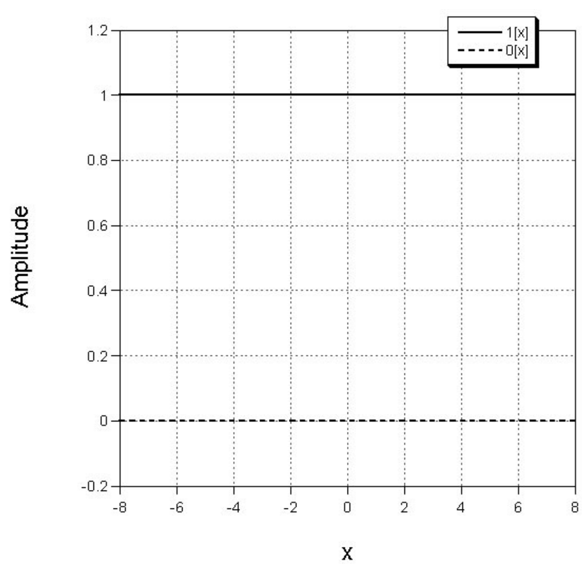
### 1.3.2 null constant $f[x] = 0[x]$:

- null support
- null area
- used as "place holder", e.g., $f[x] = 1[x] + i \cdot 0[x]$ reminds us that function is complex with null imaginary part.

$$0[x] \equiv 0 \text{ for all } x$$

- A constant function with any desired complex-valued amplitude is generated trivially by multiplying 1[x] by that complex constant.
- Translation or scaling applied to argument of any constant function has no effect on amplitude at any coordinate:

$$\begin{aligned} 1\left[\frac{x-x_0}{b_0}\right] &= 1[x] = 1 \text{ for all } x \\ 0\left[\frac{x-x_0}{b_0}\right] &= 0[x] = 0 \text{ for all } x \end{aligned}$$




Unit constant function $1[x]$

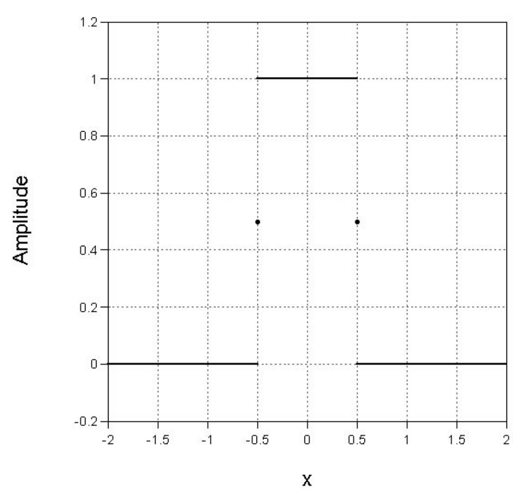
## 1.4 Rectangle Function

- Sum of pieces of unit constant and null constant
- amplitude $\propto x^0$, except at discontinuous transitions
- Many uses in imaging
  - multiply ("modulate", and thus truncate) a "wide" (or infinite-support) function
- Rectangle function has unit amplitude within its finite support
  - endpoint amplitudes are the average of neighboring amplitudes
  - endpoint amplitudes are VERY important, particularly when sampling

$$RECT[x] \equiv \begin{cases} 1 & \text{for} \quad |x| < \frac{1}{2} \\ \frac{1}{2} & \text{for} \quad |x| = \frac{1}{2} \\ 0 & \text{for} \quad |x| > \frac{1}{2} \end{cases} \Longrightarrow RECT[x] \equiv \begin{cases} 0 & \text{for} \quad x > +\frac{1}{2} \\ \frac{1}{2} & \text{for} \quad x = +\frac{1}{2} \\ 1 & \text{for} \quad -\frac{1}{2} < x < +\frac{1}{2} \\ \frac{1}{2} & \text{for} \quad x = -\frac{1}{2} \\ 0 & \text{for} \quad x < -\frac{1}{2} \end{cases}$$

- symmetric (even) with respect to the origin of coordinates: $RECT[-x] = RECT\left[\frac{x}{-1}\right] = RECT[x]$
  - reversed replica is identical to original function
- discontinuous range: $f = 0, +\frac{1}{2}$, or $+1$
- More general form of RECT includes parameters for location of center of symmetry $x_0$ and real-valued width $b_0$:

$$RECT\left[\frac{x-x_0}{b_0}\right] \equiv \begin{cases} 1 & \text{for} \quad |x-x_0| < \frac{|b_0|}{2} \\ \frac{1}{2} & \text{for} \quad |x-x_0| = \frac{|b_0|}{2} \\ 0 & \text{for} \quad |x-x_0| > \frac{|b_0|}{2} \end{cases} \Longrightarrow RECT\left[\frac{x-x_0}{b_0}\right] \equiv \begin{cases} 0 & \text{for} \quad x > \frac{b_0}{2} + x_0 \\ \frac{1}{2} & \text{for} \quad x = \frac{b_0}{2} + x_0 \\ 1 & \text{for} \quad -\frac{b_0}{2} + x_0 < x < \frac{b_0}{2} + x_0 \\ \frac{1}{2} & \text{for} \quad x = -\frac{b_0}{2} + x_0 \\ 0 & \text{for} \quad x < -\frac{b_0}{2} + x_0 \end{cases}$$




Rectangle function $RECT[x]$

- Rectangle with negative amplitude is $f[x] = -RECT\left[\frac{x}{b_0}\right]$

## 1.5 Triangle Function

- Unit amplitude at origin
- Unit area
- even (symmetric) function if translation parameter $x_0 = 0$
- Nonnull amplitudes vary as $\pm x^1$ within region of support

$$TRI[x] \equiv \left\{ \begin{array}{ccc} 0 & \text{for} & x \le -1 \\ 1+x & \text{for} & -1 \le x \le 0 \\ 1-x & \text{for} & 0 \le x \le 1 \\ 0 & \text{for} & x \ge 1 \end{array} \right\}$$

$$= \left\{ \begin{array}{ccc} 1-|x| & \text{for} & |x| < 1 \\ \\ 0 & \text{for} & |x| \ge 1 \end{array} \right\}$$

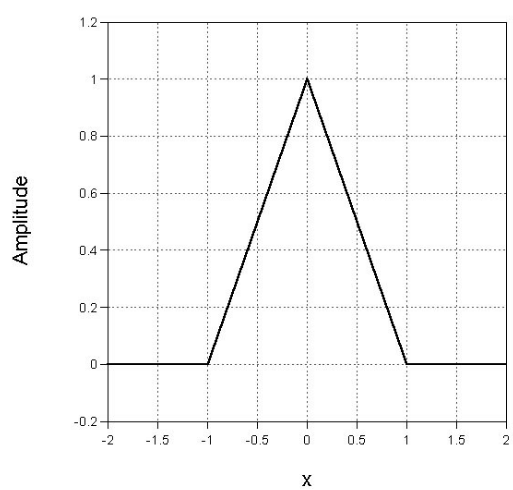
$$= (1-|x|) \cdot RECT\left[\frac{x}{2}\right]$$

- support is two units
- slope of sides are $+1$ for $-1 \le x \le 0$ and $-1$ for $0 \le x \le 1$
- More general expression for triangle centered at $x = x_0$ with real-valued scale parameter $b_0$:

$$TRI\left[\frac{x-x_0}{b_0}\right] \equiv \left\{ \begin{array}{ccc} 1-\left|\frac{x}{b_0}\right| & \text{for} & |x-x_0| < |b_0| \\ \\ 0 & \text{for} & |x-x_0| \ge |b_0| \end{array} \right\}$$

$$= \left(1 - \frac{|x-x_0|}{b_0}\right) RECT\left[\frac{x-x_0}{2b_0}\right]$$

- Support is $2 \cdot |b_0|$
- Area is $|b_0|$
- often used to modulate other functions in imaging situations
  - particularly useful as apodizing (or "multiplicative weighting") function.



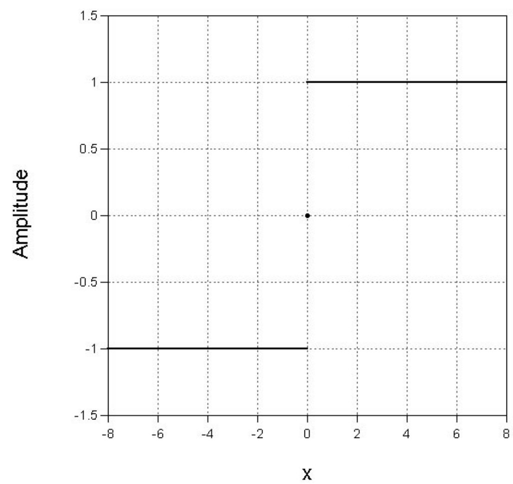

Triangle function $TRI[x]$

## 1.6 Signum Function

(pronounced *sig'num)* assigns numerical values 0 and ±1 to dependent variable based on algebraic sign of the argument. Its name is intended to prevent any confusion of the homonyms "*sign*" and "*sine*". Our definition differs slightly from that of some authors, e.g., Bracewell does not explicitly include null value at $x = 0$

$$SGN[x] \equiv \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 0 \end{cases}$$

- infinite support
- null area
- odd ("antisymmetric" function since $SGN[-x] = -SGN[+x]$)



"SIGNUM" function $SGN[x]$

- Location of abscissa where amplitude transition occurs may be translated via additive constant factor into argument:

$$SGN[x - x_0] \equiv \begin{cases} 1 & \text{for} \quad x - x_0 > 0 \Longrightarrow x > x_0 \\ 0 & \text{for} \quad x - x_0 = 0 \Longrightarrow x = x_0 \\ -1 & \text{for} \quad x - x_0 < 0 \Longrightarrow x < x_0 \end{cases}$$

- Scaling of argument of $SGN[x]$ by a real-valued factor $b_0 > 0$ has no effect on amplitude at any location:

$$SGN\left[\frac{x - x_0}{b_0}\right] \equiv SGN[x - x_0]$$

- If $b_0 < 0$, the function is reversed, e.g.,

$$SGN\left[\frac{x-x_0}{-2}\right] \equiv \begin{cases} 1 & \text{for} \quad \frac{x-x_0}{-2} > 0 \Longrightarrow \frac{x}{-2} > \frac{x_0}{-2} \Longrightarrow x < x_0 \\ 0 & \text{for} \quad \frac{x-x_0}{-2} = 0 \Longrightarrow x = x_0 \\ -1 & \text{for} \quad \frac{x-x_0}{-2} < 0 \Longrightarrow x > x_0 \end{cases}$$

## 1.7 Step Function

- scaled and biased replica of $SGN[x]$
- amplitude "switches" between two extrema (0 and +1) at origin, and defined as the average value at the transition:

$$STEP[x] \equiv \begin{cases} 1 & \text{for} \quad x > 0 \\ \frac{1}{2} & \text{for} \quad x = 0 \\ 0 & \text{for} \quad x < 0 \end{cases}$$

- translated to arbitrary coordinate $x_0$:

$$STEP[x - x_0] \equiv \begin{cases} 1 & \text{for} \quad x - x_0 > 0 \Longrightarrow x > x_0 \\ \frac{1}{2} & \text{for} \quad x - x_0 = 0 \Longrightarrow x = x_0 \\ 0 & \text{for} \quad x - x_0 < 0 \Longrightarrow x < x_0 \end{cases}$$

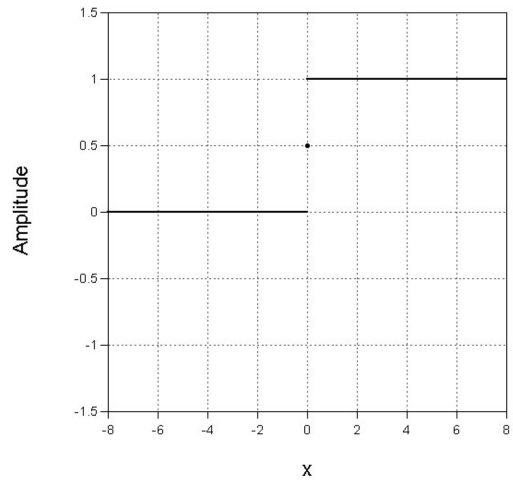
- Again, a positive valued scale factor $b_0 > 0$ has no effect on the amplitudes, e.g.,

$$STEP\left[\frac{x - x_0}{+2}\right] \equiv \begin{cases} 1 & \text{for} \quad \frac{x-x_0}{+2} > 0 \Longrightarrow x > x_0 \\ \frac{1}{2} & \text{for} \quad \frac{x-x_0}{+2} = 0 \Longrightarrow x = x_0 \\ 0 & \text{for} \quad \frac{x-x_0}{+2} < 0 \Longrightarrow x < x_0 \end{cases}$$

  but a negative scale factor "reverses" the function:

$$STEP\left[\frac{x - x_0}{-2}\right] \equiv \begin{cases} 1 & \text{for} \quad \frac{x-x_0}{-2} > 0 \Longrightarrow x < x_0 \\ \frac{1}{2} & \text{for} \quad \frac{x-x_0}{-2} = 0 \Longrightarrow x = x_0 \\ 0 & \text{for} \quad \frac{x-x_0}{-2} < 0 \Longrightarrow x > x_0 \end{cases}$$

- Other authors use other notations.
  - Bracewell ignores amplitude of $STEP[0] = \frac{1}{2}$
  - $H[x]$ as notation for "Heaviside unit step"




Step function $STEP[x]$

- Step function has:
  - semi-infinite support
  - infinite area
- Step function is conveniently expressed in terms of signum function:

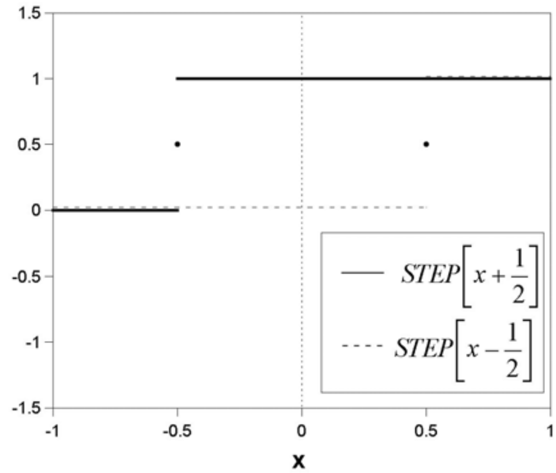
$$\begin{aligned} STEP[x] &= \frac{1}{2}(1[x] + SGN[x]) \\ &= \frac{1}{2} \cdot 1[x] + \frac{1}{2} \cdot SGN[x] \end{aligned}$$

  - this is the decomposition into the *even part* $\frac{1}{2} \cdot 1[x]$ and *odd part* $\frac{1}{2} \cdot SGN[x]$
- May write $RECT[x]$ as difference of two $STEP$ functions

$$RECT[x] = STEP\left[x - \left(-\frac{1}{2}\right)\right] - STEP\left[x - \left(+\frac{1}{2}\right)\right] = STEP\left[x + \frac{1}{2}\right] - STEP\left[x - \frac{1}{2}\right]$$

$$RECT\left[\frac{x}{b_0}\right] = STEP\left[x + \frac{b_0}{2}\right] - STEP\left[x - \frac{b_0}{2}\right]$$

$$RECT\left[\frac{x - x_0}{b_0}\right] = STEP\left[x - x_0 + \frac{b_0}{2}\right] - STEP\left[x - x_0 - \frac{b_0}{2}\right]$$

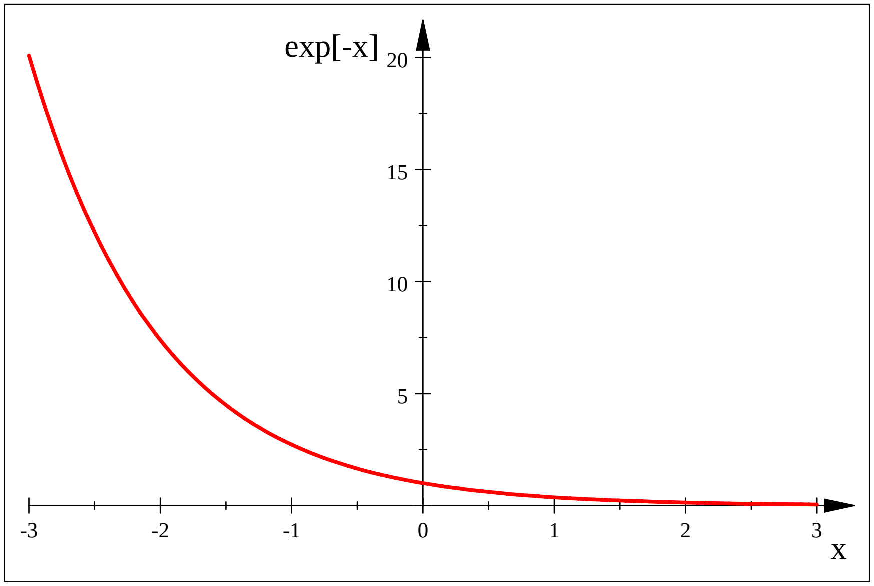



$RECT\,[x]$ constructed from the difference of two translated $STEP$ functions,
$RECT\,[x] = STEP\left[x+\frac{1}{2}\right] - STEP\left[x-\frac{1}{2}\right]$.

## 1.8 Exponential Function

- Most frequently occurring function in physical and engineering problems, other than perhaps the sinusoid

$$f_1[x] = e^{-x}$$



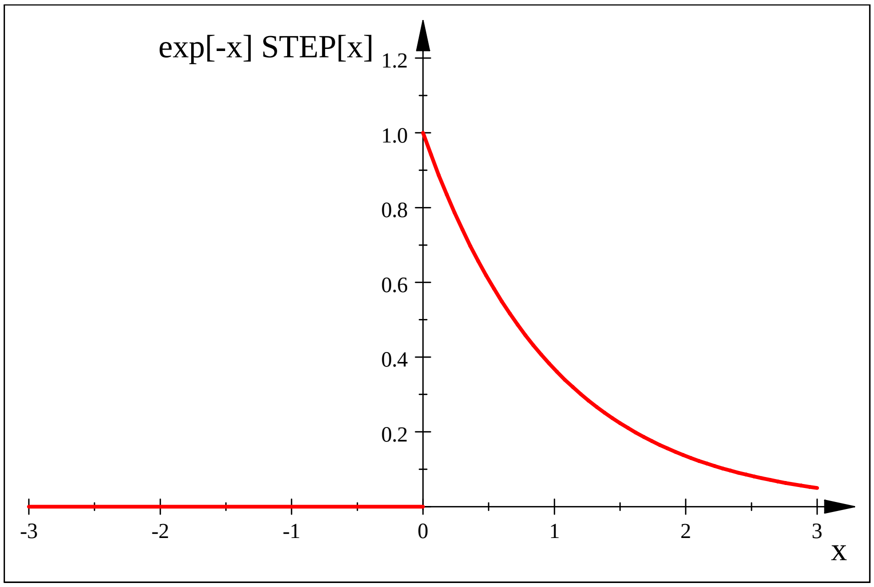

$f[x] = \exp[-x]$ over $-3 \leq x \leq +3$; it approaches 0 as $x \to +\infty$ and $+\infty$ as $x \to -\infty$

- $f_1[0] = 1$
- $f_1[1] = e^{-1} \simeq 0.3679$
- $f_1[x \to +\infty] = 0$
- $f_1[x \to -\infty] = +\infty$
- area is infinite:

$$\int_{-\infty}^{+\infty} e^{-x} dx = +\infty$$

Modify $\exp[-x]$ by multiplying ("modulating") by $STEP[x]$:

$$f_2[x] = e^{-x} \cdot STEP[x] \equiv \begin{cases} e^{-x} & \text{for} \quad x > 0 \\ \frac{1}{2} \cdot e^0 = \frac{1}{2} & \text{for} \quad x = 0 \\ 0 & \text{for} \quad x < 0 \end{cases}$$



$f[x] = \exp[-x] \cdot STEP[x]$ over $-3 \leq x \leq +3$

- Unit area

$$\int_{-\infty}^{+\infty} e^{-x}\, STEP[x]\ dx = \int_{0}^{+\infty} e^{-x}\ dx = -\ e^{-x}\Big|_{x=0}^{x=+\infty} = 1$$

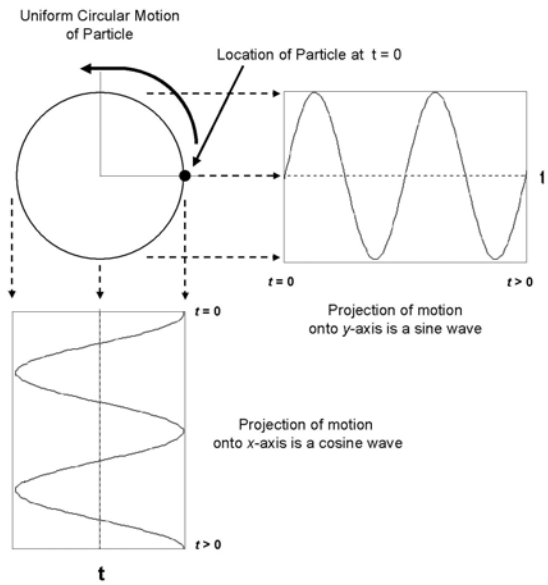
- $f_2[0] \equiv \frac{1}{2}$
- useful when modeling dynamic systems that lose energy or amplitude due to losses (e.g., friction)

## 1.9 Sinusoid (harmonic function)

- most pervasive function in physical science
- form derived in many ways
  - solution to second-order linear differential equation:

$$\frac{d^2}{dx^2}(f[x]) + \alpha^2 f[x] = 0$$

- sinusoid is projection of endpoint of vector that rotates about origin at uniform rate




Projection of uniform circular motion onto orthogonal axes generates two sinusoidal functions in quadrature.

- distance from tip of vector to origin projected onto line perpendicular to distance of closest approach is sinusoidal function of time
- Often more convenient to define form of general sinusoid in terms of symmetric cosine:

$$\begin{aligned} f[x] &= \cos[\alpha_0 x + \phi_0] \\ \phi_0 &= 0 \implies f[x] = \cos[\alpha_0 x] \text{ (even)} \\ \phi_0 &= \pm\pi \implies f[x] = -\cos[\alpha_0 x] \text{ (even)} \\ \phi_0 &= \pm\frac{\pi}{2} \implies f[x] = \mp\sin[\alpha_0 x] \text{ (odd)} \end{aligned}$$

- argument of sinusoid is an *angle* (the *phase angle*, or just the *phase*), measured in *radians* (rarely do we talk about angular degrees)
- General sinusoidal function in the space domain is:

$$A_0 \cos[\Phi[x]] = A_0 \cos\left[2\pi\frac{x}{X_0} + \phi_0\right] = A_0 \cos[2\pi\xi_0 x + \phi_0]$$

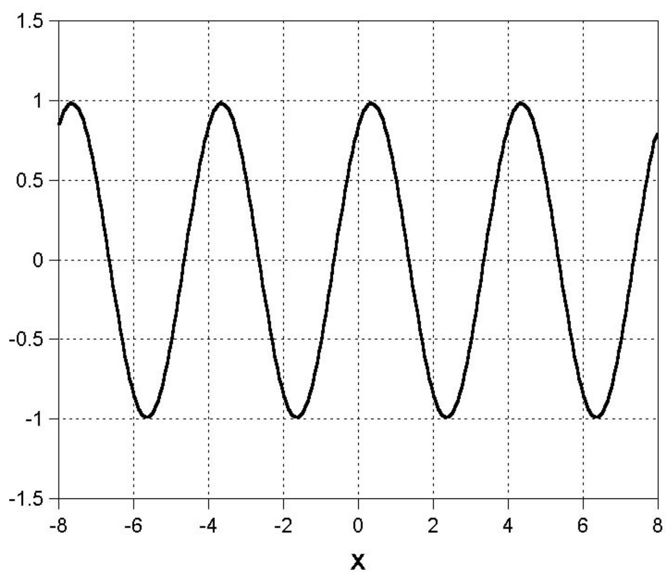
- 
  - phase changes by $2\pi$ radians as $x$ increments by $X_0$, so $X_0$ is the ("spatial") *period*, measured in same units as argument $x$

- $\phi_0$ is the phase angle if $x = 0$ (at the origin of coordinates), called the *initial phase.*

- sinusoidal functions may be rewritten in terms of complex exponential function

$$A_0 \cos[2\pi\xi_0 x + \phi_0] = \frac{1}{2}\left(e^{+i(2\pi\xi_0 x+\phi_0)} + e^{-i(2\pi\xi_0 x+\phi_0)}\right)$$

- Proportionality constant applied to the argument is $\frac{2\pi}{X_0}$
  - describes number of radians traversed by argument per unit distance
  - "angular spatial frequency", sometimes labeled $k_0$
  - Reciprocal of period $X_0$ is "spatial frequency" $\xi_0$
  - describes number of cycles traversed by sinusoid in unit distance.
- "cycle" describes a single period of any periodic function
  - (strictly speaking) applies to sinusoidal functions only
  - analogous term for square waves is "line pairs per mm"
- Amplitude $A_0$ measured in appropriate units
- cosine may be computed for any real-valued argument, including higher-order functions of $x$
- sinusoid is symmetric cosine when initial phase $\phi_0$ is integer multiple of $\pi$ radians.




Sinusoidal function $\cos[2\pi\xi_0 x + \phi_0]$ with $\xi_0 = \frac{1}{4}$ cycle per unit length and $\phi_0 = -\frac{\pi}{6}$ radians.

- *Spatial frequency* $\xi$ is the "rate of change of phase"

$$\xi = \frac{1}{2\pi}\frac{\partial\Phi[x]}{\partial x} = \frac{1}{2\pi}\frac{\partial}{\partial x}[2\pi\xi_0 x + \phi_0] = \left(\frac{1}{2\pi}\right)2\pi\xi_0 = \xi_0$$

- Positive and negative lobes of sinusoid have equal areas of opposite sign

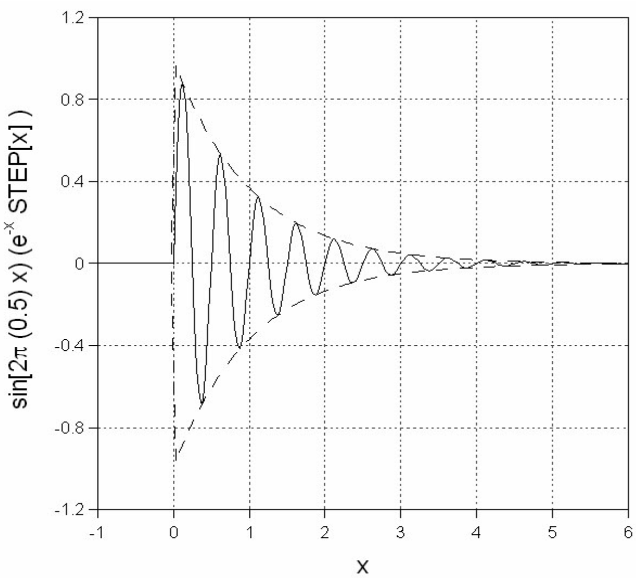
- $\Longrightarrow$ Area of harmonic sinusoid is zero:

$$\int_{-\infty}^{+\infty} \cos\left[2\pi\xi_0 x + \phi_0\right]\ dx = 0 \text{ if } \xi_0 \neq 0$$

- Applied as modulations (multipliers) of other functions with compact support
  - particularly common in electrical engineering to describe temporal signals.
  - Example:

$$f\left[x\right] = \cos\left[2\pi\xi_0 x - \frac{\pi}{2}\right]\ e^{-x}\ STEP\left[x\right] = \sin\left[2\pi\xi_0 x\right]\ e^{-x}\ STEP\left[x\right]$$

  - Odd sinusoid is modulating function because $\sin[0] = 0$, increases "slowly" with increasing $x$




Sinusoid modulating a decaying exponential: $f\left[x\right] = COS\left[2\pi\xi_0 x\right]\ \left(e^{-x}\ STEP\left[x\right]\right)$

- squared magnitude of sinusoid is *spatial power* or *intensity*
  - nonnegative
  - recast into sum of constant and sinusoidal part by squaring the complex-exponential expression for cosine

$$\begin{aligned}\left(\cos\left[2\pi\xi_0 x\right]\right)^2 = \cos^2\left[2\pi\xi_0 x\right] &= \left(\frac{e^{2\pi i\xi_0 x} + e^{-2\pi i\xi_0 x}}{2}\right)^2 \\ &= \frac{1}{4}\left(2 + e^{4\pi i\xi_0 x} + e^{-4\pi i\xi_0 x}\right) \\ &= \frac{1}{2} + \frac{1}{2}\cos\left[4\pi\xi_0 x\right] = \frac{1}{2}\left(1 + \cos\left[2\pi\left(2\xi_0\right)x\right]\right)\end{aligned}$$

    * $\cos^2\left[2\pi\xi_0 x\right]$ is equivalent to sum of half-unit additive constant and cosine function at doubled spatial frequency
    * additive constant is the *bias* that ensures that amplitude is nonnegative.

### 1.9.1 Modulation of Sinusoid

- Relative sizes of maxima and minima of nonnegative sinusoid give a metric used in optics and image processing
- For $f[x]$ a biased **nonnegative** sinusoid with maximum and minimum amplitudes $f_{\max}$ and $f_{\min}$,

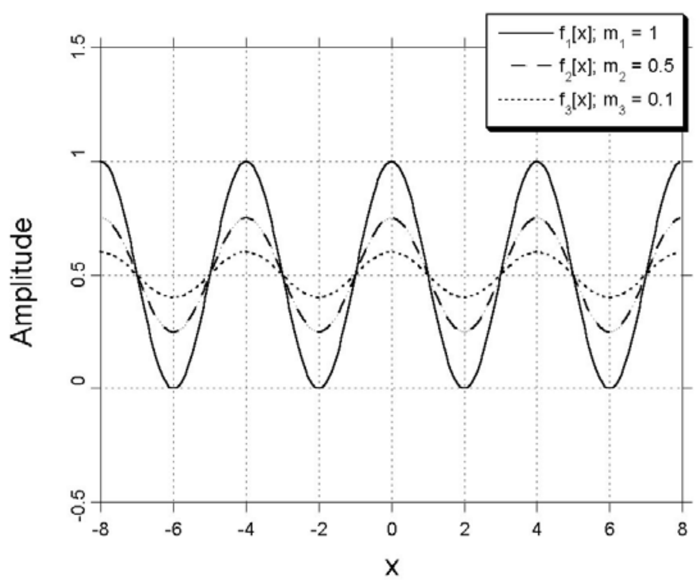
$$m_f \equiv \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}} \text{ if } f_{\max} \geq f_{\min} \geq 0$$

- The range of possible $m_f$ for nonnegative sinusoids is $0 \leq m_f \leq 1$.
- $m_f = 1$ for all functions with $f_{min} = 0$
- $m_f < 1$ if $f_{min} > 0$
- $m_f = 0$ if $f_{max} = f_{min}$ (amplitude of sinusoid is zero).
- modulation of $\cos^2[2\pi\xi_0 x]$ is unity, since $0 \leq \cos^2[2\pi\xi_0 x] \leq +1$
- modulation of

$$f[x] = A_0 + A_1 \cos[2\pi\xi_0 x + \phi_0] \quad \text{where } A_0 \geq A_1$$

$$m_f = \frac{(A_0 + A_1) - (A_0 - A_1)}{(A_0 + A_1) + (A_0 - A_1)} = \frac{A_1}{A_0}$$

  - modulation of nonnegative sinusoid is ratio of amplitude to bias
  - provides measure of relative "brightnesses" of maxima and minima.




Nonnegative sinusoidal functions with modulation factors of 1, 0.5, and 0.2.

- Note that we have introduced two different definitions of "modulation"
  1. multiplication of two functions ("apply a modulation to a sinusoidal carrier")
  2. as a quality factor of sinusoid
- Different names for sinusoidal quality factor in other applications:

  - Michelson: $m_f$ is "visibility" of sinusoid
  - "contrast" (we use to refer to nonnegative square-wave signals rather than sinusoids)

- *modulation transfer function* (MTF) for sine waves plots the measured modulation $m$ of a sinusoidal wave vs. the spatial frequency $\xi$

- *contrast transfer function* (CTF).plots the measured contrast of a square wave as a function of spatial frequency of the fundamental sinusoid.

## 1.10 $SINC$ Function

- product of odd sinusoid and $(\pi x)^{-1}$, which is not continuous at $x = 0$
- amplitude of $(\pi x)^{-1} \to 0$ as $x \to \pm\infty$:

$$SINC[x] \equiv \frac{\sin[\pi x]}{\pi x} = \frac{\cos\left[\frac{2\pi x}{2} - \frac{\pi}{2}\right]}{\pi x}$$

  - $SINC[x] = 0$ for $x = \pm n$, $(n = 1, 2, 3, \cdots)$.

- Some authors define $SINC[x]$ without factors of $\pi$ in arguments, e.g., $\text{sinc}[x] \equiv \frac{\sin x}{x}$
  - $\text{sinc}[x] = 0$ for $x = \pm n\pi$, $(n = 1, 2, 3, \cdots)$
- Both numerator and denominator are zero at the origin $\Longrightarrow$ the amplitude of $SINC[0]$ determined via L'Hôspital's rule:

$$SINC[0] = \frac{\lim\limits_{x\to 0}\left\{\frac{d}{dx}\sin[\pi x]\right\}}{\lim\limits_{x\to 0}\left\{\frac{d}{dx}[\pi x]\right\}} = \frac{\pi\cos[0]}{\pi} = 1$$

- values of $SINC\left[\pm\frac{1}{2}\right]$

$$SINC\left[\pm\frac{1}{2}\right] = \frac{\sin\left[\pm\frac{\pi}{2}\right]}{\pm\frac{\pi}{2}} = \frac{\pm 1}{\pm\frac{\pi}{2}} = +\frac{2}{\pi} \cong 0.637$$

- Between zeros, local extrema of amplitude are in vicinity of (though not coincident with) half-integer values of $x$.
  - First few local extrema of the magnitudes are approximately:

$$SINC[\pm 1.428] \simeq -0.2172$$

$$SINC[\pm 2.459] \simeq +0.1284$$

$$SINC[\pm 3.471] \simeq -0.09132$$

  - Amplitudes at half-integer arguments are:

$$SINC\left[\tfrac{3}{2}\right] = -\tfrac{2}{3\pi} \simeq -0.2122$$

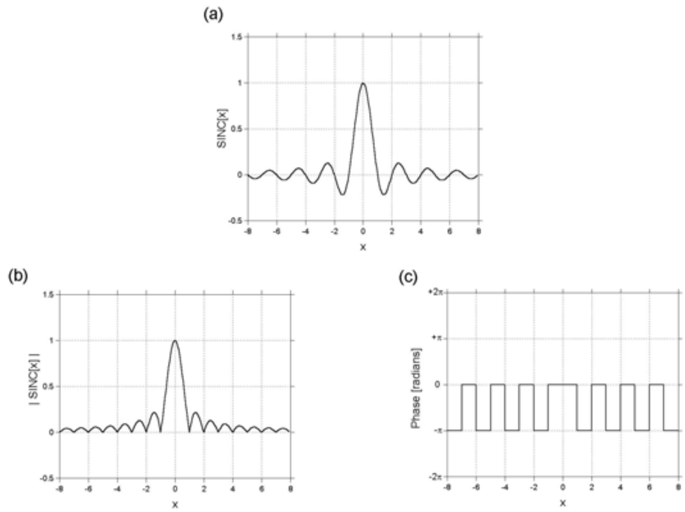
$$SINC\left[\tfrac{5}{2}\right] = +\tfrac{2}{5\pi} \simeq 0.1273$$

$$SINC\left[\tfrac{7}{2}\right] = -\tfrac{2}{7\pi} \simeq -0.09095$$

$$\vdots$$

$$SINC\left[\tfrac{m}{2}\right] = (-1)^m \tfrac{2}{(2m+1)\pi}$$

- Phase of real-valued $SINC$ is 0 or $-\pi$ radians




Representations of $SINC[x]$: (a) real part (imaginary part is $0[x]$); (b) magnitude; (c) phase.

### 1.10.1 Power-Series Representation of $SINC[x]$:

$$
\begin{aligned}
\frac{\sin[\pi x]}{\pi x} &= \frac{1}{\pi x}\left(\frac{(\pi x)^1}{1!} - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots\right) \\
&= \left(1 - \left(\frac{\pi^2}{6}\right)x^2 + \left(\frac{\pi^4}{120}\right)x^4 - \left(\frac{\pi^6}{5040}\right)x^6 + \cdots\right) \\
&= \sum_{n=0}^{+\infty} (-1)^n \frac{(\pi x)^{2n}}{(2n+1)!}
\end{aligned}
$$

- demonstrates that $SINC[0]$ must be unity
- Note denominators of sequence increase rapidly with order

### 1.10.2 Area of $SINC[x]$

- determined rigorously by evaluating appropriate contour integral in the complex plane
- simpler method uses *central-ordinate theorem* of Fourier transform.
  - statement without proof: area of unscaled SINC function is unity:

$$
\int_{-\infty}^{+\infty} SINC[x]\ dx = 1
$$

### 1.10.3 Translated and Scaled $SINC$

$$
SINC\left[\frac{x-x_0}{b_0}\right] = \frac{\sin\left[\pi\left(\frac{x-x_0}{b_0}\right)\right]}{\left[\pi\left(\frac{x-x_0}{b_0}\right)\right]}
$$

- amplitude vanishes at $x = n|b_0| + x_0$ $(n = \pm1, \pm2, \ldots)$
- area is $|b_0|$ regardless of the translation $x_0$

- support of $SINC\left[\frac{x}{b_0}\right]$ is infinite
- magnitude $\left|SINC\left[\frac{x-x_0}{b_0}\right]\right| < |x|^{-1}$

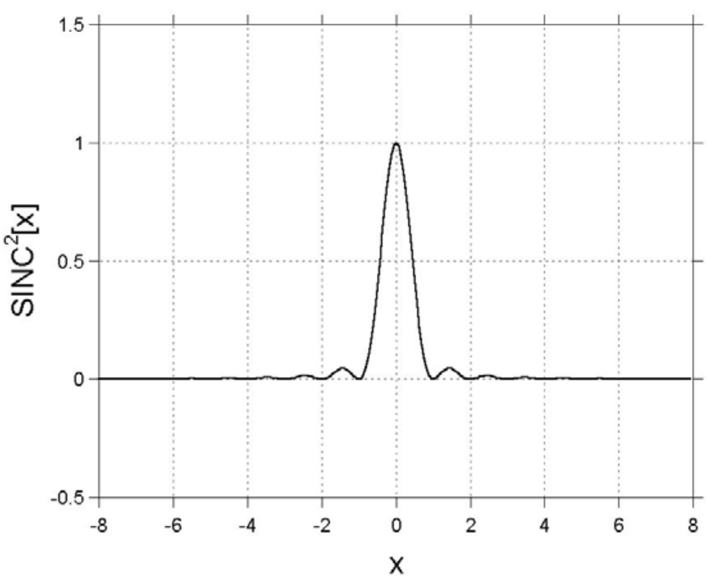
## 1.11 $SINC^2$ Function

- Appears in many imaging contexts
  - e.g., describes action of optical imaging system with rectangular aperture used in natural (incoherent) light
- zeros of $SINC^2[x]$ and $SINC[x]$ occur at the same locations ($x = \pm n$ for $n \neq 0$)
- $SINC^2[x]$ varies "smoothly" in vicinity of zeros (novices often *incorrectly* visualize cusps in $SINC^2[x]$ similar to those of $|SINC[x]|$)
- $SINC^2[x] \geq 0$ everywhere $\Longrightarrow$ phase $= 0$ radians for all $x$.
- area of $SINC^2[x]$ evaluated via appropriate integral in the complex plane or by central-ordinate theorem
  - areas of $SINC[x]$ and $SINC^2[x]$ are identical

$$\int_{-\infty}^{+\infty} SINC^2[x]\ dx = \int_{-\infty}^{+\infty} SINC[x]\ dx = 1$$

  - areas of $SINC^2\left[\frac{x}{b_0}\right]$ and $SINC\left[\frac{x}{b_0}\right]$ are identical:

$$\int_{-\infty}^{+\infty} SINC^2\left[\frac{x}{b_0}\right]\ dx = \int_{-\infty}^{+\infty} SINC\left[\frac{x}{b_0}\right]\ dx = |b_0|$$




$SINC^2[x]$

## 1.12 Gamma Function $\Gamma[x]$

- $\Gamma[x]$ is a "different animal;" it does not represent useful signal or descriptive function of useful imaging system
- rather, $\Gamma[x]$ is a computational tool for solving imaging problems that involve other special functions
  - avenue for deriving the areas and Fourier transforms of quadratic-phase sinusoid and Gaussian functions
- Gamma function evaluated at positive argument $x_0$ is area of decaying exponential $e^{-\alpha}\, STEP[\alpha]$ modulated by $\alpha^{x_0-1}$:

$$\Gamma[x] = \int_0^{+\infty} e^{-\alpha}\ \alpha^{x-1}\ d\alpha = \int_{-\infty}^{+\infty} STEP[\alpha]\ e^{-\alpha}\ \alpha^{x-1}\ d\alpha,\ for\ x > 0$$

- Argument $x$ appears only in exponential term $\alpha^{x-1}$.
  - $\Gamma[1]$ is area of $STEP[\alpha]\ e^{-\alpha}$, which was shown to be unity

$$\Gamma[1] = 1$$

  - for fixed positive finite value of $x$, $e^{-\alpha}$ decreases with increasing $\alpha$ while $\alpha^{x-1}$ increases rapidly
  - combined conflicting behaviors ensure that area remains finite for finite and positive values of $x$
  - area of product function increases rapidly with $x$ for $x > 1$.
- Change variable of integration from $\alpha$ to $e^{-\alpha}$

$$d\left(e^{-\alpha}\right) = -e^{-\alpha}\ d\alpha$$

$$\Gamma[x] = \int_0^{+\infty} \left(-\alpha^{x-1}\right)\ d\left(e^{-\alpha}\right)$$

$$u = \alpha^{x-1}$$
$$du = (x-1)\ \alpha^{x-2}\ d\alpha$$
$$v = -e^{-\alpha}$$
$$dv = +e^{-\alpha}\ d\alpha$$

  - Substitute into integration-by-parts formula to obtain a recursion relation for $\Gamma[x]$ for positive values of $x$:

$$\begin{aligned}\Gamma[x] &= \int_0^{+\infty} u\,dv = uv|_{u=0}^{u=+\infty} - \int_{u=0}^{u=+\infty} v\ du \\ &= \alpha^{x-1}e^{-\alpha}\Big|_{\alpha=0}^{\alpha=+\infty} + (x-1)\int_0^{+\infty} \alpha^{x-2}\ e^{-\alpha}\ d\alpha = 0 + (x-1)\ \Gamma[x-1] \\ \Gamma[x] &= (x-1)\ \Gamma[x-1]\end{aligned}$$

  - Repeat to demonstrate:

$$\begin{aligned}\Gamma[x] &= (x-1)\ \Gamma[x-1] \\ &= (x-1)\left((x-2)\ \Gamma[x-2]\right) \\ &= (x-1)(x-2)(x-3)\cdots(x-INT[x])\ \Gamma[x-INT[x]]\end{aligned}$$

  where $INT[x]$ is integer part of positive real number $x$

– Relationship valid because $x - INT[x] > 0$.

– If $x = n$, expression simplifies to:

$$\Gamma[n] = (n-1)(n-2)(n-3)\cdots(1)\ \Gamma[1] = (n-1)!$$

– *Gamma function = factorial function*

$$\Gamma[1] = \Gamma[2] = 1$$

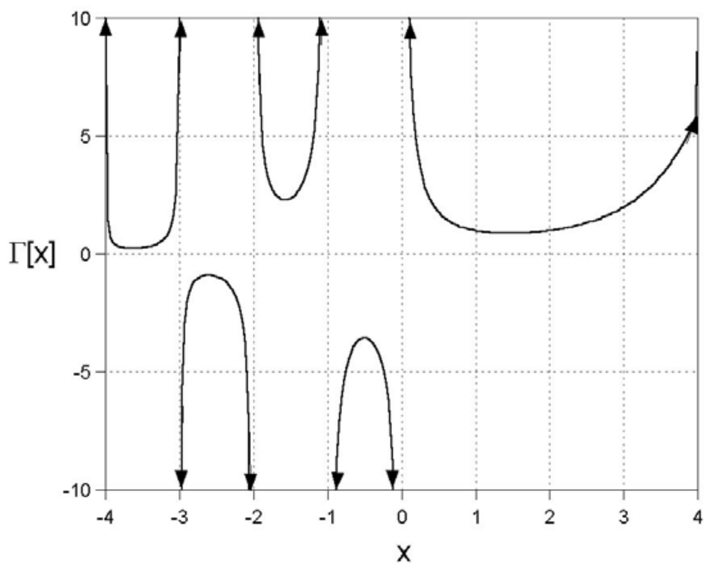
$$0! = 1$$

– use gamma function to generalize definition of "factorial" to nonintegers

$$x! \equiv \Gamma[x+1] = \int_0^{+\infty} \alpha^x\ e^{-\alpha}\ d\alpha$$

- Extend gamma function to negative arguments

$$\Gamma\left[-\frac{1}{2}\right] = -2\ \Gamma\left[+\frac{1}{2}\right]$$




Graphical representation of $\Gamma[x]$, showing that the amplitude is undefined at all integer values of $x \leq 0$.

- $\Gamma[x]$ has no zeros

**Case 1** – $\Longrightarrow (\Gamma[x])^{-1}$ *has no singularities*

– expressed as Taylor series valid for all $x$

$$(\Gamma[x])^{-1} \cong 1 + 0.57721\ x - 0.65587\ x^2 - 0.04200\ x^3 + 0.16654\ x^4 + \cdots$$

- Other expansions converge more efficiently to correct value

### 1.12.1 Gamma function for half-integer arguments:

- $\left(x = \frac{1}{2}, \frac{3}{2}, \text{etc.}\right)$ obtained by applying recursion relation to $\Gamma\left[\frac{1}{2}\right]$
  - Evaluate $\Gamma\left[\frac{1}{2}\right]$ by recasting into form of easily evaluated "error function"

$$\Gamma\left[\frac{1}{2}\right] = \int_0^{+\infty} \alpha^{\left(\frac{1}{2}-1\right)}\, e^{-\alpha}\, d\alpha = \int_0^{+\infty} \alpha^{-\frac{1}{2}}\, e^{-\alpha}\, d\alpha$$

1. Change the variable of integration to $\beta = \sqrt{\alpha}$ to obtain an integral that is defined as $I$:

$$\begin{aligned}\Gamma\left[\frac{1}{2}\right] &= \int_{\beta=0}^{\beta=+\infty} \beta^{-1}\, e^{-\beta^2}\, 2\beta\, d\beta = 2 \cdot \int_{\beta=0}^{\beta=+\infty} e^{-\beta^2}\, d\beta \\ &= \int_{\beta=-\infty}^{\beta=+\infty} e^{-\beta^2} d\beta \equiv I\end{aligned}$$

2. Evaluate $I$ by constructing square as product of two integrals with independent variables, convert to polar coordinates:

$$\begin{aligned}I^2 &= \left(\int_{\beta=-\infty}^{\beta=+\infty} e^{-\beta^2} d\beta\right)\left(\int_{\gamma=-\infty}^{\gamma=+\infty} e^{-\gamma^2} d\gamma\right) \\ &= \int_{\beta=-\infty}^{\beta=+\infty} \int_{\gamma=-\infty}^{\gamma=+\infty} e^{-\left(\beta^2+\gamma^2\right)} d\beta\, d\gamma \\ &= \int_{\theta=-\pi}^{\theta=+\pi} \int_{\rho=0}^{\rho=+\infty} e^{-\rho^2} \rho\, d\rho\, d\theta \\ &= 2\pi \int_{\rho=0}^{\rho=+\infty} e^{-\rho^2}\, \rho\, d\rho\end{aligned}$$

3. Change integration variable again to $u = e^{-\rho^2}$

$$du = -2\rho\, e^{-\rho^2}\, d\rho$$

$$I^2 = 2\pi \int_{u=1}^{u=0} \left(-\frac{1}{2}\right) du = 2\pi \left(\frac{1}{2}\right) = \pi$$

Required result is:

$$\Gamma\left[\frac{1}{2}\right] = \sqrt{I^2} = \sqrt{\pi} \simeq 1.7725$$

Apply recursion relation:

$$\begin{aligned}\Gamma\left[\frac{5}{2}\right] &= \frac{3}{2} \cdot \Gamma\left[\frac{3}{2}\right] = \frac{3 \cdot \sqrt{\pi}}{4} \simeq 1.3293 \\ \Gamma\left[\frac{3}{2}\right] &= \frac{1}{2} \cdot \Gamma\left[\frac{1}{2}\right] = \frac{\sqrt{\pi}}{2} \simeq 0.8862 \\ \Gamma\left[-\frac{1}{2}\right] &= -2 \cdot \Gamma\left[\frac{1}{2}\right] = -2 \cdot \sqrt{\pi} \simeq -3.5449 \\ \Gamma\left[-\frac{3}{2}\right] &= -\frac{2}{3} \cdot \Gamma\left[-\frac{1}{2}\right] = -\frac{4}{3} \cdot \sqrt{\pi} \simeq +2.3633\end{aligned}$$

Note growth in $\Gamma[x]$ for increasing values of $x > 1$.

### 1.12.2 Gamma Function for "Reciprocal Integer" Arguments

- $\left(x = \frac{1}{n}\text{, where } n = 1, 2, 3, \cdots\right)$evaluated via series expansion
- First 5 examples are:

$$\Gamma\left[\frac{1}{1}\right] = 0! = 1$$
$$\Gamma\left[\frac{1}{2}\right] = \sqrt{\pi} \simeq 1.7725$$
$$\Gamma\left[\frac{1}{3}\right] \simeq 2.6789$$
$$\Gamma\left[\frac{1}{4}\right] \simeq 3.6256$$
$$\Gamma\left[\frac{1}{5}\right] \simeq 4.5908$$

$\Gamma\left[\frac{1}{n}\right]$ increases as $n$ increases $(\frac{1}{n} \to 0_+)$

- $\Gamma\left[\frac{1}{n}\right]$ in limit of large $n$ evaluated from recursion relation:

$$\lim_{n\to\infty}\left\{\frac{1}{n}\,\Gamma\left[\frac{1}{n}\right]\right\} = \lim_{n\to\infty}\left\{\Gamma\left[1+\frac{1}{n}\right]\right\} \simeq \Gamma[1] = 1 \Longrightarrow \lim_{n\to\infty}\left\{\Gamma\left[\frac{1}{n}\right]\right\} = n$$

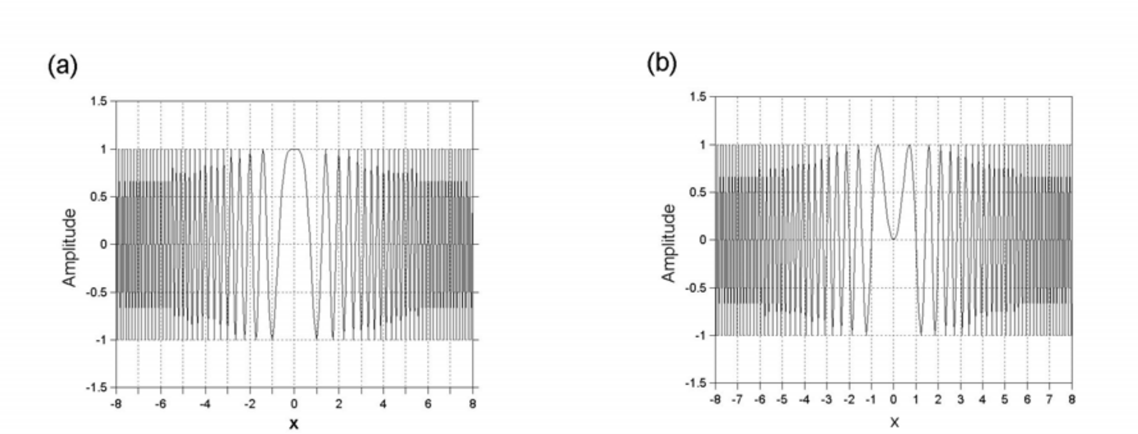
## 1.13 Quadratic-Phase Sinusoid – "CHIRP" Function

- Phase dependence includes term proportional to $x^2$
- Phase is "dimensionless" or "unitless" (measured in radians) ensures that argument $x^2$ includes factors with aggregate dimensions of length$^{-2}$.

$$f[x] = A\cos[\Phi[x]] = A\cos\left[\frac{\pi x^2}{\alpha_0^2} + \phi_0\right]$$

  - scale parameter $\alpha_0$ has dimensions of length
  - $\alpha_0$ specifies "closest" coordinates ($x = \pm\alpha_0$) where phase differs from $\phi_0$ (at origin) by $\pi$ radians.
- Amplitude symmetric with respect to origin regardless of $\phi_0$
  - only term involving $x$ in argument of sinusoid appears as $x^2$ $\Longrightarrow$symmetric ("even")
- center of symmetry may be translated by adding a constant to the argument:

$$f[x] = A\cos\left[\pi\frac{(x-x_0)^2}{\alpha_0^2} + \phi_0\right]$$




Quadratic-phase sinusoidal functions: (a) $\cos[\pi x^2]$, (b) $\cos\left[\pi x^2 - \frac{\pi}{2}\right] = \sin[\pi x^2]$

- spatial frequency is:

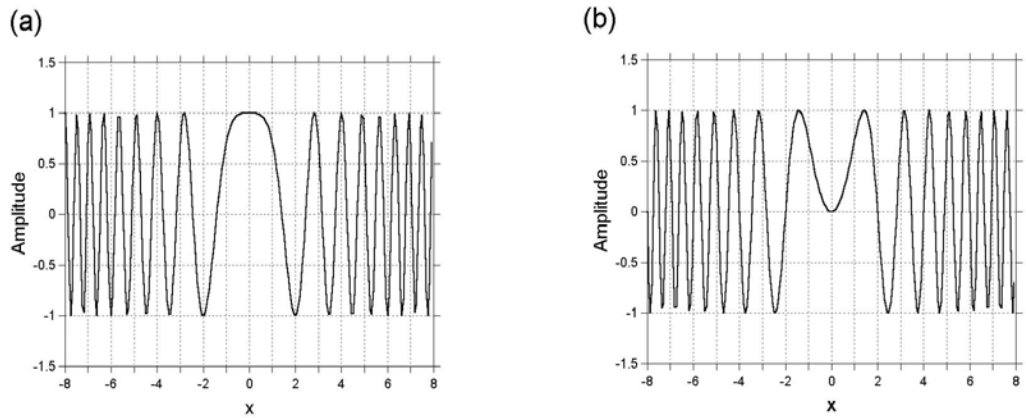
$$\xi[x] = \frac{1}{2\pi}\frac{\partial\Phi[x]}{\partial x} = \frac{1}{2\pi}\left[\frac{2\pi x}{\alpha_0^2}\right] = +\frac{x}{\alpha_0^2}$$

spatial frequency $\xi$ is negative for $x < 0$

spatial frequency $\xi$ is positive for $x > 0$

- spatial frequency $\xi$ depends on $x$, $\xi[x]$ is the *instantaneous spatial frequency* of quadratic-phase sinusoid
- linear dependence of $\xi$ on $x$ leads to names "linear frequency modulation" or "linear FM signal"
- "chirp"

- $\alpha_0$ is "chirp rate", smaller $\alpha_0 \Longrightarrow$more rapid change in spatial frequency




Effect of scale factor on quadratic-phase sinusoids: (a) $\cos\left[\pi\left(\frac{x}{2}\right)^2\right]$, (b) $\sin\left[\pi\left(\frac{x}{2}\right)^2\right]$

- Areas of quadratic-phase sinusoids are not zero
- Areas of linear-phase sinusoids are zero (because areas of adjacent positive and negative lobes cancel)
  - phase is linear function of coordinate
  - rate of change of phase of linear-phase sinusoid is constant
  - Phase of chirp function (or of any sinusoidal function whose phase is a nonlinear function of $x$) changes with $x$ at a variable rate, adjacent positive and negative lobes have different "widths" and thus different (and noncancelling) areas.

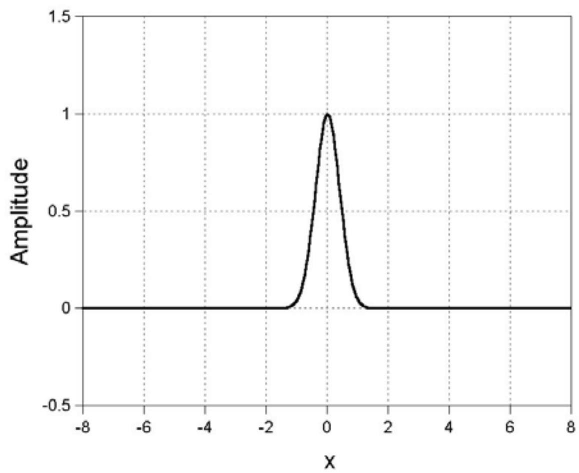
$$\int_{-\infty}^{+\infty} \cos\left[\pi x^2\right]\, dx = \int_{-\infty}^{+\infty} \sin\left[\pi x^2\right]\, dx = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \simeq 0.7071$$

## 1.14 Gaussian Function

- proportional to familiar "bell curve" of probability theory
- appears in many imaging contexts.

$$GAUS[x] = e^{-\pi x^2}$$

- scale factor of $\pi$ not used by some authors
  - Peak amplitude is unity (at origin)
  - Decays smoothly as $|x|$ increases, decreases to $e^{-\pi} \simeq 0.043$ at $x = \pm 1$
  - Approaches zero as $|x| \to \infty$.
- Infinite support.




Gaussian function $e^{-\pi x^2}$

- Area is unity (not proven as yet)

$$\int_{-\infty}^{+\infty} GAUS[x]\ dx = \int_{-\infty}^{+\infty} e^{-\pi x^2} dx = 1$$

- More general form has area $= |b_0|$

$$GAUS\left[\frac{x-x_0}{b_0}\right] = \exp\left[-\pi\frac{(x-x_0)^2}{b_0^2}\right] \Longrightarrow \int_{-\infty}^{+\infty} GAUS\left[\frac{x-x_0}{b_0}\right]\ dx = |b_0|$$

- Relate to Gaussian Distribution in Probability
- Random variable $n$ is *normally distributed* with mean $\langle n\rangle$ and standard deviation $\sigma$ (variance$=$ $\sigma^2$) probability density function $p[n]$ is:

$$p[n] = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{(n-\langle n\rangle)^2}{2\sigma^2}\right)}$$

$$b_0 = \left(\sqrt{2\pi}\right)\sigma \simeq 2.5\sigma \to \sigma \simeq \frac{b_0}{2.5} = 0.4 b_0$$

$$b_0^2 = 2\pi\sigma^2 \simeq 6.28\sigma^2 \Longrightarrow \sigma^2 \simeq \frac{b_0^2}{6.28} \simeq 0.16 b_0^2$$

### 1.15 "SuperGaussian" Function

- Same expression as Gaussian but with integer exponents other than 2:

$$GAUS\left[x;n\right] \equiv e^{-\pi|x|^n}$$

- parameter $n$ is a positive integer
- Absolute value of coordinate ensures that $GAUS\left[x;n\right]$ remains finite and even for negative $x$ and odd values of $n$
- SuperGaussian for $n=1$ is sum of decaying exponential $e^{-\pi x}\ STEP\left[x\right]$ and "reversed" replica $e^{-\pi|x|}\ STEP\left[-x\right]$.
- Amplitude remains near 1 near origin for larger $n$ In the limit $n \to \infty$, the amplitudes of the function in the various regions are:

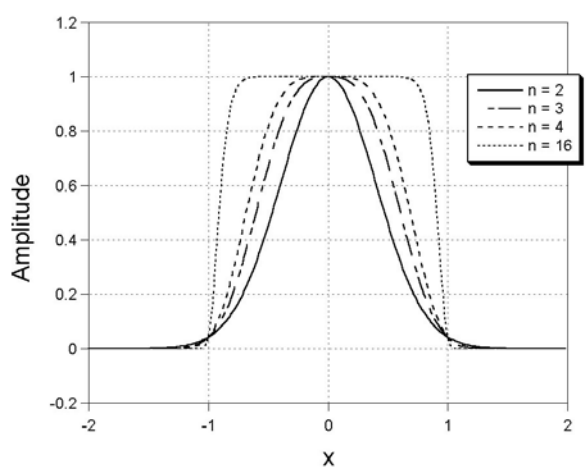
$$\text{for } |x|<1\text{:}\quad \lim_{n\to 0}\left\{|x|^n\right\} = 0 \Longrightarrow \lim_{n\to\infty}\left\{e^{-\pi|x|^n}\right\} = 1$$
$$\text{for } |x|>1\text{:}\quad \lim_{n\to 0}\left\{|x|^n\right\} = +\infty \Longrightarrow \lim_{n\to\infty}\left\{e^{-\pi|x|^n}\right\} = 0$$
$$\text{for } |x|=1\text{:}\quad \lim_{n\to 0}\left\{|x|^n\right\} = 1 \Longrightarrow \lim_{n\to\infty}\left\{e^{-\pi|1|^n}\right\} = e^{-\pi} \simeq 0.04321$$

- Resembles rectangle function with width $b_0 = 2$ for large values of $n$:

$$\lim_{n\ \to\infty}\left\{GAUS\left[x,n\right]\right\} \simeq RECT\left[\frac{x}{2}\right]$$

- "Endpoint" amplitudes not identical, but isolated values have no effect on any integrals of superGaussian.

$$RECT\left[x\right] \simeq \lim_{n\ \to\infty}\left\{GAUS\left[2x,n\right]\right\} = \lim_{n\ \to\infty}\left\{e^{-\pi|2x|^n}\right\}$$




Supergaussian functions $f\left[x\right] = e^{-\pi x^n}$ for $n = 2$, 3, 4, and 16.

### 1.15.1 Area of superGaussian

- from gamma function

$$\Gamma[x] = \int_0^{+\infty} \alpha^{x-1}\ e^{-\alpha}\ d\alpha$$

- Change variable of integration from $\alpha$ to $\pi u^n$:

$$\begin{aligned}
\Gamma[x] &= \int_0^{+\infty} (\pi u^n)^{x-1}\ e^{-\pi u^n}\ d(\pi u^n) \\
&= \int_0^{+\infty} \pi^{x-1}\ u^{nx-n}\ e^{-\pi u^n}\ n\pi\ u^{n-1}\ du \\
&= \int_0^{+\infty} n\pi^x\ u^{nx-n+n-1}\ e^{-\pi u^n}\ du \\
&= n\pi^x \int_0^{+\infty} u^{nx-1}\ e^{-\pi u^n}\ du
\end{aligned}$$

- Divide both sides of this equation by $n\pi^x$:

$$\frac{1}{n}\ \pi^{-x}\ \Gamma[x] = \int_0^{+\infty} u^{nx-1}\ e^{-\pi u^n}\ du$$

- Set $x = n^{-1}$ to ensure that $u^{nx-1} = u^0 = 1$

$$\int_0^{+\infty} e^{-\pi u^n}\ du = \frac{1}{n}\ \pi^{-n^{-1}}\ \Gamma\left[\frac{1}{n}\right]$$

- Area of symmetric function $e^{-\pi|x|^n}$ is twice:

$$\boxed{\int_{-\infty}^{+\infty} e^{-\pi|x|^n} dx = \frac{2}{\pi^{\frac{1}{n}}}\ \frac{1}{n}\ \Gamma\left[\frac{1}{n}\right]}$$

- Area of the superGaussian is proportional to gamma function with a reciprocal-integer argument.

- $n = 1 - 4$ :

$$\begin{aligned}
n = 1: \int_{-\infty}^{+\infty} e^{-\pi|x|}\ dx &= 2\ \pi^{-1}\ \Gamma[1] = \left(\frac{2}{\pi}\right) 0! = \frac{2}{\pi} \simeq 0.6366 \\
n = 2: \int_{-\infty}^{+\infty} e^{-\pi|x|^2}\ dx &= \frac{2}{2}\ \frac{1}{\sqrt{\pi}}\ \Gamma\left[\frac{1}{2}\right] = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1 \\
n = 3: \int_{-\infty}^{+\infty} e^{-\pi|x|^3}\ dx &= \frac{2}{3}\ \left(\frac{1}{\pi^{\frac{1}{3}}}\right)\ \Gamma\left[\frac{1}{3}\right] = \frac{2}{\pi^{\frac{1}{3}}}\ \Gamma\left[\frac{4}{3}\right] \simeq 1.2194 \\
n = 4: \int_{-\infty}^{+\infty} e^{-\pi|x|^4}\ dx &= \frac{2}{4}\ \frac{1}{\pi^{\frac{1}{4}}}\ \Gamma\left[\frac{1}{4}\right] = \frac{2}{\pi^{\frac{1}{4}}}\ \Gamma\left[\frac{5}{4}\right] \simeq 1.3616
\end{aligned}$$

- $n = 2$ confirms that area of "normal" Gaussian function is unity

- Area increases with $n$

- Limiting value of the area as $n \to +\infty$ :

$$\lim_{n\to\infty}\left\{\int_{-\infty}^{+\infty} e^{-\pi|x|^n} dx\right\} = \lim_{n\to\infty}\left\{\frac{2}{\pi^{\frac{1}{n}}}\ \frac{1}{n}\ \Gamma\left[\frac{1}{n}\right]\right\} = \left(\frac{2}{\pi^0}\right) 1 = 2$$

## 1.16 Bessel Functions $J_n[x]$

- Linear-phase sinusoid are solutions of second-order linear differential equation
- Other useful functions are solutions of other 1-D linear differential equations:

$$x^2 \frac{d^2}{dx^2}(Z_\nu[x]) + x\frac{d}{dx}(Z_\nu[x]) + \left(x^2 - \nu^2\right) Z_\nu[x] = 0,\ x \geq 0,\ \nu \in \Re$$

- Solutions $Z_\nu[x]$ are *Bessel functions*
- Often appears in physical problems involving planar circular symmetry or cylindrical coordinates
  - descriptions of imaging systems constructed from optics with circular cross-sections
- Three independent types of solutions are recognized.
  1. The "Bessel functions of the first kind" with integer and half-integer order ($\nu = n$ or $\frac{n}{2}$ for $n = 0, 1, 2, 3, \cdots$)
     - most relevant in imaging
     - labeled $J_n[x]$
     - finite amplitude for positive $x$.
  2. "Bessel functions of second kind"
     - also called "Neumann functions"
     - denoted by $N_\nu[x]$
     - indeterminate amplitude at $x = 0$.
  3. "Hankel function"
     - (not to be confused with the "Hankel transform" that will be discussed later)
     - complex-valued linear combination of first two types using scheme analogous to Euler relation:
     - $H_\nu[x] = J_\nu[x] \pm i\ N_\nu[x]$
- Numerical values for the Bessel functions via:
  - "generating function"
  - contour integrals
  - series solution of differential equation for integer indices ($\nu = n$).

### 1.16.1 Series Solution:

1. Assume that $J_n[x]$ has form of power series in $x$ with unknown coefficients:

$$J_n[x] = \sum_{\ell=0}^{+\infty} a_\ell\ x^\ell$$

2. Insert series into differential equation
3. Evaluate derivatives
4. Equate terms of same power of $x$
5. Juggle terms

6. Gives two series solutions for $J_n[x]$, one each for $n > 0$ and $n < 0$
7. Most interested in $J_0[x]$ and $J_1[x]$ $\Longrightarrow$series for $n > 0$ considered here

$$J_0[x] = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$
$$= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{147,456} + \cdots$$

- Compare coefficients to those of $SINC[x]$

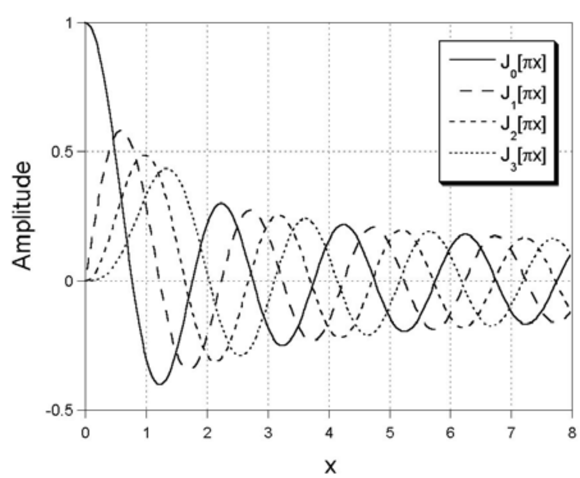
$$\begin{aligned} SINC[x] &= 1 - \frac{(\pi x)^2}{3!} + \frac{(\pi x)^4}{5!} - \frac{(\pi x)^6}{7!} + \cdots \\ &= 1 - \frac{x^2}{\left(\frac{6}{\pi^2}\right)} + \frac{x^4}{\left(\frac{120}{\pi^4}\right)} - \frac{x^6}{\left(\frac{5040}{\pi^6}\right)} + \cdots \\ &= 1 - \frac{x^2}{0.6079} + \frac{x^4}{1.2319} - \frac{x^6}{5.242} + \cdots \end{aligned}$$

  - Coefficients of same order have same algebraic sign
  - Absolute values of coefficients of $J_0[x]$ decrease more quickly with order than $SINC[x]$
  - Extrema of local amplitude of $J_0[x]$ fall off more slowly.

- Amplitude and slope of $J_0$ at $x = 0$ are unity and zero, respectively
- resembles cosine oscillation modulated by decaying function that happens to be $x^{-\frac{1}{2}}$ instead of $x^{-1}$ in $SINC[x]$.

$$\begin{aligned} \lim_{x \to +\infty} \{J_0[x]\} &= \sqrt{\frac{2}{\pi x}} \cos\left[x - \frac{\pi}{4}\right] \\ &= x^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} \cos\left[2\pi \left(\frac{1}{2\pi}\right) x - \frac{\pi}{4}\right] \end{aligned}$$

- Period of asymptotic form is $X_0 = 2\pi$



The Bessel function of the first kind $J_n[\pi x]$ for $n = 0 - 3$.

- zeros are only approximately uniformly spaced

- three zeros nearest to origin occur at:

| Location of zero of $J_0[x]$ | Difference $\Delta x$ |
|---|---|
| $x_1 \simeq 2.4048 \simeq 0.7655\ \pi$ | |
| | $\} x_2 - x_1 \simeq 0.9916\ \pi$ |
| $x_2 \simeq 5.5201 \simeq 1.7571\ \pi$ | |
| | $\} x_3 - x_2 \simeq 0.9974\ \pi$ |
| $x_3 \simeq 8.6537 \simeq 2.7546\ \pi$ | |
| $\vdots$ | $\vdots$ |
| $x_{M-1}$ | |
| | $\} x_M - x_{M-1} \lesssim 1.0\ \pi$ |
| $x_M$ | |

- Intervals between first and second pair of adjacent zeros are approximately $0.9916\pi$ and $0.9974\pi$, respectively
- Interval between $x_2$ and $x_3$ is larger than that between $x_1$ and $x_2$.
- Incremental distance between successive pairs of zeros asymptotically approaches $\pi$ as $x \to +\infty$.

**Series Solution for $J_1[x]$**

- $$\begin{aligned} J_1[x] &= \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots \\ &= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18,432} + \cdots = \frac{x}{2}\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9,216} + \cdots\right) \end{aligned}$$

- Amplitude and slope at origin are equal to coefficients $\alpha_0 = 0$ and $\alpha_1 = \frac{1}{2}$ respectively
- As $x$ increases from zero, amplitude increases from zero and reaches local maximum at $x \simeq 1.8412 \simeq 0.586\pi$.
- Absolute values of the numerical coefficients of $J_1[x]$ decrease more slowly than those of $SINC[x]$
  - absolute values of successive local maxima of $J_1[x]$ decrease more slowly than those of $SINC[x]$ as $x \to \infty$
  - same behavior exhibited by $J_0[x]$.
- First three zeros of $J_1[x]$:

| Location of zero of $J_1[x]$ | Difference $\Delta x$ |
|---|---|
| $x_1 \simeq 3.8317 \simeq 1.219\ \pi$ | |
| | $\} x_2 - x_1 \simeq 1.0135\ \pi$ |
| $x_2 \simeq 7.0156 \simeq 2.2331\ \pi$ | |
| | $\} x_3 - x_2 \simeq 1.0052\ \pi$ |
| $x_3 \simeq 10.1735 \simeq 3.2383\ \pi$ | |
| $\vdots$ | $\vdots$ |
| $x_{M-1}$ | |
| | $\} x_M - x_{M-1} \gtrsim 1.0\ \pi$ |
| $x_M$ | |

- Intervals between first pairs of zeros are approximately $1.0135\ \pi$ and $1.0052\ \pi$
- Interval between adjacent zeros of $J_1[x]$ *decreases* for increasing $x$.
  - Complementary to $J_0[x]$
- Interval between zeros for both $J_0[x]$ and $J_1[x]$ asymptotically approach $\pi$ as $|x| \to \infty$, though from different directions.
- Asymptotic behavior of $J_1[x]$ is *in quadrature* to $J_0[x]$
  - phase difference of oscillations of two functions is $-\frac{\pi}{2}$ radians:

$$\begin{aligned}
\lim_{x\to+\infty} \{J_1[x]\} &= \sqrt{\frac{2}{\pi x}}\ \cos\left[x - \frac{3\pi}{4}\right] \\
&= \sqrt{\frac{2}{\pi x}}\ \cos\left[\left(x - \frac{\pi}{4}\right) - \frac{\pi}{2}\right] = \sqrt{\frac{2}{\pi x}}\ \sin\left[x - \frac{\pi}{4}\right]
\end{aligned}$$

### 1.16.2 General Series Solution for $J_n[x]$:

$$J_n[x] = \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{\ell!\,(n+\ell)!}\left(\frac{x}{2}\right)^{n+2\ell}$$

$$\begin{aligned}
\lim_{x\to+\infty} \{J_2[x]\} &= \sqrt{\frac{2}{\pi x}}\ \cos\left[x - \frac{5\pi}{4}\right] \\
\lim_{x\to+\infty} \{J_3[x]\} &= \sqrt{\frac{2}{\pi x}}\ \cos\left[x - \frac{7\pi}{4}\right]
\end{aligned}$$

## 1.17 Lorentzian Function

- Named after Dutch physicist Hendrik Lorentz
  - demonstrated significance of curve in study of atomic radiation in early 1900s
- Lorentzian curve is theoretical shape of spectral lines created by atomic absorption or emission
- Our definition:

$$LOR[x] = \frac{2}{1+(2\pi x)^2} = \frac{2}{1+4\pi^2 x^2}$$

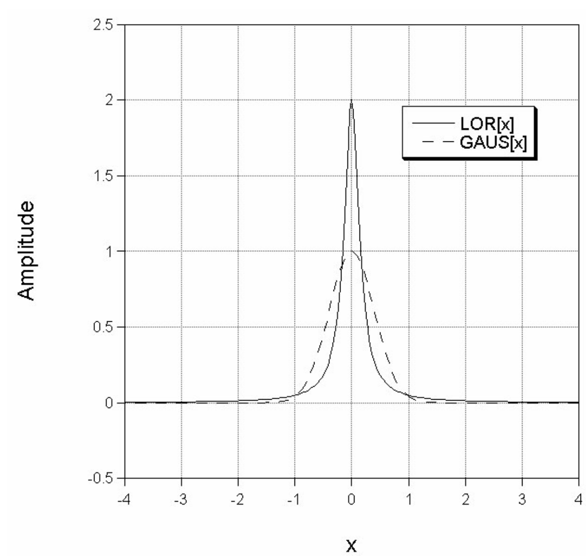
- Amplitude proportional to reciprocal of sum of unit constant and quadratic function of coordinate.
- Quadratic dependence on $x$ ensures that $LOR[x]$ is even.
- Multiplicative factor of 2 ensures that $LOR[x]$ has unit area:

$$Set\ u \equiv 2\pi x,\ dx = \frac{1}{2\pi}du$$

$$\int_{-\infty}^{+\infty} \frac{2}{1+(2\pi x)^2}dx = \int_{-\infty}^{+\infty} \frac{2}{1+u^2}\frac{du}{2\pi} = \frac{1}{\pi}\tan^{-1}[u]\Big|_{u=-\infty}^{u=+\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

- Amplitude decays from $LOR[0] = 2$ at origin through $LOR[\pm 1] \simeq 0.0494$ and on to zero at $x = \pm\infty$.
- Note similarity in amplitude of Lorentzian and Gaussian evaluated at $x = 1$
  - $GAUS[1] = e^{-\pi} \simeq 0.0432$
  - $LOR[\pm 1] \simeq 0.0494$
- $LOR[x]$ decays to zero more rapidly than $GAUS[x]$ for $|x| < 1$, and more slowly for $|x| > 1$

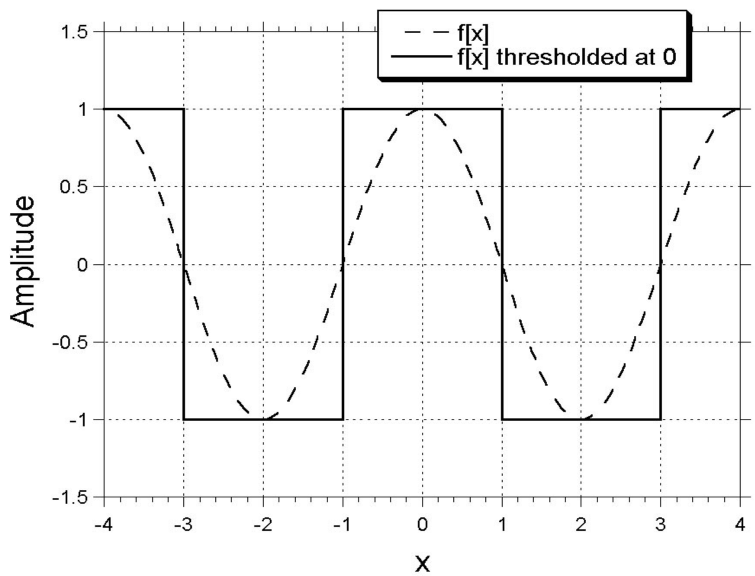



Comparison of the Lorentzian and Gaussian functions. Note that $LOR[x]$ "decays" more quickly than the Gaussian for larger values of $|x|$.

## 1.18 Thresholded Functions

- Square Wave

$$SGN\left[b_0+\cos\left[2\pi\xi_0 x\right]\right]=\begin{cases}+1 & \text{for} \quad b_0+\cos\left[2\pi\xi_0 x\right]>0\\ 0 & \text{for} \quad b_0+\cos\left[2\pi\xi_0 x\right]=0\\ -1 & \text{for} \quad b_0+\cos\left[2\pi\xi_0 x\right]<0\end{cases}$$




"Thresholding" of a sinusoidal function produces a "square wave".

- Thresholding process interpreted as action of nonlinear imaging system which generates discrete three-state output from continuous input
- Form of lookup table.
- Allowed output amplitudes of 0, 1/2, and 1 by addition and multiplication of the output in fashion analogous to that used to apply the *STEP* function in place of the *SIGNUM* function.

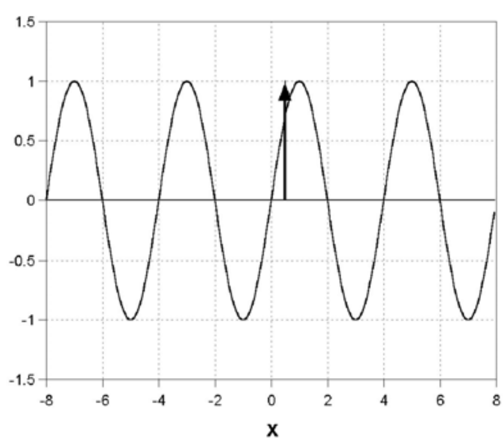
$$\frac{1}{2}\left(1+SGN\left[b_0+\cos\left[2\pi\xi_0 x\right]\right]\right)=\begin{cases}1 & \text{for} \quad b_0+\cos\left[2\pi\xi_0 x\right]>0\\ \frac{1}{2} & \text{for} \quad b_0+\cos\left[2\pi\xi_0 x\right]=0\\ 0 & \text{for} \quad b_0+\cos\left[2\pi\xi_0 x\right]<0\end{cases}$$

- Thresholding process applied to quadratic phase yields 1-D "zone plate"

$$f[x]=\frac{1}{2}\left(1+SGN\left[\cos\left[\frac{\pi x^2}{\alpha^2}+\phi_0\right]\right]\right)=STEP\left[\cos\left[\frac{\pi x^2}{\alpha^2}+\phi_0\right]\right]$$

## 1.19 1-D Dirac Delta Function

- Commonly called *impulse* function
- Name honors physicist P.A.M. Dirac, who introduced notation in quantum mechanics
- Dirac delta function does not have "proper" definition that assigns a specific finite amplitude to each coordinate $x$.
- Strictly $\delta[x]$ is not a function at all
  - notation of $\delta[x]$ is meaningful only within integrand of integral
  - More properly called *delta distribution*.
- Properties of Dirac delta function make an essential tool for solving problems involving many types of physical systems
  - classical mechanics
  - quantum mechanics
  - electrodynamics
- Conditions are:
  1. infinitesimal support
  $$\delta[x - x_0] = 0 \text{ for } x \neq x_0$$
  2. unit area.
  $$\int_{-\infty}^{+\infty} \delta[x - x_0]\, dx = 1$$
- Cannot be depicted in conventional graphical way
- "Image" of characteristics conveyed by arrow (or "spike") located at $x_0$ with height of the arrowhead's tip above the $x$-axis equal to area
- Base of the arrow *always* rests on $x$-axis, even in those cases where $\delta[x - x_0]$ has been added to "proper" finite-amplitude function $f[x]$




Graphical depiction of $f[x] = \delta[x] + COS\left[2\pi\frac{x}{4} - \frac{\pi}{2}\right]$. Because the "height" of the tip of the arrow represents the area of the Dirac delta function, its "base" always lies on the x-axis.

- Fairly common error misconstrues Dirac delta function and a finite discontinuous function $f_0[x]$:

$$\text{(a)}\quad f_0[x] = 0 \text{ for } x \neq 0$$
$$\text{(b)}\quad f_0[x] = 1 \text{ for } x = 0$$

  - (probably results from early exposure to "discrete Dirac delta function" – sampled approximation.

$$\int_{x_0-\frac{1}{2}}^{x_0+\frac{1}{2}} \left(f[x-x_0] + 1[x]\right)\, dx = 1$$
$$\int_{x_0-\frac{1}{2}}^{x_0+\frac{1}{2}} \left(\delta[x-x_0] + 1[x]\right)\, dx = 2$$

- Details of the functional form of continuous $\delta[x]$ do not matter in most situations
- $\delta[x]$ often defined as limit of any of several sequences of functions, differ in some details
- Requirement of unit area is satisfied by several of special functions already defined.
- Scaled rectangle:

$$\int_{-\infty}^{+\infty} \frac{1}{|b_0|} RECT\left[\frac{x}{b_0}\right]\, dx = 1$$

- In limit $b_0 \to 0$

$$\lim_{b_0\to 0}\left\{\frac{1}{|b_0|} RECT\left[\frac{x}{b_0}\right]\right\} = 0 \text{ for } x \neq 0$$

- Symmetry of $RECT\left[\frac{x}{b_0}\right]$ implies that Dirac delta function is symmetric:

$$\delta[x] = \lim_{b_0\to 0}\left\{\frac{1}{|b_0|} RECT\left[\frac{x}{b_0}\right]\right\} = \lim_{b_0\to 0}\left\{\frac{1}{|-b_0|} RECT\left[\frac{x}{-b_0}\right]\right\} = \delta[-x]$$

- Leads to observation that order of arguments of shifted Dirac delta function is immaterial:

$$\delta[x-x_0] = \delta[-(x-x_0)] = \delta[x_0-x]$$

- Other function sequences that converge to Dirac delta function (based on unit-area special functions)

$$\begin{aligned}
\delta[x] &= \lim_{b_0\to 0}\left\{\frac{1}{|b_0|} TRI\left[\frac{x}{b_0}\right]\right\} \\
&= \lim_{b_0\to 0}\left\{\frac{1}{|b_0|} SINC\left[\frac{x}{b_0}\right]\right\} \\
&= \lim_{b_0\to 0}\left\{\frac{1}{|b_0|} SINC^2\left[\frac{x}{b_0}\right]\right\} \\
&= \lim_{b_0\to 0}\left\{\frac{1}{|b_0|} GAUS\left[\frac{x}{b_0}\right]\right\} \\
&= \lim_{b_0\to 0}\left\{\frac{1}{|b_0|} LOR\left[\frac{x}{b_0}\right]\right\}
\end{aligned}$$

$$f[x] = \lim_{b_0\to 0}\left\{\frac{1}{|b_0|} SINC\left[\frac{x}{b_0}\right]\right\} = \lim_{b_0\to 0}\left\{\frac{1}{|b_0|}\left(\frac{\sin\left[\frac{\pi x}{b_0}\right]}{\left[\frac{\pi x}{b_0}\right]}\right)\right\} = \lim_{b_0\to 0}\left\{\frac{\sin\left[\frac{\pi x}{b_0}\right]}{\pi x}\right\}$$

– Scale factor $b_0$ may be expressed in terms of reciprocal $N = \frac{1}{b_0}$,

$$f[x] = \lim_{N\to\infty}\left\{\frac{\sin[N\pi x]}{\pi x}\right\} = \lim_{N\to\infty}\left\{\frac{\sin\left[2\pi\left(\frac{N}{2}\right)x\right]}{\pi x}\right\}$$

– Apply L'Hôspital's Rule to show that amplitude at origin is undefined:

$$f[x] = \frac{\lim\limits_{N\to\infty}\{N\pi\}}{\lim\limits_{N\to\infty}\{\pi\}} = \lim_{N\to\infty}\{N\} = \infty$$

- Limiting behavior of Lorentzian:

$$\delta[x] = \lim_{b_0\to 0}\left\{\frac{1}{|b_0|}LOR\left[\frac{x}{b_0}\right]\right\} = \lim_{b_0\to 0}\left\{\frac{1}{|b_0|}\frac{2}{1+\left(2\pi\left(\frac{x}{b_0}\right)\right)^2}\right\}$$

$$= \lim_{b_0\to 0}\left\{\frac{\left(\frac{b_0}{2\pi}\right)\frac{1}{\pi}}{\left(\frac{b_0}{2\pi}\right)^2+x^2}\right\} = \lim_{\epsilon\to 0}\left\{\frac{\frac{\epsilon}{\pi}}{\epsilon^2+x^2}\right\} \text{ where } \epsilon \equiv \frac{b_0}{2\pi}.$$

### 1.19.1 Area of $\delta[x]$ Over Indefinite Limit

$$\int_{-\infty}^{x}\delta[\alpha]\;d\alpha = \left\{\begin{array}{ccc} 0 & \text{for} & x<0 \\ 1 & \text{for} & x>0 \end{array}\right\}$$

- Apply Cauchy Principal Value at origin:

$$\int_{-\infty}^{x}\delta[\alpha]\;d\alpha = \left\{\begin{array}{ccc} 1 & \text{for} & x>0 \\ \frac{1}{2} & \text{for} & x=0 \\ 0 & \text{for} & x<0 \end{array}\right\} = STEP[x]$$

- Differentiate both sides and apply fundamental theorem of calculus:

$$\frac{d}{dx}STEP[x] = \frac{d}{dx}\int_{-\infty}^{x}\delta[\alpha]\,d\alpha = \delta[x]-\delta[-\infty]$$
$$= \delta[x]-0 = \delta[x]$$

- Means that derivative of step function is Dirac delta function :.

$$\frac{d}{dx}STEP[x-x_0] = \delta[x-x_0]$$

- Very (most?) important representation is obtained by summing complex sinusoids with unit amplitudes and zero phase over all spatial frequencies:

$$\int_{-\infty}^{+\infty}e^{+2\pi i\xi x}d\xi = \int_{-\infty}^{+\infty}\cos[2\pi\xi x]\;d\xi + i\int_{-\infty}^{+\infty}\sin[2\pi\xi x]\;d\xi$$

– Demonstrate by evaluating integral over arbitrary finite and symmetric limits:

$$\int_{-b_0}^{+b_0}e^{+2\pi i\xi x}d\xi = \frac{1}{2\pi i x}\left(e^{+2\pi\xi x}\right)\Big|_{\xi=-b_0}^{\xi=b_0} = \frac{1}{\pi x}\frac{\left(e^{+2\pi i x b_0}-e^{-2\pi i x b_0}\right)}{2i}$$
$$= \frac{1}{\pi x}\;\sin[2\pi b_0 x] = \frac{2b_0}{2\pi b_0 x}\;\sin[2\pi b_0 x] = 2b_0\;SINC[2b_0 x]$$

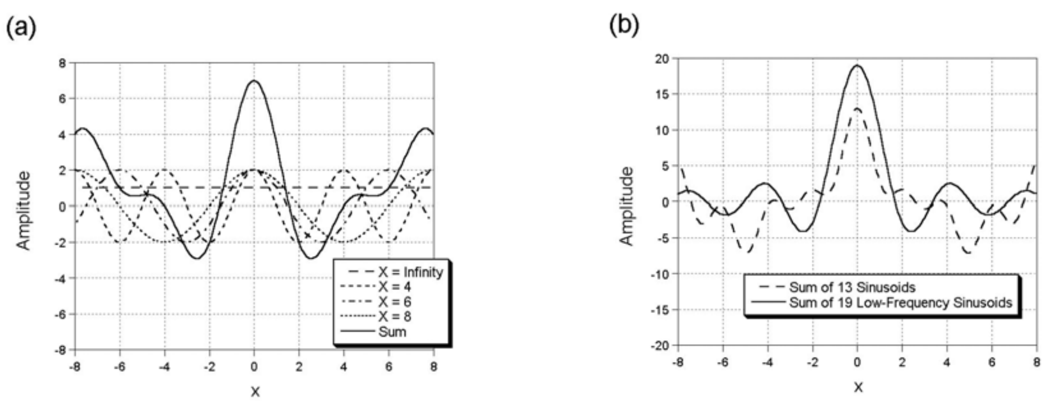
- In limit $b_0 \to +\infty$, this is a valid representation of Dirac delta function
- Integral of original complex Hermitian function over symmetric limits yields real-valued result due to cancellation of areas in the antisymmetric imaginary part for positive and negative $x$.

$$\delta[-x] = \int_{-\infty}^{+\infty} e^{-2\pi i\xi[-x]} d\xi = \int_{-\infty}^{+\infty} e^{+2\pi i\xi x} d\xi = \delta[x]$$

- Integral form of $\delta[x]$ used to derive equivalent expression for Dirac delta function scaled by "width parameter" $b_0$.

$$\begin{aligned}\delta\left[\frac{x}{b_0}\right] &= \int_{-\infty}^{+\infty} e^{-2\pi i\xi\left(\frac{x}{b_0}\right)} d\xi = \int_{-\infty}^{+\infty} e^{-2\pi i\left(\frac{\xi}{b_0}\right)x} d\xi \\ &= \int_{-\infty}^{+\infty} e^{-2\pi i\alpha x}\ |b_0|\ d\alpha; (\text{for } \xi \equiv \alpha b_0 \implies d\xi = |b_0|\ d\alpha) \\ &= |b_0| \int_{-\infty}^{+\infty} e^{-2\pi i\alpha x} d\alpha = |b_0|\ \delta[x]\end{aligned}$$

- "scaling property" of 1-D Dirac delta function
- Scaling "width" and scaling "amplitude" of $\delta[x]$ by factor $b_0$ are equivalent operations.




Approximations of $\delta[x]$ obtained by summing sinusoidal functions with small spatial frequencies.

## 1.20 Sifting Property of the Dirac Delta Function

- Most significant property of Dirac delta function is ability to evaluate amplitude of another function at any coordinate
- Responsible for most important applications.
- Some authors use mathematical statement of sifting property as definition of Dirac delta function.
- Infinitesimal support of $\delta[x]$ allows area of product of $\delta[x]$ and $f[x]$ to be evaluated by method similar to approximation of integrals as summations of rectangular areas.

$$\int_{-\infty}^{+\infty} f[x]\ \delta[x]\ dx = \lim_{b_0 \to 0} \left\{ \frac{1}{|b_0|} \int_{-\infty}^{+\infty} RECT\left[\frac{x}{b_0}\right]\ f[x]\ dx \right\}$$

- Area of limiting rectangle is width $b_0$ multiplied by amplitude evaluated at origin:

$$\int_{-\infty}^{+\infty} f[x]\ \delta[x]\ dx = \lim_{b_0 \to 0} \left\{ \frac{1}{|b_0|} (|b_0|\ f[0]) \right\} = f[0]$$

- Integral of product of $f[x]$ and $\delta[x]$ has "sifted" out specific amplitude $f[0]$ from $f[x]$
- Reason for name *sifting* property
- Straightforward to translate Dirac delta function by adding a term of $-x_0$ to argument of $RECT$ function;
- Analogous integral "sifts" out amplitude of $f[x]$ at $x_0$:

$$\int_{-\infty}^{+\infty} f[x]\ \delta[x - x_0]\ dx = \int_{-\infty}^{+\infty} f[x]\ \delta[x_0 - x]\ dx = f[x_0]$$

- Sifting property may be used to derive yet another useful result for well-behaved $f[x]$:
  - so-called "property of Dirac delta function in products"

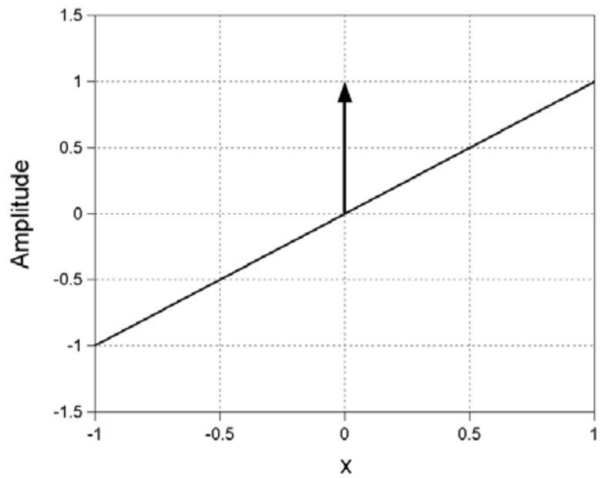
$$\int_{-\infty}^{+\infty} f[x]\ \delta[x - x_0]\ dx = f[x_0] = f[x_0] \times 1$$

$$= f[x_0] \int_{-\infty}^{+\infty} \delta[x - x_0]\ dx = \int_{-\infty}^{+\infty} f[x_0]\ \delta[x - x_0]\ dx$$

$$\boxed{f[x]\ \delta[x - x_0] = f[x_0]\ \delta[x - x_0]}$$

- Substitution of $f[x] = x$ and $x_0 = 0$ produces the special case:

$$x\ \delta[x] = 0\ \delta[x] = 0$$



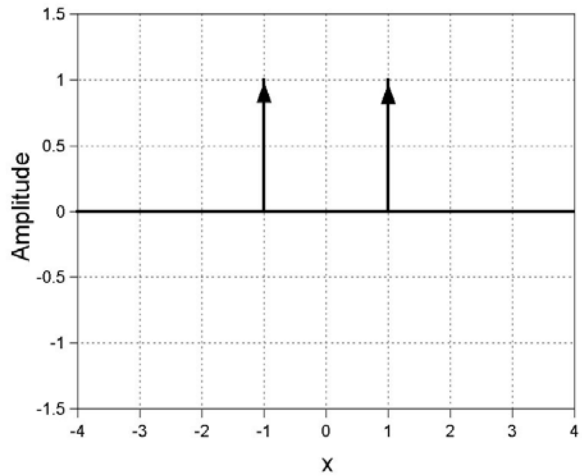

Demonstration of sifting property of $\delta[x]$ applied to $f[x] = x$.

## 1.21 Relatives of Dirac Delta Function

Three additional special functions based on Dirac delta function

1. even pair of Dirac delta functions

   - Unit-amplitude Dirac delta functions located at $x = \pm 1$
   - Notation: $\delta\delta[x] \equiv \delta[x+1] + \delta[x-1]$

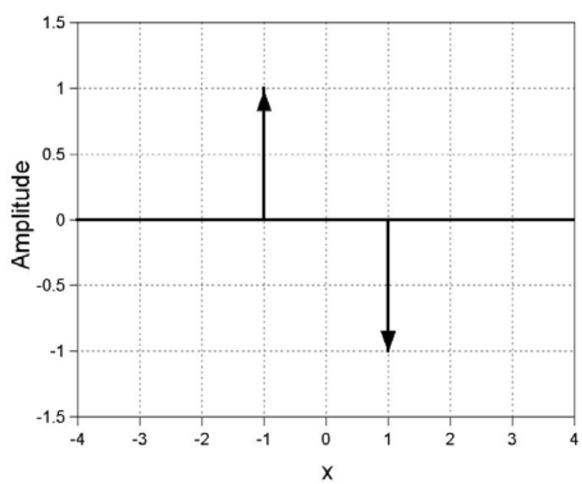



Even pair of Dirac delta functions: $f[x] = \delta[x+1] + \delta[x-1]$.

   - $\int_{-\infty}^{+\infty} \delta\delta[x] \; dx = 2$
   - $\delta\delta\left[\frac{x}{b_0}\right] = \delta\left[\frac{x}{b_0} + 1\right] + \delta\left[\frac{x}{b_0} - 1\right] = \delta\left[\frac{x+b_0}{b_0}\right] + \delta\left[\frac{x-b_0}{b_0}\right] = |b_0| \;\; (\delta[x+b_0] + \delta[x-b_0])$

2. odd pair of Dirac delta functions

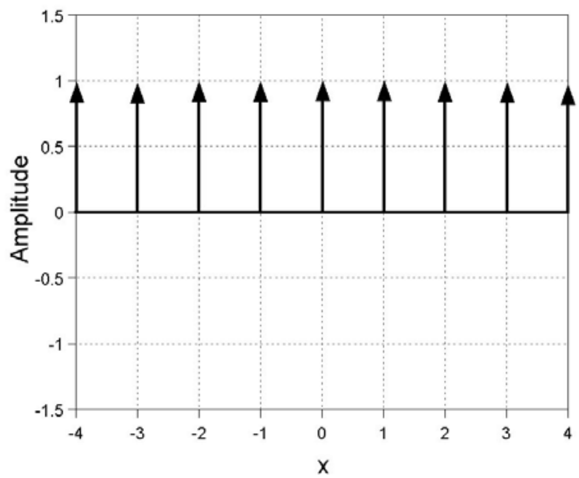
   - Unit-amplitude Dirac delta functions located at $x = \pm x$ with areas $\mp 1$
   - Notation:${}^{\delta}{}_{\delta}[x] = \delta[x+1] - \delta[x-1]$
   - ${}^{\delta}{}_{\delta}\left[\frac{x}{b_0}\right] = |b_0| \;\; (\delta[x+b_0] - \delta[x-b_0])$
   - $\frac{1}{|b_0|}{}^{\delta}{}_{\delta}\left[\frac{x}{b_0}\right] = \delta[x+b_0] - \delta[x-b_0]$




Odd pair of Dirac delta functions: $f[x] = \delta[x+1] - \delta[x-1]$ .

3. *COMB* function

- infinite set of uniformly spaced Dirac delta functions
- $COMb_0[x] \equiv \sum_{n=-\infty}^{+\infty} \delta[x-n]$
- $\int_{-\infty}^{+\infty} COMb_0[x]\ dx = \infty$
- $\sum_{n=-\infty}^{+\infty} \delta[x-x_0-n] = COMb_0[x-x_0]$




$$f[x] = COMB[x] \equiv \sum_{n=-\infty}^{+\infty} \delta[x-n]$$

$$\begin{aligned} COMb_0\left[\frac{x}{b_0}\right] &= \sum_{n=-\infty}^{+\infty} \delta\left[\frac{x}{b_0}-n\right] = \sum_{n=-\infty}^{+\infty} \delta\left[\frac{x-nb_0}{b_0}\right] \\ &= \sum_{n=-\infty}^{+\infty} |b_0|\ \delta[x-nb_0] = |b_0| \sum_{n=-\infty}^{+\infty} \delta[x-nb_0] \end{aligned}$$

$$\frac{1}{|b_0|}\ COMb_0\left[\frac{x}{b_0}\right] = \sum_{n=-\infty}^{+\infty} \delta[x-nb_0]$$

- Use *COMB* to "sample" a continuous function

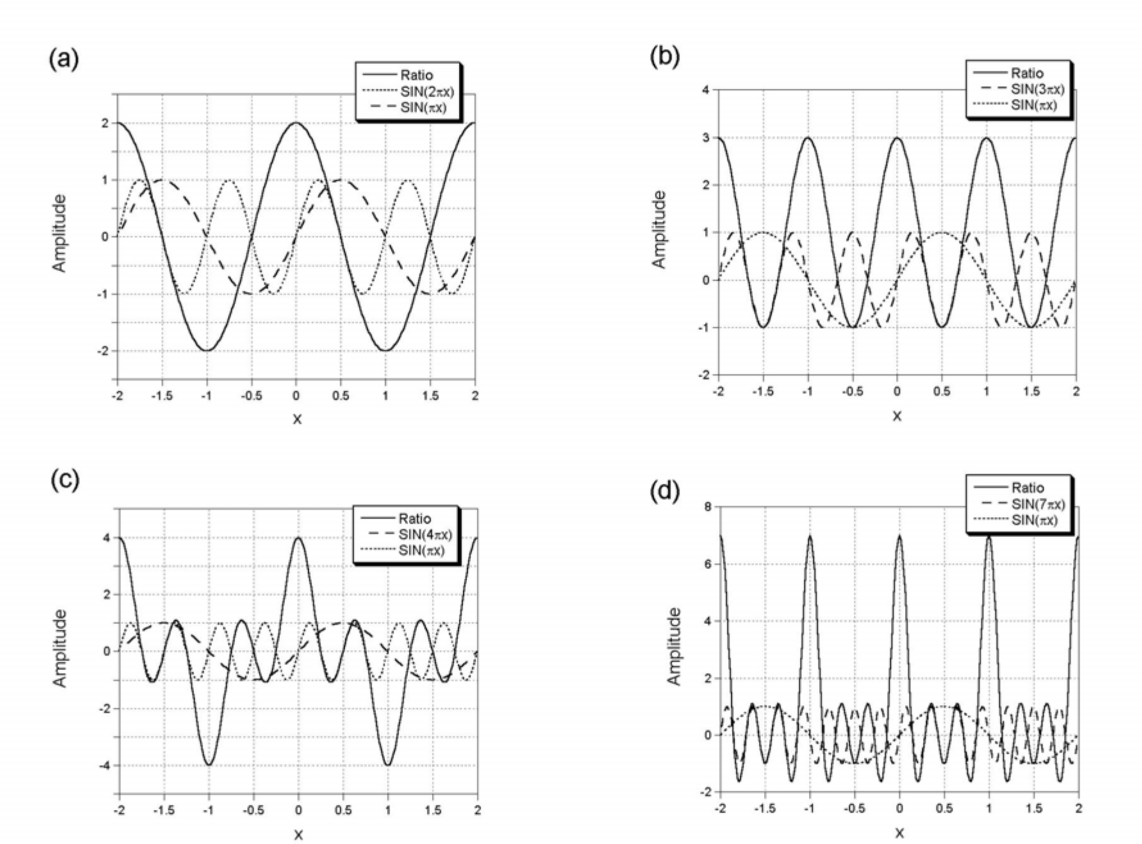
$$\begin{aligned} \cos[2\pi\xi_0 x+\phi_0]\ \left(\frac{1}{\Delta x}\ COMb_0\left[\frac{x}{\Delta x}\right]\right) &= \cos[2\pi\xi_0 x+\phi_0]\left(\sum_{n=-\infty}^{+\infty} \delta[x-n\ \Delta x]\right) \\ &= \sum_{n\ =\ -\infty}^{+\infty} \cos[2\pi\xi_0 x+\phi_0]\ \ \delta[x-n\ \Delta x] \\ &= \sum_{n\ =\ -\infty}^{+\infty} \cos[2\pi\xi_0(n\ \Delta x)+\phi_0]\ \ \delta[x-n\ \Delta x] \end{aligned}$$

- *COMB* function expressed as ratio of functions:

$$\delta\left[x\right]=\lim_{N\to\infty}\left\{\frac{\sin\left[N\pi x\right]}{\pi x}\right\}=\lim_{N\to\infty}\left\{\frac{\sin\left[2\pi\left(\frac{N}{2}\right)x\right]}{\pi x}\right\}$$

- Ratio of two terms that have null amplitude and unit slope at all integer values of $x$.

$$COMb_0\left[x\right]=\lim_{odd\ N\to\infty}\left\{\frac{\sin\left[N\pi x\right]}{\sin\left[\pi x\right]}\right\}$$




Approximations for $COMB\left[x\right]$ as ratios of sine waves: (a) $\frac{\sin[2\pi x]}{\sin[\pi x]}=2\ \cos\left[\pi x\right]$; (b) $\frac{\sin[3\pi x]}{\sin[\pi x]}=2\ \cos\left[2\pi x\right]+1$; (c) $\frac{\sin[4\pi x]}{\sin[\pi x]}$; (d) $\frac{\sin[7\pi x]}{\sin[\pi x]}$. Note that the numerator must be an odd multiple of $\pi x$ to obtain an approximation of $COMB\left(x\right)$.

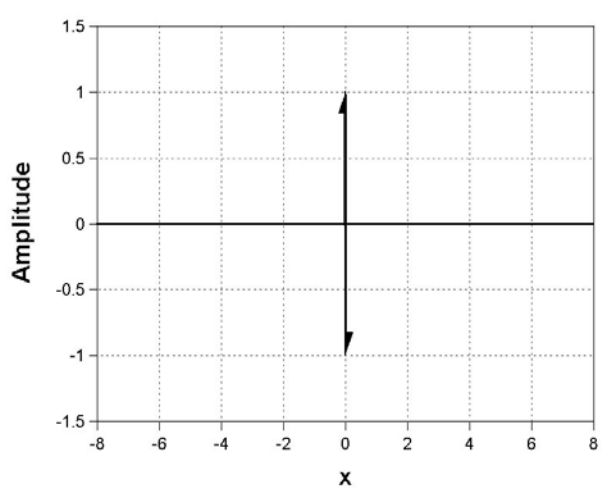
## 1.22 Derivatives of the Dirac Delta Function

- $$\begin{aligned}\frac{d\delta[x]}{dx} &\equiv \delta^{(1)}[x] = \lim_{b_0\to 0}\left\{\frac{d}{dx}\left(\frac{1}{|b_0|}RECT\left[\frac{x}{b_0}\right]\right)\right\}\\ &= \lim_{b_0\to 0}\left\{\frac{1}{|b_0|}\frac{d}{dx}\left(STEP\left[x+\frac{b_0}{2}\right]-STEP\left[x-\frac{b_0}{2}\right]\right)\right\}\\ &= \lim_{b_0\to 0}\left\{\frac{1}{|b_0|}\left(\delta\left[x+\frac{b_0}{2}\right]-\delta\left[x-\frac{b_0}{2}\right]\right)\right\}\\ &= \lim_{b_0\to 0}\left\{\frac{1}{|b_0|}\left(\delta\left[\frac{\left(\frac{2x}{b_0}+1\right)}{\left(\frac{2}{b_0}\right)}\right]-\delta\left[\frac{\left(\frac{2x}{b_0}-1\right)}{\left(\frac{2}{b_0}\right)}\right]\right)\right\}\\ &= \lim_{b_0\to 0}\left\{\frac{1}{|b_0|}\frac{2}{|b_0|}\delta_\delta\left[\frac{2x}{b_0}\right]\right\}\\ &= \lim_{b_0\to 0}\left\{\frac{2}{|b_0|^2}\delta_\delta\left[\frac{x}{\left(\frac{b_0}{2}\right)}\right]\right\}\end{aligned}$$
- $\delta^{(1)}[x]$ is odd because $\delta[x]$ is even
- Integral of $\delta^{(1)}[x]$ over symmetric limits must be zero.
- Differentiate $f[x]$ within sifting property:

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}\left(\int_{-\infty}^{+\infty} f[\alpha]\;\delta[x-\alpha]\;d\alpha\right) = \int_{-\infty}^{+\infty} f[\alpha]\;\frac{d}{dx}(\delta[x-\alpha])\;d\alpha\\ &= \int_{-\infty}^{+\infty} f[\alpha]\;\delta^{(1)}[x-\alpha]\;d\alpha\end{aligned}$$

- Generalize to $n^{th}$ order

$$\int_{-\infty}^{+\infty} f[\alpha]\;\frac{d^n}{dx^n}(\delta[x-\alpha])\;dx = \int_{-\infty}^{+\infty} f[x]\;\left(\delta^{(n)}[x-\alpha]\right)\;dx = \frac{d^n f}{dx^n} = f^{(n)}[x]$$




$\delta'[x]$ represented as "doublet" of Dirac delta functions.

## 1.23 Dirac Delta Function with Functional Argument: $\delta[g[x]]$

- $\delta\left[\frac{x}{b_0}\right]$ , where $g[x] = \frac{x}{b_0}$
- Functional form may be evaluated when $g[x]$ satisfies certain conditions.
- Expression appears frequently in some imaging applications
  - (e.g., computed tomography to derive inverse Radon transform)
- First criterion for Dirac delta function $\Longrightarrow \delta[g[x]] = 0$ wherever $g[x] \neq 0$
  - Example: $g[x] = 2 + \cos(2\pi x) \Longrightarrow \delta[g[x]] = 0[x]$ because $g[x] \neq 0$ .
- Second criterion of unit area considered in case where $g[x]$ has a single zero located at $x_0$ $\Longrightarrow g[x_0] = 0$
  - If derivatives exist and are finite in vicinity of $x_0$, then $g[x]$ may be expanded in Taylor series:

$$\begin{aligned} g[x] &= \sum_{n=0}^{+\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n g[x]}{dx^n}\right)\Bigg|_{x=x_0} (x-x_0)^n \\ &= g[x_0] + (x-x_0)\left(\frac{dg}{dx}\right)\Bigg|_{x=x_0} + \frac{(x-x_0)^2}{2}\left(\frac{d^2 g}{dx^2}\right)\Bigg|_{x=x_0} + \cdots \\ &= 0 + (x-x_0)\left(\frac{dg}{dx}\right)\Bigg|_{x=x_0} + \frac{(x-x_0)^2}{2}\left(\frac{d^2 g}{dx^2}\right)\Bigg|_{x=x_0} + \cdots, \textit{ because } g[x_0]=0 \end{aligned}$$

  - Amplitude of $\delta[g[x]]$ is zero except at $x = x_0 \Longrightarrow$ area of $\delta[g[x]]$ is concentrated in vicinity of $x_0$
  - Evaluate area by restricting domain of integral to $x_0 \pm \varepsilon$, where $\varepsilon$ is arbitrarily small positive real number:

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta[g[x]]\; dx &\simeq \int_{x_0-\epsilon}^{x_0+\epsilon} \delta[g[x]]\; dx \\ &= \int_{x_0-\epsilon}^{x_0+\epsilon} \delta\left[(x-x_0)\left(\frac{dg}{dx}\right)\Bigg|_{x=x_0} + \frac{(x-x_0)^2}{2}\left(\frac{d^2 g}{dx^2}\right)\Bigg|_{x=x_0} + \cdots\right] dx \end{aligned}$$

  - Restriction of domain of $x$ to neighborhood of $x_0$ ensures that only smallest-order term of Taylor series with nonzero amplitude need be retained
    - If first derivative of $g[x]$ is nonzero and finite at $x_0$, then $(x-x_0) >> (x-x_0)^n$ for $n > 1$
  - Simplify:

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta[g[x]]\; dx &\simeq \int_{x_0-\epsilon}^{x_0+\epsilon} \delta\left[(x-x_0)\left(\frac{dg}{dx}\right)\Bigg|_{x=x_0}\right] dx \\ &= \int_{x_0-\epsilon}^{x_0+\epsilon} \frac{\delta[x-x_0]}{\left(\left|\frac{dg}{dx}\right|\right)\Big|_{x=x_0}}\, dx \\ &= \left(\left|\frac{dg}{dx}\right|^{-1}\right)\Bigg|_{x=x_0} \int_{x_0-\epsilon}^{x_0+\epsilon} \delta[x-x_0]\; dx \\ &= \left(\left|\frac{dg}{dx}\right|^{-1}\right)\Bigg|_{x=x_0} \end{aligned}$$

- Requirements for Dirac delta function are satisfied if :

$$\delta[g[x]] = \frac{\delta[x-x_0]}{\left(\left|\frac{dg}{dx}\right|\right)\Big|_{x=x_0}}, \text{ where } g[x_0]=0 \text{ and } \left.\frac{dg}{dx}\right|_{x=x_0} \neq 0.$$

- If both $g[x_0]=0$ and $\left.\frac{dg}{dx}\right|_{x=x_0}=0$, but second derivative is nonzero and finite, then all terms in Taylor series of order three and higher are neglected:

$$\begin{aligned}\delta[g[x]] &= \delta\left[\frac{(x-x_0)^2}{2}\left(\frac{d^2g}{dx^2}\right)\Big|_{x=x_0}\right] \\ &= \frac{2\delta\left[(x-x_0)^2\right]}{\left|\left(\frac{d^2g}{dx^2}\right)\Big|_{x=x_0}\right|}\end{aligned}$$

- Now stuck; further simplification requires evaluation of $\delta\left[x^2\right]$, which has same qualities used to derive expression that $g[x_0]=g[0]=0$ and $\left(\frac{dg}{dx}\right)\Big|_{x_0=0}=0$.
- In more general case where $g[x]$ has $N$ zeros and derivative is nonzero and finite at each, evaluate series at each zero and sum:

$$\delta[g[x]] = \sum_{n=1}^{\infty} \frac{\delta[x-x_n]}{\left|\left(\left.\frac{dg}{dx}\right|_{x=x_n}\right)\right|} \text{ where } g[x_n]=0 \text{ and } \left.\frac{dg}{dx}\right|_{x=x_n} \neq 0$$

1. $g_1[x] = x^2$
   - one zero at $x=0$
   - derivative is zero at $x=0$
   - produces worthless result $\delta\left[x^2\right] = \delta\left[x^2\right]$.
2. $g_2[x] = x^2-1$
   - two zeros located at $x=\pm 1$
   - respective slopes are $\pm 2$:

$$\begin{aligned}\delta\left[x^2-1\right] &= \frac{\delta[x+1]}{|-2|} + \frac{\delta[x-1]}{|+2|} \\ &= \frac{1}{2}\left[\delta[x+1]+\delta[x-1]\right] \\ &= \frac{1}{2}\delta\delta[x]\end{aligned}$$

3. $g_3[x] = \sin[2\pi\xi_0 x]$
   - zeros at integer multiples of $\frac{1}{2\xi_0}$
   - slopes evaluated at zeros are $\pm 2\pi\xi_0$

- Equivalent expression is:

$$\delta\left[\sin\left[2\pi\xi_0 x\right]\right] = \frac{\cdots + \delta\left[x + \frac{2}{2\xi_0}\right] + \delta\left[x + \frac{1}{2\xi_0}\right] + \delta\left[x\right] + \delta\left[x - \frac{1}{2\xi_0}\right] + \delta\left[x - \frac{2}{2\xi_0}\right] + \cdots}{\left|\pm 2\pi\xi_0\right|}$$

$$= \frac{1}{2\pi\left|\xi_0\right|} \sum_{n=-\infty}^{+\infty} \delta\left[x - \frac{n}{2\xi_0}\right]$$

$$= \frac{1}{2\pi\xi_0} COMb_0\left[\frac{2\xi_0 x - n}{2\xi_0}\right]$$

$$= \frac{1}{\pi}\ COMb_0\left[2\xi_0 x\right]$$

- Useful equivalent expression for $COMb_0\left[x\right]$ in terms of $\sin\left(\pi x\right)$ by setting $\xi_0 = \frac{1}{2}$:

$$COMb_0\left[x\right] = \pi\ \delta\left[\sin\left[\pi x\right]\right] = \pi\ \delta\left[\sin\left[2\pi\frac{x}{2}\right]\right]$$

# 2 1-D Complex-Valued Special Functions

## 2.1 Sum of Weighted 1-D Functions

- Complex-valued 1-D function may be created by assigning examples of individual real-valued special functions as real and imaginary parts.

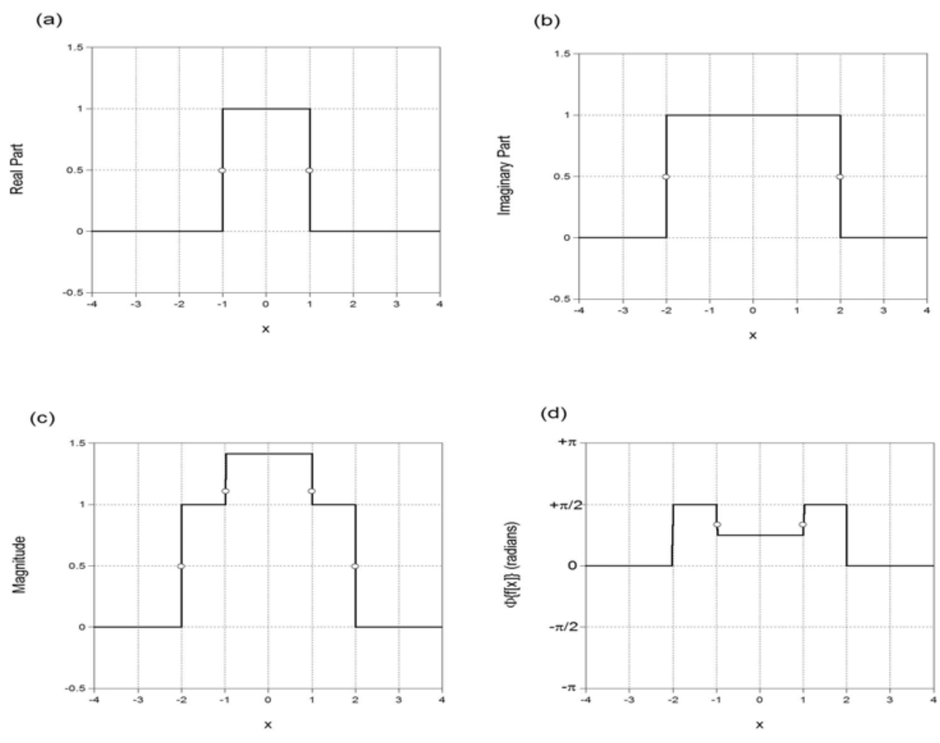
$$f[x] = f_R[x] + i\ f_I[x] = RECT[x] + i\ RECT\left[\frac{x}{2}\right]$$

  - Magnitude and phase via:

$$|f[x]| \equiv \sqrt{(f_R[x])^2 + (f_I[x])^2} = RECT\left[\frac{x}{2}\right] + \left(\frac{\sqrt{2}-1}{2}\right) RECT[x]$$

$$\Phi\{f[x]\} \equiv \tan^{-1}\left[\frac{f_I[x]}{f_R[x]}\right]$$

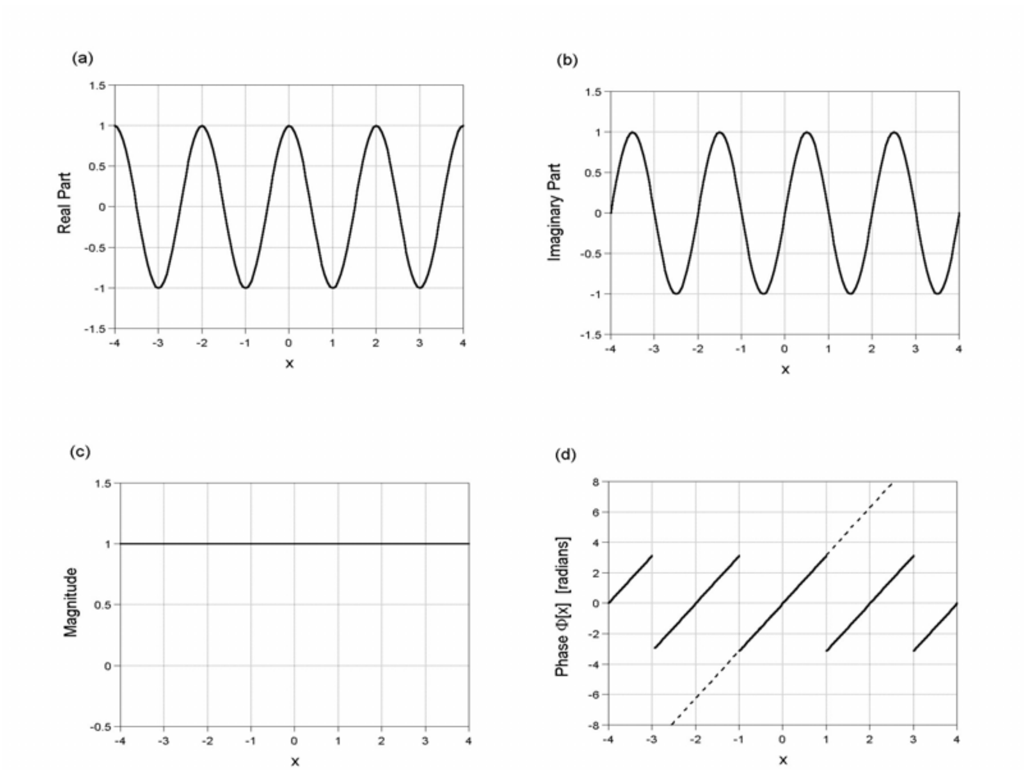
  - Particularly note algebraic sign of phase angles




Complex function constructed from real functions: $f[x] = RECT[x] + i\ RECT\left[\frac{x}{2}\right]$, (a) Real part; (b) imaginary part; (c) magnitude; and (d) phase.

## 2.2 Complex Linear-Phase Sinusoid

$$
\begin{aligned}
e^{\pm 2\pi i \xi_0 x} &= \cos[2\pi\xi_0 x] \pm i\ \sin[2\pi\xi_0 x] \\
&= \cos[2\pi\xi_0 x] \pm i\ \cos\left[2\pi\xi_0 x - \frac{\pi}{2}\right]
\end{aligned}
$$




Complex sinusoid $f[x] = e^{+2\pi\frac{x}{2}}$: (a) $\Re\left\{e^{+2\pi\frac{x}{2}}\right\} = \cos[\pi x]$, (b)$\Im\left\{e^{+2\pi\frac{x}{2}}\right\} = \sin[\pi x]$, (c) $\left|e^{+2\pi\frac{x}{2}}\right| = 1[x]$, and (d) phase with both range $[-\pi, +\pi)$ and $[-\infty, +\infty)$.

## 2.3 Complex Quadratic-Phase Exponential – Complex "Chirp"

$$e^{\pm i\pi\left(\frac{x}{\alpha}\right)^2} = \cos\left[\frac{\pi x^2}{\alpha^2}\right] \pm i\ \sin\left[\frac{\pi x^2}{\alpha^2}\right]$$

$$\xi[x] = \pm\frac{1}{2\pi}\frac{\partial\Phi}{\partial x} = \pm\frac{x}{\alpha^2}$$

- area is complex valued.
- Gaussian and complex chirp functions are defined by single functional form throughout domain
- "widths" may be scaled by complex-valued analogues of $b_0$.

$$e^{\pm i\pi\left(\frac{x}{\sqrt{\mp i}}\right)^2} = e^{\pm i\pi\left(\frac{x^2}{\mp i}\right)} = e^{-\pi x^2} = GAUS[x]$$

- Also possible to express chirp function in form of Gaussian:

$$\begin{aligned} e^{\pm i\pi x^2} &= e^{-(\mp i)\pi x^2} = e^{-\pi\left(x\sqrt{\mp i}\right)^2} \\ &= GAUS\left[\left(\sqrt{\mp i}\right)x\right] \\ &= GAUS\left[\frac{x}{\sqrt{\pm i}}\right] \end{aligned}$$

## 2.4 "Superchirp" Function

$$e^{\pm i\pi x^n} = \cos[\pi x^n] \pm i \ \sin[\pi x^n]$$

- Even for even n
- Hermitian for odd n
- Symmetry assured if use argument $|x|$

$$f[x] = e^{\pm i\pi |x|^n} = \cos[\pi |x|^n] \pm i \ \sin[\pi |x|^n] \Longrightarrow f[-x] = f[x]$$

- Construct a Hermitian superchirp for even value of $n$ by forcing imaginary part to be odd via multiplication by $SGN[x]$:

$$f[x] = \cos[\pi x^n] \pm \ SGN[x] \ \sin[\pi |x|^n] \Longrightarrow f^*[-x] = f[x]$$

- Symmetric superchirp may be expressed in terms of corresponding $n^{th}$-order superGaussian:

$$e^{\pm i\pi |x|^n} = e^{-\pi\left((\mp i)^{\frac{1}{n}}|x|\right)^n} = GAUS\left[\left((\mp i)^{\frac{1}{n}} |x|\right); n\right]$$

- Area

$$\begin{aligned}\int_0^{+\infty} e^{\pm i\pi x^n} dx &= \frac{1}{n} \Gamma\left[\frac{1}{n}\right] \pi^{-\frac{1}{n}} e^{\pm\frac{i\pi}{2n}} \\ &= \left(\frac{1}{\pi}\right)^{\frac{1}{n}} e^{\pm\frac{i\pi}{2n}} \Gamma\left[1+\frac{1}{n}\right]\end{aligned}$$

$$\int_{-\infty}^{+\infty} e^{\pm i\pi |x|^n} dx = 2\Gamma\left[1+\frac{1}{n}\right] \pi^{-\frac{1}{n}} e^{\pm\frac{i\pi}{2n}}$$

- Area of Hermitian superchirp (odd n) must be real valued because area of odd imaginary part is zero:

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{\pm i\pi x^n} dx &= \Re\left\{2 \pi^{-\frac{1}{n}} \Gamma\left[1+\frac{1}{n}\right] e^{\pm\frac{i\pi}{2n}}\right\} \\ &= 2\Gamma\left[1+\frac{1}{n}\right] \pi^{-\frac{1}{n}} \cos\left[\frac{\pi}{2n}\right] \quad \text{(odd n)}\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{\pm i\pi x^2} dx &= e^{\pm\frac{i\pi}{4}} \pi^{-\frac{1}{2}} \frac{2}{|2|} \Gamma\left[\frac{1}{2}\right] \\ &= e^{\pm\frac{i\pi}{4}} \left(\frac{1}{\sqrt{\pi}}\right) \times 1 \times \sqrt{\pi} = e^{\pm\frac{i\pi}{4}} \\ &= \left(\frac{1}{\sqrt{2}}\right)(1 \pm i) \\ &\simeq 0.707 \ (1 \pm i)\end{aligned}$$

$$\int_{-\infty}^{+\infty} e^{\pm i\pi x^3} dx = \Re\left\{2\ e^{\pm\frac{i\pi}{6}}\ \pi^{-\frac{1}{3}}\ \Gamma\left[\frac{4}{3}\right]\right\}$$
$$\simeq \cos\left[\frac{\pi}{6}\right]\ \frac{1}{\pi^{\frac{1}{3}}} \cdot 2 \cdot 0.893$$
$$\simeq 1.056$$

$$\int_{-\infty}^{+\infty} e^{\pm i\pi |x|^3} dx = 2\ e^{\pm\frac{i\pi}{6}}\ \pi^{-\frac{1}{3}}\ \Gamma\left[\frac{4}{3}\right]$$
$$\simeq 1.219\ e^{\frac{\mp i\pi}{6}}$$
$$\simeq 1.056 \pm i\ 0.609$$

$$\int_{-\infty}^{+\infty} e^{\pm i\pi x^4} dx = 2\ e^{+\frac{i\pi}{8}}\ \pi^{-\frac{1}{4}}\ \Gamma\left[\frac{5}{4}\right]$$
$$= \frac{1}{2} \cdot 3.6256$$
$$\simeq 1.2580 \pm i\ 0.5211$$

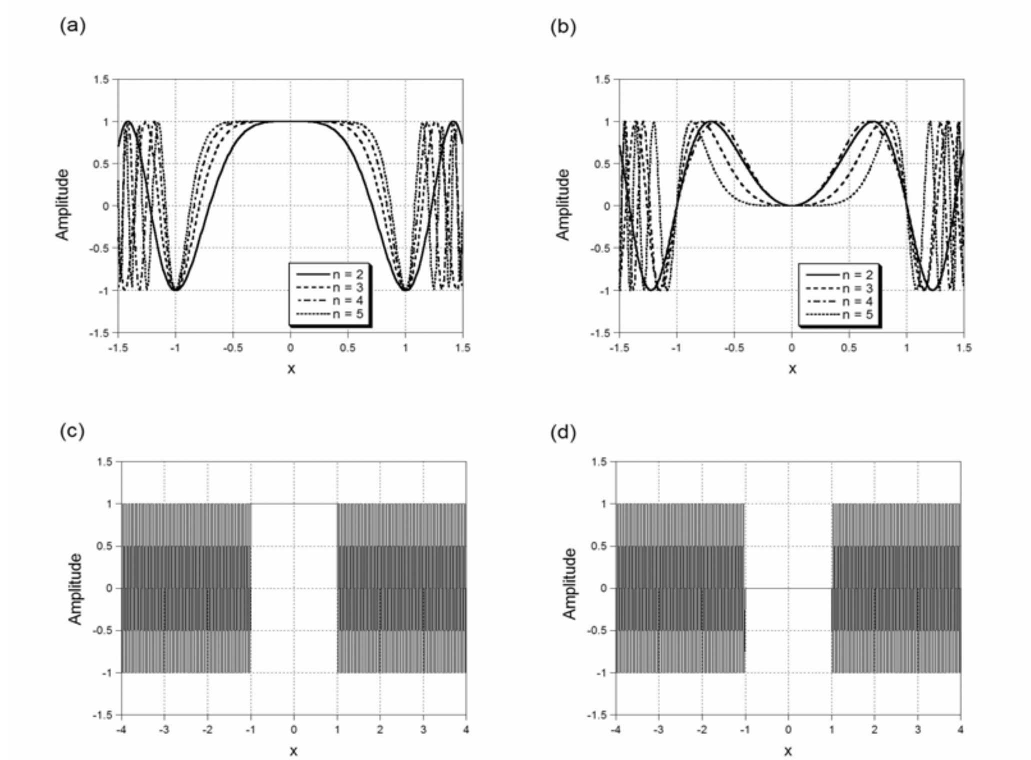
$$\int_{-\infty}^{+\infty} e^{\pm i\pi x^5} dx = \Re\left\{2 \cdot e^{\pm\frac{i\pi}{10}}\ \pi^{-\frac{1}{5}}\ \Gamma\left[\frac{6}{5}\right]\right\}$$
$$= \cos\left[\frac{\pi}{10}\right] \cdot \pi^{-\frac{1}{5}} \cdot 2 \cdot 0.9182$$
$$\simeq 1.3891$$

$$\int_{-\infty}^{+\infty} e^{\pm i\pi |x|^5} dx = 2\ e^{\pm\frac{i\pi}{10}}\ \pi^{-\frac{1}{5}}\ \Gamma\left[\frac{6}{5}\right]$$
$$\simeq 1.2891 \pm i\ 0.4513$$

- Trends:
  - real part of area increases with $n$,
  - magnitude of area of imaginary parts of symmetric chirps decrease with increasing $n$.

$$\lim_{n\to\infty}\left\{\int_{-\infty}^{+\infty} e^{-i\pi x^n} dx\right\} = \lim_{n\to\infty}\left\{e^{+\frac{i\pi}{n}}\ \pi^{-\frac{1}{n}}\ \frac{2}{n}\ \Gamma\left[\frac{1}{n}\right]\right\}$$
$$\Longrightarrow e^0 \cdot \pi^0 \cdot 2 \cdot \lim_{n\to\infty}\left\{\frac{1}{n}\ \Gamma\left[\frac{1}{n}\right]\right\} = 2 + i\ 0$$

| $n$ | $\int_{-\infty}^{+\infty} e^{\pm i\pi x^n} dx$ | $\int_{-\infty}^{+\infty} e^{\pm i\pi \lvert x\rvert^n} dx$ |
|---|---|---|
| 2 | $\left(\frac{1}{\sqrt{2}}\right)(1 \pm i) \simeq 0.707\ (1 \pm i)$ | $\left(\frac{1}{\sqrt{2}}\right)(1 \pm i) \simeq 0.707\ (1 \pm i)$ |
| 3 | $\simeq 1.056$ | $\simeq 1.056 \pm i\ 0.609$ |
| 4 | $\simeq 1.2580 \pm i\ 0.5211$ | $\simeq 1.2580 \pm i\ 0.5211$ |
| 5 | $\simeq 1.2891$ | $\simeq 1.2891 \pm i\ 0.4513$ |




"Superchirp" functions $e^{+i\pi x^n}$ for various values of $n$: (a) $\Re\left\{e^{+i\pi x^n}\right\} = \cos\left[\pi x^n\right]$, (b) $\Im\left\{e^{+i\pi x^n}\right\} = \sin\left[\pi x^n\right]$, (c) $\lim_{n\to\infty}\left(\Re\left\{e^{+i\pi x^n}\right\}\right) \simeq RECT\left[\frac{x}{2}\right]$, (d) and $\lim_{n\to\infty}\left(\Im\left\{e^{+i\pi x^n}\right\}\right) \simeq 0\,[x]$.

## 2.5 Complex-Valued Lorentzian Function

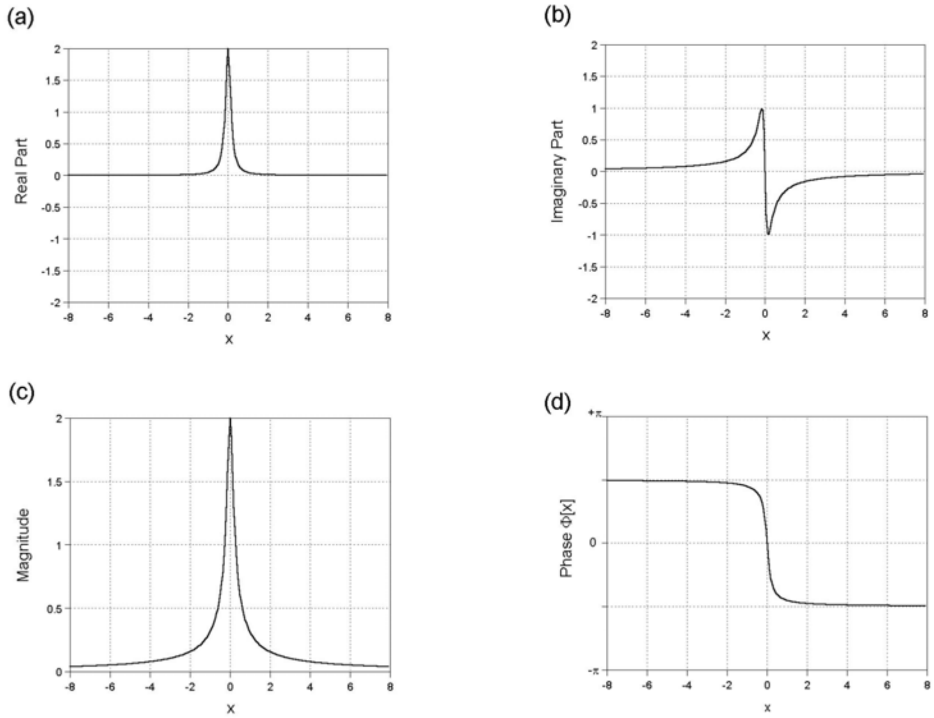
$$\begin{aligned} f[x] &= LOR[x] \;\; (1-2\pi i x) \\ &= \frac{2\;(1-2\pi i x)}{1+(2\pi x)^2} \\ &= \frac{2}{1+(2\pi x)^2} - i\frac{4\pi x}{1+(2\pi x)^2} \\ &\equiv CLOR[x] \end{aligned}$$

$$\begin{aligned} |CLOR[x]| &= \left|\frac{2\;(1-2\pi i x)}{1+(2\pi x)^2}\right| \\ &= \sqrt{\frac{4\;\left(1+(2\pi x)^2\right)}{\left(1+(2\pi x)^2\right)^2}} \\ &= \sqrt{\frac{4}{\left(1+(2\pi x)^2\right)}} \\ &= \sqrt{2\;LOR[x]} \end{aligned}$$

- Magnitude of complex Lorentzian “decays” more slowly than real-valued Lorentzian.
- Phase is arctangent of ratio of imaginary and real parts.
- Identical denominators cancel to leave inverse tangent of $-2\pi x$:.

$$\begin{aligned} \Phi\{CLOR[x]\} &= \Phi\left\{\frac{2\;(1-2\pi i x)}{1+(2\pi x)^2}\right\} \\ &= \tan^{-1}\left[-\frac{\left(\frac{4\pi x}{1+(2\pi x)^2}\right)}{\left(\frac{2}{1+(2\pi x)^2}\right)}\right] \\ &= \tan^{-1}[-2\pi x] \\ &= -\tan^{-1}[2\pi x] \end{aligned}$$

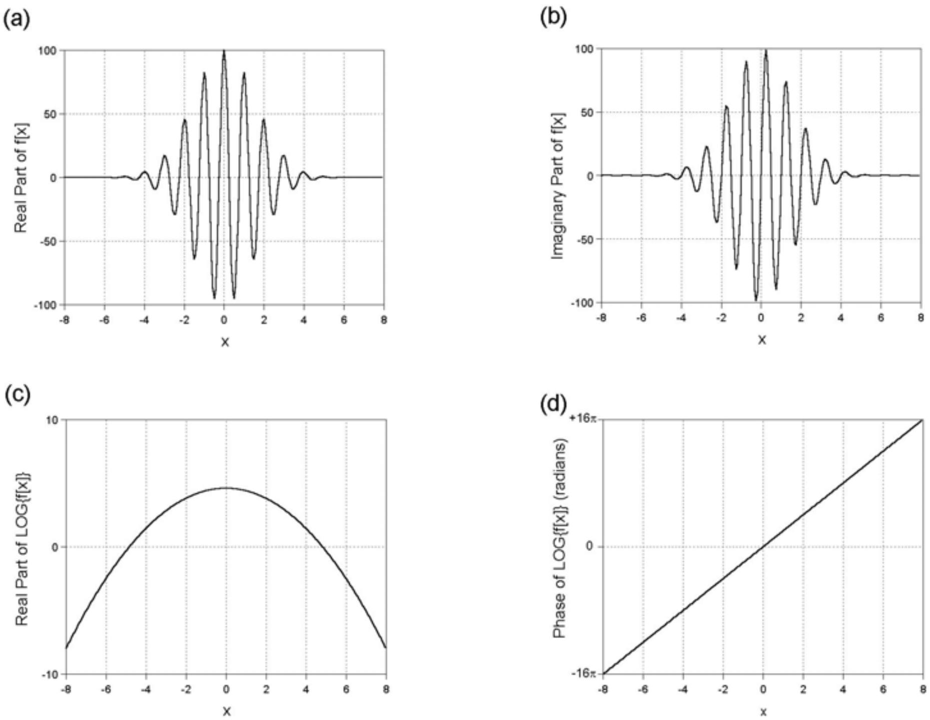
- Phase of complex Lorentzian function follows curve of inverse tangent function.




Complex Lorentzian function $CLOR[x] = \frac{2\ (1-2\pi i x)}{1+(2\pi x)^2}$: (a) $\Re\{CLOR[x]\} = \frac{2}{1+(2\pi x)^2}$; (b) $\Im\{CLOR[x]\} = \frac{-2\pi x}{1+(2\pi x)^2}$ ; (c) $|CLOR[x]| = \sqrt{\frac{4}{1+(2\pi x)^2}}$; and (d) phase $\Phi\{CLOR[x]\} = -TAN^{-1}[2\pi x]$.

## 2.6 Logarithm of Complex Amplitude

$$\begin{aligned} LOG\left[f\left[x\right]\right] &= LOG\left[\left|f\left[x\right]\right|\ e^{i\Phi\{f[x]\}}\right] \\ &= LOG\left[\left|f\left[x\right]\right|\right] + LOG\left[e^{i\Phi\{f[x]\}}\right] \\ &= LOG\left[\left|f\left[x\right]\right|\right] + i\ \Phi\left\{f\left[x\right]\right\} \end{aligned}$$




$\log\left(f\left[x\right]\right) = \log\left(100\ e^{(+2\pi i x)}\ e^{-\pi\left(\frac{x}{4}\right)^2}\right)$: (a) $\Re\left\{\left(f\left[x\right]\right)\right\} = 100\ \cos\left[2\pi x\right] e^{-\pi\left(\frac{x}{4}\right)^2}$, (b) $\Im\left\{\left(f\left[x\right]\right)\right\} = 100\ \sin\left[2\pi x\right]\ e^{-\pi\left(\frac{x}{4}\right)^2}$, (c) $\Re\left\{\log_e\left(f\left[x\right]\right)\right\} = \log_e\left|f\left[x\right]\right| = \log_e\left|100\pi\left(\frac{x}{4}\right)^2\right|$, (d) $\Im\left\{\log_e\left(f\left[x\right]\right)\right\} = \Phi\left\{f\left[x\right]\right\} = 2\pi x$.

# 3 A. APPENDIX: Series Solution for Bessel Functions

## 3.1 A.1. Series Solution for $J_0[x]$

1. Substitute $\nu = 0$ into differential equation:

$$x^2 \frac{d^2}{dx^2}(J_0[x]) + x \frac{d}{dx}(J_0[x]) + x^2 J_0[x] = 0 \tag{A1}$$

   The power-series solution for $J_0[x]$ has the form:

$$J_0[x] = \sum_{\ell=0}^{+\infty} a_\ell \, x^\ell \tag{A2}$$

2. After substitution of eq.(A2) in eq.(A1) and evaluating the derivatives of each term:relationship satisfied by coefficient $a_\ell$ for each power of $x$:

$$\begin{aligned} x^2 \frac{d^2}{dx^2}\left(\sum_{\ell=0}^{+\infty} a_\ell \, x^\ell\right) + x \frac{d}{dx}\left(\sum_{\ell=0}^{+\infty} a_\ell \, x^\ell\right) + x^2 \sum_{\ell=0}^{+\infty} a_\ell \, x^\ell \\ = x^2 \sum_{\ell=0}^{+\infty} a_\ell \, \left(\ell(\ell-1)\right) \, x^{\ell-2} + \left(\sum_{\ell=0}^{+\infty} a_\ell \, \ell \, x^\ell\right) + \left(\sum_{\ell=0}^{+\infty} a_\ell \, x^{\ell+2}\right) \\ = \sum_{\ell=0}^{+\infty} a_\ell \, \left(\ell(\ell-1)\right) x^\ell + a_\ell \, \ell \, x^\ell + a_\ell \, x^{\ell+2} = 0 \end{aligned} \tag{A3}$$

3. Collect coefficients of identical powers of $x$:

$$\sum_{\ell=0}^{+\infty} a_\ell \left(\ell(\ell-1)\right) x^\ell + a_\ell \, \ell \, x^\ell + a_\ell \, x^{\ell+2} = \sum_{\ell=0}^{+\infty} \left(a_\ell \left[\ell(\ell-1)\right] + a_\ell \, \ell + a_{\ell-2}\right) x^\ell = 0 \tag{A4}$$

   Coefficient of each power of $x$ must be zero.

4. Result is recursion relation for $a_\ell$:

$$a_\ell \left[\ell(\ell-1)\right] + a_\ell \, \ell + a_{\ell-2} = 0 \Longrightarrow a_\ell = -\frac{a_{\ell-2}}{\left[\ell(\ell-1)\right] + \ell} = -\frac{a_{\ell-2}}{\ell^2} \tag{A5}$$

- Relates only coefficients that differ in power by 2
- Sign of each coefficient is the opposite of the next one in the series.
- Coefficients determined by boundary conditions.
- Zeroth-order coefficient is amplitude of function at origin, $a_0 = J_0[0]$, assumed to be unity
- Subsequent coefficients for even powers are:

$$a_2 = -\frac{1}{2^2} = -\frac{1}{4} \tag{A6a}$$

$$a_4 = -\frac{a_2}{4^2} = -\frac{\left(-\frac{1}{4}\right)}{16} = +\frac{1}{64} \tag{A6b}$$

$$a_6 = -\frac{a_4}{6^2} = -\frac{\left(+\frac{1}{64}\right)}{36} = -\frac{1}{2304} \tag{A6c}$$

$$a_8 = -\frac{a_6}{8^2} = -\frac{\left(-\frac{1}{2304}\right)}{64} = -\frac{1}{147,456} \tag{A6d}$$

$$\vdots$$

$$\begin{aligned} a_{2\ell} &= (-1)^\ell \frac{1}{\left((2\ell)^2 \;\; [2(\ell-1)]^2 \;\; (2[\ell-2])^2 \cdots 2^2\right)} \\ &= (-1)^\ell \frac{1}{(2^2)^\ell (\ell!)^2} = \frac{(-1)^\ell}{2^{2\ell} (\ell!)^2} \end{aligned} \tag{A6e}$$

- Coefficients of odd powers assumed to be zero
- $J_0[x]$ is even function.
- The power series for zero-order Bessel function of the first kind is:

$$\begin{aligned} J_0[x] &= \sum_{\ell=0}^{\infty} (-1)^\ell \frac{1}{(\ell!)^2} \left(\frac{x}{2}\right)^{2\ell} \\ &= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147,456} - \cdots \end{aligned} \tag{A7}$$

- Magnitudes decrease very rapidly with order
  - Extrema of $J_0[x]$ decrease much more slowly with increasing $x$ than $SINC[x]$.

### 3.2 A.2. Series Solution for $J_1[x]$

- Substitute $v = 1$:

$$x^2 \frac{d^2}{dx^2}(J_1[x]) + x\frac{d}{dx}(J_1[x]) + \left(x^2 - 1\right) J_1[x] = 0 \tag{A8}$$

- Power-series solution for $J_1[x]$ has form:

$$J_1[x] = \sum_{\ell=0}^{+\infty} b_{0\ell}\, x^{\ell} \tag{A9}$$

$$\begin{aligned}
&x^2 \frac{d^2}{dx^2}\left(\sum_{\ell=0}^{+\infty} b_{0\ell}\, x^{\ell}\right) + x\frac{d}{dx}\left(\sum_{\ell=0}^{+\infty} b_{0\ell}\, x^{\ell}\right) + \left(x^2-1\right)\sum_{\ell=0}^{+\infty} b_{0\ell}\, x^{\ell} \\
&= x^2 \sum_{\ell=0}^{+\infty} b_{0\ell}\left(\ell\left(\ell-1\right)\right) x^{\ell-2} + \left(\sum_{\ell=0}^{+\infty} b_{0\ell}\,\ell\, x^{\ell}\right) + \left(\sum_{\ell=0}^{+\infty} b_{0\ell}\, x^{\ell+2} - \sum_{\ell=0}^{+\infty} b_{0\ell}\, x^{\ell}\right) \\
&= \sum_{\ell=0}^{+\infty}\left(b_{0\ell}\left(\ell\left(\ell-1\right)\right) + b_{0\ell}\,\ell + b_{0\ell-2} - b_{0\ell}\right) x^{\ell} = 0 \\
&= \sum_{\ell=0}^{+\infty}\left(b_{0\ell}\left(\left(\ell\left(\ell-1\right)\right) + \ell - 1\right) + b_{0\ell-2}\right) x^{\ell} = 0
\end{aligned} \tag{A10}$$

- Recursion relation:

$$b_{0\ell} = -\frac{b_{0\ell-2}}{\ell^2 - \ell + \ell - 1} = -\frac{b_{0\ell-2}}{\ell^2 - 1} \tag{A11}$$

- Boundary conditions determine first two coefficients
  - $J_1[0] = 0 \Longrightarrow$all even-order coefficients vanish
  - First-order coefficient is slope at origin, set to $\frac{1}{2}$

Remaining odd-order coefficients determined by recursion relation in eq.(A10):

$$b_{01} = \frac{1}{2} \tag{A12a}$$

$$b_{03} = -\frac{b_{01}}{(3^2-1)} = -\frac{\left(\frac{1}{2}\right)}{2\cdot 4} = -\frac{1}{16} \tag{A12b}$$

$$b_{05} = -\frac{b_{03}}{(5^2-1)} = \frac{(-1)^2}{(5^2-1)\cdot(3^2-1)\cdot 2} = \frac{(-1)^2}{(4\cdot 6)\cdot(4\cdot 2)\cdot 2} = \frac{(-1)^2}{(2^2\cdot 4^2\cdot 6)} = +\frac{1}{384} \tag{A12c}$$

$$\begin{aligned}
b_{07} &= -\frac{b_{05}}{(7^2-1)} = \frac{(-1)^3}{(7^2-1)\cdot(5^2-1)\cdot(3^2-1)\cdot 2} = \frac{(-1)^3}{(8\cdot 6)\cdot(4\cdot 6)\cdot(4\cdot 2)\cdot 2} \\
&= -\frac{\left(\frac{1}{384}\right)}{48} = -\frac{1}{18,432}
\end{aligned} \tag{A12d}$$

$$\vdots$$

$$\begin{aligned}
b_{02\ell+1} &= (-1)^{\ell}\frac{1}{\left((2\ell+1)^2-1\right)\cdot\left((2\ell-1)^2-1\right)\cdot\left((2\ell-3)^2-1\right)\cdot\cdots\cdot 2} \\
&= \frac{(-1)^{\ell}}{\left(2^2\cdot 4^2\cdot\cdots\cdot(2\ell)^2\cdot(2\ell+2)\right)}
\end{aligned} \tag{A12e}$$

- Power series for the first-order Bessel function of the first kind:

$$J_1[x] = \frac{x}{2} - \frac{x^3}{(2^2 \cdot 4)} + \frac{x^5}{(2^2 \cdot 4^2 \cdot 6)} - \frac{x^7}{(2^2 \cdot 4^2 \cdot 6^2 \cdot 8)} + \cdots$$
$$= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18,432} + \cdots \qquad \text{(A13)}$$

- Magnitudes decrease rapidly as power of $x$ increases.
- Comparing series for $J_1[x]$ and $J_0[x]$ in eq.(A7):

$$\frac{d}{dx}(J_0[x]) = 0 - \frac{x}{2} + \frac{x^3}{16} - \frac{x^5}{384} + \frac{x^7}{18,432} - \cdots = J_1[x] \qquad \text{(A14)}$$

- General expression valid for all positive integer values of $n$:

$$J_n[x] = \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{\ell!\,(n+\ell)!} \left(\frac{x}{2}\right)^{n+2\ell} \qquad \text{(A15)}$$