SIMG-716
Linear Imaging Mathematics I
03

1 Representations of Functions

1.1 Functions

- Function: rule that assigns one numerical value (*dependent variable*) to each of a set of other numerical values (*independent variable* or input *coordinate*).
- used to describe all aspects of imaging process: input scene f, imaging system $\mathcal{O}$, and output image g.
- Number of independent variables required to specify a unique location in that domain is the *dimensionality* of the function
- set of possible values of the independent variable(s) is the *domain* of the function.
- Set of possible values of the dependent variable f is the *range* of the function.

1.1.1 Multidimensional Functions

- More than one independent variable
- If independent variables have same "units" (e.g., $[x, y]$) then may be expressed in various equivalent coordinate systems
  - 2-D Cartesian coordinates
  - 2-D polar coordinates
  - 3-D Cartesian
  - 3-D cylindrical
  - 3-D spherical
  - ...
- Choice of coordinate system depends on conditions of specific imaging problem; the art is choosing most convenient representation
- Conventions for notation:
  - Cartesian coordinates in brackets, e.g., $[x, y]$ and $[x, y, z]$.
  - domains with at least one angle enclosed in parentheses; e.g., (r, θ).
    * (note potential for confusion with common notation for "open-ended" and "closed" intervals of real-valued coordinates)
    * combinations of brackets and parentheses specify intervals in the domain.
    * Bracketed coordinates $[-1, +1]$, specify a domain that includes the indicated endpoints
    * parentheses, $(-1, +1)$ indicate that the interval does not include the endpoints
    * infinite domain is $(-\infty, +\infty)$.
    * bracket and a parenthesis used together to specify interval with one included endpoint (the "closed" coordinate) but not the other (the "open" coordinate): $[0, 1)$ indicates an interval of unit length on the real line that includes the origin but not $x = 1$, so that $0 \leq x < 1$.

1.2 Classes of Functions

- based on characteristics of its domain and/or range.

Examples:

1. domain may be specified either over continuous domain or at discrete set of "samples"
2. coordinates $[x, y, \cdots]$ in domain may be real, imaginary, or complex-valued (always real-valued for us)
   - finite (compact) support
   - infinite support
3. range of dependent variable f may be real-, imaginary-, or complex-valued (often complex for us)
4. dependent variable f may repeat at regular intervals of domain to create a *periodic* function.
5. set of "zeros" of function: locations where amplitude f is zero.
6. "shape" of function over its domain
   - amplitude is a power of coordinate x
     - *linear* $\Longrightarrow$ amplitude is proportional to coordinate ($f[x] = \alpha x \propto x^1$).
     - *quadratic* function of x has the form $f[x] = \beta x^2 \propto x^2$, a parabolic shape.
     - *arbitrary* shapes – may often be expressed as a sum of "simpler" shapes
       * basis for power-series expansions, e.g.,

$$f[x] = (f[x_0]) \cdot x^0 + \left(\frac{1}{1!} \left. \frac{df}{dx} \right|_{x=x_0} \right) \cdot (x - x_0)^1 + \left(\frac{1}{2!} \left. \frac{d^2 f}{dx^2} \right|_{x=x_0} \right) \cdot (x - x_0)^2 + \cdots + \left(\frac{1}{n!} \left. \frac{d^n f}{dx^n} \right|_{x=x_0} \right) \cdot (x - x_0)^n + \cdots$$

which "decomposes" $f[x]$ into the sum of constant, linear, quadratic, ... functions of x.

These classifications for functions (and others yet to be introduced) will be useful in the course of this discussion. We begin by considering the classes based on the continuity of the domain and range.

1.3 Examples of Functions with Continuous and Discrete Domains

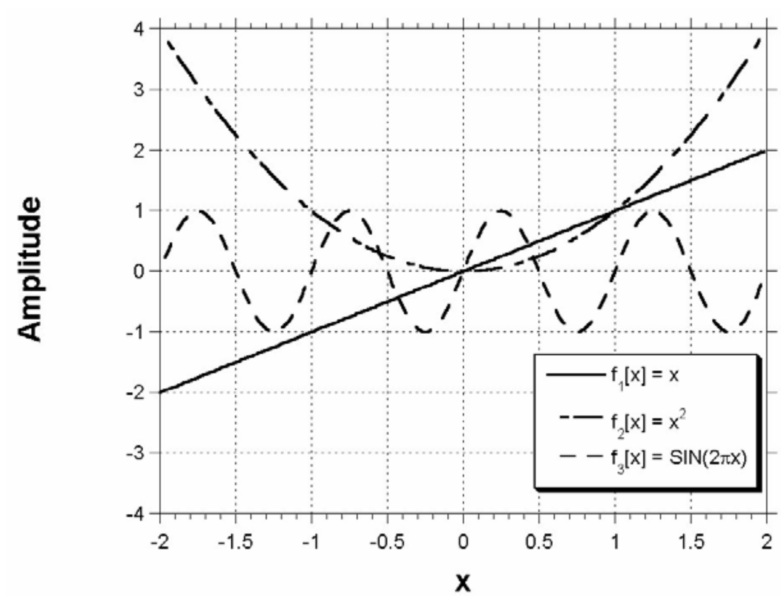
1. $$f_1[x] \equiv y = 4x$$
   - amplitude of f_1 proportional to input coordinate, and therefore $f_1[x]$ is a linear function
   - domain and range include all real numbers, both domain and range are $(-\infty, +\infty)$.
   - range is "bipolar" because it includes positive and negative values.
   - one zero, located at $x = 0$.
2. $$f_2[x] = x^2$$
   - quadratic (not linear).

- domain and range of $f_2[x]$ are continuous
  - domain is $(-\infty, +\infty)$
  - range is $[0, +\infty)$.
- One zero at $x = 0$.

3. 1-D sinusoid with period X_3:

$$f_3[x] = \sin\left[\frac{2\pi x}{X_3}\right]$$

- domain and range are continuous
  - domain is $(-\infty, +\infty)$
  - range is $[-1, +1]$.
- amplitude f_3 repeats for coordinates x separated by intervals of X_3
  - $f_3[x]$ is *periodic*
  - infinite number of isolated zeros uniformly spaced at intervals of width $\Delta x = \frac{X_3}{2}$.

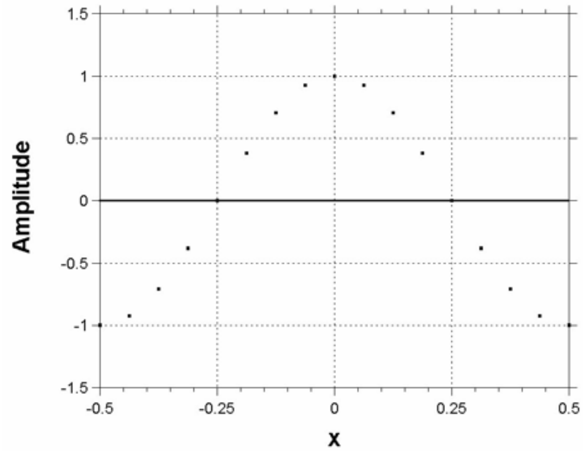


Graphs of different functions, all with continuous domains but different ranges. Though shown with different line styles, all represent continuous functions.

4. Discrete functions: amplitude is defined only at discrete set of uniformly spaced coordinates

$$f_4[x] = \cos[2\pi n\ \Delta x],\ n = 0, \pm 1, \pm 2, \cdots$$

- may be constructed from continuous functions by *sampling*, considered later.



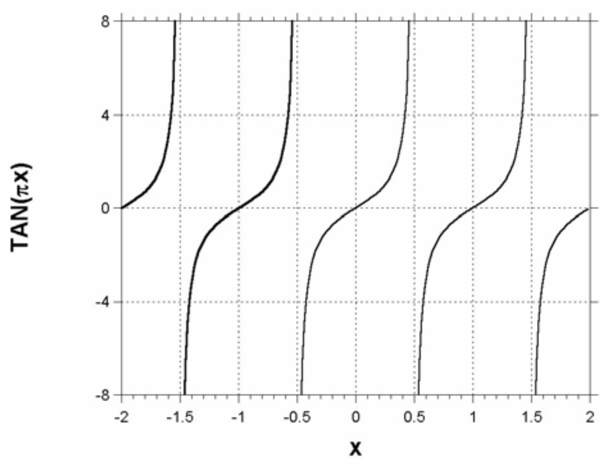
Discrete ("sampled") function, with discrete domain and continuous range.

5. Discontinuous Range

- "transition" coordinates where derivative is not defined

$$f_5[x] = \tan\left[\frac{2\pi x}{X}\right]$$

- amplitude "jumps" from $f_5 \to +\infty$ at $x = +\frac{X}{4} - \epsilon$ to $f_5 \to -\infty$ at $x = +\frac{X}{4} + \epsilon$, where ϵ is a small positive real number $\epsilon \simeq 0$
- amplitude not defined at $x = \frac{X}{4}$ and its derivative is not finite there.



"Discontinuous" function $\tan[\pi x]$ with continuous domain and continuous range.

1.4 Continuous and Discrete Ranges

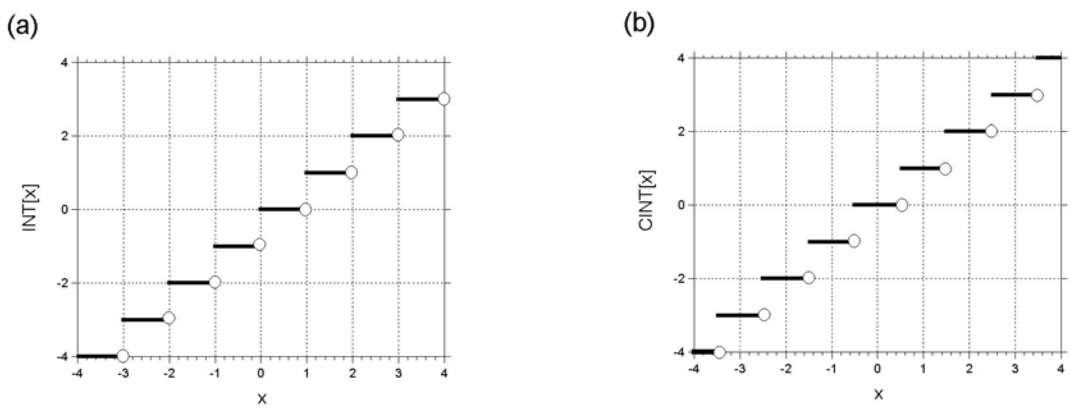
- set of all allowed values of dependent variable f
- different "flavors" of range exist for different functions
  - may have an infinite or finite extent
  - may have all real values between extrema or only some discrete set.
  - may be "continuous" or "discrete"

6.

$$f_6[x] = INT[x]$$

- domain is entire real line $(-\infty, +\infty)$
- range includes discrete (though infinite) set of integers $(0, \pm 1, \pm 2, \cdots)$
- common variant is $CINT[\ ]$, which "rounds" the value to the nearest integer.

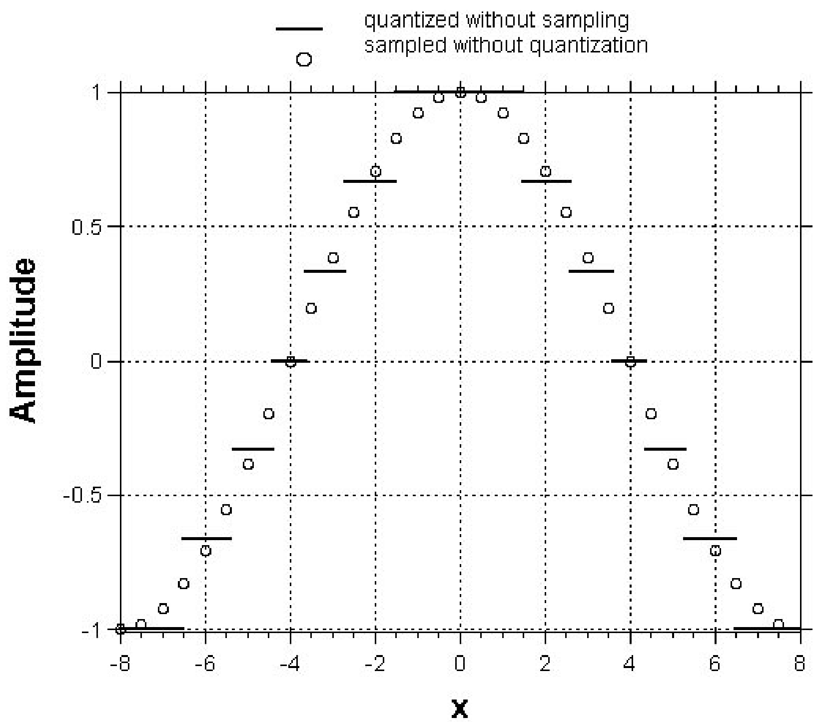
$$f_7[x] = CINT[x] \equiv INT\left[x + \frac{1}{2}\right]$$



(a) "Greatest integer" function (truncation of amplitude) compared to (b) "closest integer" function $CINT[x]$ that "rounds" the amplitude to the nearest integer.

- truncation or rounding may be applied to amplitude of any function $f[x]$ to convert from a continuous to discrete range

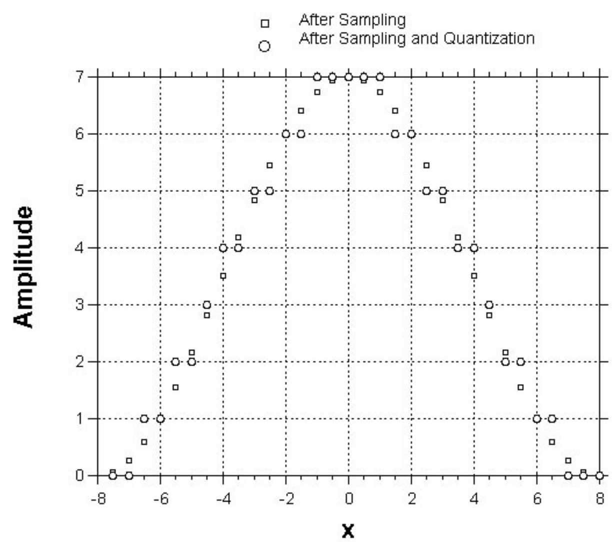
$$g[x] = CINT\{f[x]\} \Longrightarrow g = CINT[f]$$



$COS\left[\frac{2\pi x}{16}\right]$ after independent quantization to 7 levels and independent sampling at integer coordinates.

1.5 Discrete Domain and Range – Digitized Functions

Both domain and range are discrete



Results from the cascade of sampling and subsequent quantization of nonnegative sinusoid; note the changes in amplitude due to the quantization.

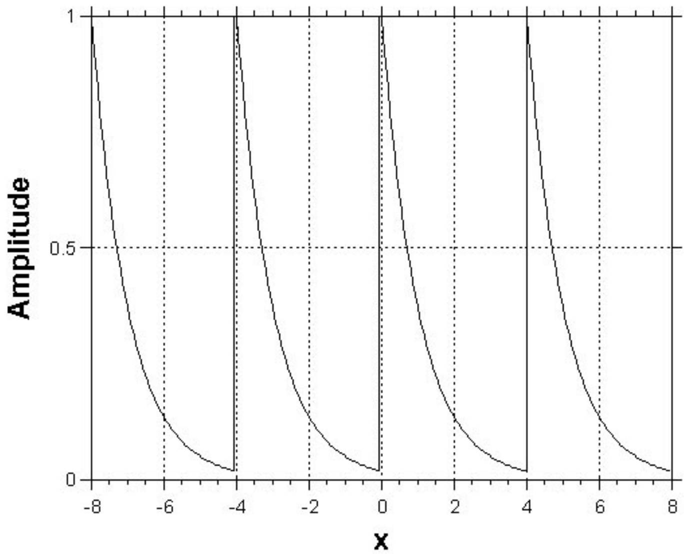
1.6 Periodic, Aperiodic, and Harmonic Functions

1. criterion for a one-dimensional (1-D) function $f[x]$ to be periodic:

$$f[x_0] = f[x_0 + nX_0]$$

where n is any integer and X_0 (the period of $f[x]$) is the smallest possible increment of the independent variable x such that the requirement is satisfied.

- a nonnull function $f[x]$ is periodic if amplitudes are identical at all coordinates separated by integer multiples of some distance X_0.
- may have very irregular form; the criterion for periodicity merely requires that the amplitude repeat at regular intervals.
- *aperiodic function* is any function that does not satisfy the criterion for periodicity
- Functions of dimension two (or larger) may be periodic over all coordinates or over just a subset.



Function that obeys requirement for "periodicity" because $f[x] = f[x+4n]$ for any integer value of n.

2. *Harmonic* functions produced by oscillatory motions

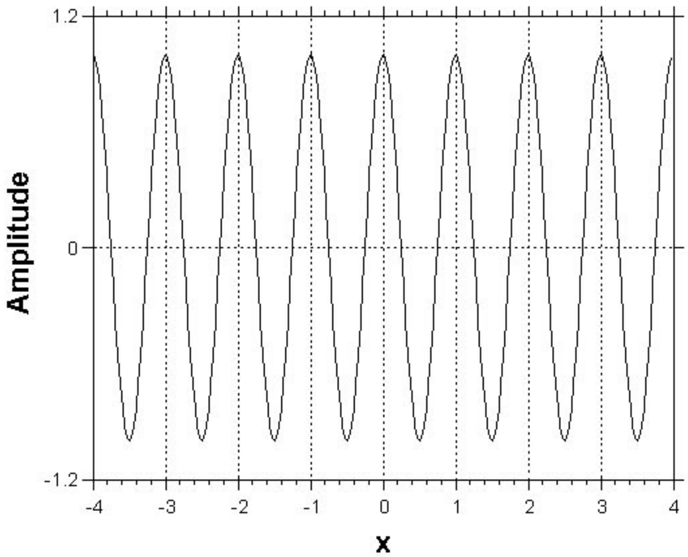
- common in electromagnetism (and therefore in optics), acoustics, classical and quantum mechanics
- harmonic function is composed of a single sinusoid
- spatially oscillating harmonic function:

$$f[x] = A_0 \cos[\Phi[x]] = A_0 \cos\left[\frac{2\pi x}{X_0} + \phi_0\right]$$

- temporally oscillating harmonic function:

$$f[t] = A_0 \cos[\Phi[t]] = A_0 \cos\left[\frac{2\pi t}{T_0} + \phi_0\right]$$

- $\Phi[x]$ or $\Phi[t]$ defines an *angle* measured in radians: the *phase angle*.
- $\Phi[x]$ must be a linear or a constant function of x for $f[x]$ to be harmonic
  - constant $\Phi[x] \implies$ oscillation frequency is 0 radians per unit length
- constant part ϕ_0 of phase angle is the angle at $x = 0$: the *initial phase*
- concept of "phase" is generalized to nonsinusoidal, and even nonperiodic, functions



“Harmonic” function $f[x] = \cos[2\pi x]$, which is composed of a single sinusoidal component.

1.6.1 Harmonic Function of Space AND Time

$$f[x,t] = A_0 \cos[\Phi[x,t]] = A_0 \cos\left[\frac{2\pi x}{X_0} - \frac{2\pi t}{T_0} + \phi_0\right]$$

- forms a “traveling wave”
- “point of constant phase” (where $\Phi[x,y] = \Phi_0$) must have x increase with increasing t, so wave moves towards $x = +\infty$
- Amplitude is A_0 wherever phase $\Phi[x,t]$ is integer multiple of 2π radians.
- Any location in 2-D space-time domain for which $\Phi[x,t] = 2\pi \times n$ is a maximum of the sinusoid.
- As t increases, the spatial position x of this particular maximum also must change to maintain the same phase.
  - x of the point with zero phase must increase as t increases
  - “point of constant phase” of the wave moves toward $x = +\infty$
  - Sinusoids of this form are called “traveling waves” in physics.
- rate of energy transfer of $f[x,t]$ is proportional to $(f[x,t])^2$, which is the “power” .
- period X_0 of traveling wave is the interval of x over which the phase changes by 2π radians
- X_0 is a “length” (e.g., mm).

Spatial Frequency:

- Often convenient to recast phase in terms of “reciprocal of period” $\xi_0 \equiv X_0^{-1}$
- specifies number of periods (“cycles”) of sinusoid in one unit of independent variable x
- ξ_0 in units of “reciprocal length,” e.g., mm^{-1} – “cycles per millimeter”

- ξ_0 is a "rate" of oscillation, the *spatial frequency*.
- Equivalent expression for sinusoid is:

$$f[x] = A_0 \cos[\Phi[x]] = A_0 \cos[2\pi\xi_0 x + \phi_0]$$

- Common to measure ξ_0 in units of "cycles per mm".

Angular Spatial Frequency

- based upon number of radians of phase within one unit of length
- denoted by k_0 or σ_0
- units of "reciprocal length" – "radians per millimeter"
- sometimes called *wavenumber* of spatial wave

$$f[x] = A_0 \cos[\Phi[x]] = A_0 \cos[k_0 x + \phi_0], \text{ where } k_0 = 2\pi\xi_0.$$

- angular spatial frequency most conveniently defined as spatial derivative of phase:

$$\text{Angular Spatial Frequency } k_0 = \frac{\partial\Phi[x]}{\partial x} \quad \text{[radians/unit length]}$$

  - angular spatial frequency is constant (function is harmonic) if $\Phi[x]$ is (at most) a linear function of x
- *spatial frequency* includes scale factor of 2π radians per cycle, or $(2\pi)^{-1}$ cycles per radian:

$$\text{Spatial Frequency } \xi_0 = \frac{1}{2\pi}\frac{\partial\Phi[x]}{\partial x} \quad \text{[cycles/unit length]}$$

Temporal Frequencies

- parameter corresponding to ξ_0 is the *temporal frequency*, often indicated by ν_0
- units of "cycles per time interval" (e.g., "cycles per second" or *Hertz*).
- units of angular temporal frequency ω_0 are "radians per unit time", so that the conversion is $\omega_0 = 2\pi\nu_0$

$$\text{Angular Temporal Frequency } \omega_0 = \frac{\partial\Phi[t]}{\partial t} \quad \text{radians per unit time (e.g., radians per sec)}$$

$$\text{Temporal Frequency } \nu_0 = \frac{1}{2\pi}\frac{\partial\Phi[t]}{\partial t} \quad \text{cycles per unit time (e.g., cycles per sec)}$$

Negative Frequencies

- ξ_0 defined as derivative $\Longrightarrow$ negative values are "allowed"
- negative frequency $\Longrightarrow$ phase *decreases* as corresponding coordinate (x or t) increases:

$$\frac{\partial\Phi}{\partial x} < 0 \Longrightarrow \xi < 0$$

- angular temporal frequency ω_0 is negative if position variable x *decreases* as t increases; the points of constant phase move toward $-\infty$.

$$\frac{\partial\Phi}{\partial t} < 0 \Longrightarrow \nu < 0$$

1.6.2 Specifying Sinusoids

- any harmonic sinusoidal function $f[x]$ may be specified by three parameters:

  1. amplitude A_0
  2. period X_0 (or spatial frequency ξ_0 or angular spatial frequency k_0)
  3. initial phase ϕ_0.

- This equivalence is basis for alternate representation of function obtained via Fourier transform.
- Independent variable x and corresponding frequency ξ_0 (or the time t and temporal frequency ν_0) are *conjugate* variables

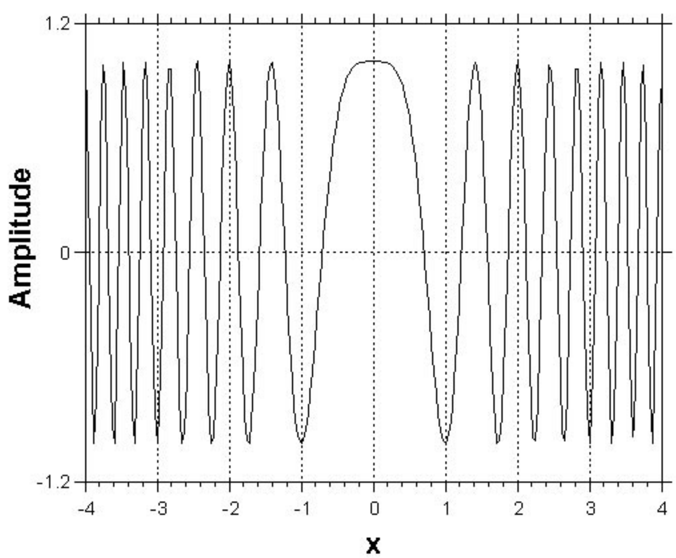
  – reciprocal units

1.6.3 Anharmonic Sinusoids

- Harmonic function is sinusoid whose frequency (spatial or temporal) does not vary with location (in space or time, or both)
- Anharmonic sinusoid: phase includes terms of order larger than 1

$$f[x] = A_0 \cos\left[\pi\left(\frac{x}{\alpha_0}\right)^2 + \phi_0\right]$$

$$\xi = \xi[x] = \frac{1}{2\pi}\frac{\partial\Phi}{\partial x} = \frac{1}{2\pi}\left(\frac{2\pi x}{\alpha_0^2}\right) = \frac{x}{\alpha_0^2} \propto x$$



Sinusoidal function with quadratic phase, $f[x] = \cos\left[\pi x^2\right]$.

1.7 Symmetry Properties of Functions

1.7.1 *even* and *odd* parts

- Every function may be "decomposed" into unique even and odd parts:

$$f[x] = f_e[x] + f_o[x]$$

- even part of $f[x]$ is *symmetric* with respect to the origin $\Longrightarrow$ obtain identical function if coordinate is "reversed"

$$f_e[x] = f_e[-x]$$

- odd part is *antisymmetric* $\Longrightarrow$ obtain negative of function if coordinate is "reversed"

$$f_o[x] = -f_o[-x]$$

- Easy to show that even and odd parts of $f[x]$ are evaluated from $f[x]$ via:

$$f_e[x] \equiv \frac{1}{2}(f[x] + f[-x])$$
$$f_o[x] \equiv \frac{1}{2}(f[x] - f[-x]).$$

- $\cos[2\pi\xi_0 x]$ and $\sin[2\pi\xi_0 x]$ are even and odd harmonic functions, respectively
- General harmonic function with spatial frequency ξ_0 and initial phase ϕ_0 is decomposed into constituent even and odd parts via:

$$\cos[\alpha \pm \beta] = \cos[\alpha]\ \ \cos[\beta] \mp \sin[\alpha]\ \ \sin[\beta]$$

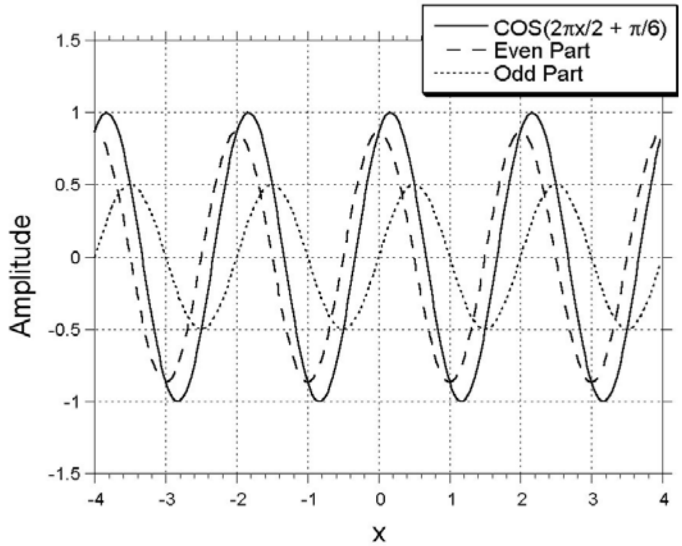
$$\begin{aligned} f[x] = A_0 \cos[2\pi\xi_0 x + \phi_0] &= A_0 (\cos[2\pi\xi_0 x]\ \ \cos[\phi_0] - \sin[2\pi\xi_0 x]\ \ \sin[\phi_0]) \\ &= (A_0 \cos[\phi_0]\ \ ) \cos[2\pi\xi_0 x] + (-A_0 \sin[\phi_0])\ \ \sin[2\pi\xi_0 x] \end{aligned}$$

$$f_e[x] = (A_0 \cos[\phi_0])\ \ \cos[2\pi\xi_0 x]$$
$$f_o[x] = (-A_0 \sin[\phi_0])\ \ \sin[2\pi\xi_0 x]$$

- amplitude of even and odd parts of general harmonic function are:

$$\begin{aligned} A_{\text{even}} &= A_0 \cos[\phi_0] \\ A_{\text{odd}} &= -A_0 \sin[\phi_0] \end{aligned}$$

- Corollary: sum of two sinusoids with same frequency and arbitrary amplitudes and phases yields a sinusoid with that same frequency.



Sinusoidal function $f[x] = \cos\left[2\pi\frac{x}{2} + \frac{\pi}{6}\right]$ decomposed into its even and odd parts: $f_e[x] = \frac{\sqrt{3}}{2}\cos[\pi x]$ and $f_o[x] = \frac{1}{2}\sin[\pi x]$.

1.7.2 Symmetry of 2-D Functions:

- symmetry determined by behavior when reversed with respect to origin

$$f[x,y] = f_e[x,y] + f_o[x,y]$$
$$f_e[x,y] \equiv \frac{1}{2}(f[x,y] + f[-x,-y])$$
$$f_o[x,y] \equiv \frac{1}{2}(f[x,y] - f[-x,-y])$$

$$f_e(r,\theta) = f_e(r,\theta \pm n\pi)$$
$$f_o(r,\theta) = -f_o(r,\theta \pm n\pi)$$