SIMG-716
Linear Imaging Mathematics I
02 - Complex Numbers and Functions

1 Complex Numbers and Functions

- convenient for describing:
  - sinusoidal functions of space and/or time (e.g., waves)
  - behavior of systems used to generate images

- Simplify representation of sinusoidal waves by using notation based on magnitude and phase angle
- Concise notation is convenient even when represented quantities are real valued
  - e.g., electric-field amplitude (voltage) of a traveling sinusoidal electromagnetic wave is a vector with real-valued amplitude that varies over both temporal and spatial coordinates. Variations in time and space are related through the *dispersion equation* that relates the frequency and velocity of the wave.

This discussion also will describe vectors constructed from complex-valued components. This extension of the vector concept will prove to be very useful when interpreting the Fourier transform.

- Complex numbers: generalization of *imaginary numbers* and often denoted by "z".
- Imaginary numbers: concept of $\sqrt{-1}$, which has no real-valued solution, symbol i was assigned the by Leonhard Euler in 1777:
$$\sqrt{-1} \equiv i \Longrightarrow i^2 = -1$$
- General complex number z is a composite number formed from sum of real and imaginary components:
$$\begin{aligned} \boxed{z \equiv a + i\ b}, \{a, b\} &\in \Re \\ a &= \Re\{z\} \\ b &= \Im\{z\} \\ a, b &\in \Re \text{ (both } a \text{ and } b \text{ are real valued!)} \end{aligned}$$
- *Complex conjugate* z^* of $z = a + ib$: multiply imaginary part by -1 :
$$z \equiv a + i\ b \Longrightarrow \boxed{z^* \equiv a - i\ b}$$
- Real/imaginary parts may be expressed in terms of z and its complex conjugate z^* via two relations that are easily confirmed:
$$\begin{aligned} z + z^* &= (a + ib) + (a - ib) = 2a = 2 \cdot \Re\{z\} \\ z - z^* &= (a + ib) - (a - ib) = 2 \cdot ib = 2i \cdot \Im\{z\} \end{aligned}$$

$$\boxed{\Re\{z\} = \frac{1}{2}(z + z^*)}$$
$$\boxed{\Im\{z\} = \frac{1}{2i}(z - z^*) = -i \cdot \frac{1}{2}(z - z^*)}$$

2 Arithmetic of Complex Numbers

Given : $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$.

2.1 Equality:

$z_1 = z_2$ if (and only if) their real parts and their imaginary parts are equal:

$$z_1 = z_2 \text{ if and only if } a_1 = a_2 \text{ and } b_1 = b_2;$$

2.2 Sum and Difference:

Add or subtract their real and imaginary parts separately:

$$\begin{aligned} z_1 \pm z_2 = (a_1 + ib_1) \pm (a_2 + ib_2) = (a_1 \pm a_2) + i\,(b_1 \pm b_2) \\ \Longrightarrow \Re\{z_1 \pm z_2\} = a_1 \pm a_2 = \Re\{z_1\} \pm \Re\{z_2\} \\ \Longrightarrow \Im\{z_1 \pm z_2\} = b_1 \pm b_2 = \Im\{z_1\} \pm \Im\{z_2\}\,; \end{aligned}$$

2.3 Multiplication:

Follow rules of arithmetic multiplication while retaining the factors of i and applying the definition that $i^2 = -1$:

$$\begin{aligned} z_1 \times z_2 &= (a_1 + ib_1) \times (a_2 + ib_2) = a_1a_2 + a_1\,(ib_2) + a_2\,(ib_1) + (ib_1)\,(ib_2) \\ &= \left(a_1a_2 + (i)^2\,b_1b_2\right) + i\,(a_1b_2 + a_2b_1) \\ &= (a_1a_2 - b_1b_2) + i\,(a_1b_2 + a_2b_1) \\ &\Longrightarrow \Re\{z_1z_2\} = a_1a_2 - b_1b_2 \\ &\Longrightarrow \Im\{z_1z_2\} = a_1b_2 + a_2b_1; \end{aligned}$$

2.4 Reciprocal:

For $z_1 \neq 0$ (i.e., $\Re\{z_1\} \neq 0$ and/or that $\Im\{z_1\} \neq 0$), reciprocal of z (denoted z_1^{-1}) is:

$$\begin{aligned} z_1^{-1} &= \frac{1}{z_1} \times \frac{z_1^*}{z_1^*} = \frac{z_1^*}{|z_1|^2} = \frac{a_1 - ib_1}{a_1^2 + b_1^2} \quad (\text{if } z_1 \neq 0) \\ \Re\left\{z_1^{-1}\right\} &= \frac{a_1}{a_1^2 + b_1^2} \\ \Im\left\{z_1^{-1}\right\} &= \frac{-b_1}{a_1^2 + b_1^2} \end{aligned}$$

This is allowed since $z_1^* = a_1 - ib_1 \neq 0$.

2.5 Ratio:

Combine definition of product and of reciprocal:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{z_1}{z_2} \times \frac{z_2^*}{z_2^*} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2} \\ &= \frac{(a_1a_2 + b_1b_2) + i\,(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2} \end{aligned}$$

$$\Re\left\{\frac{z_1}{z_2}\right\} = \frac{(a_1a_2+b_1b_2)}{a_2^2+b_2^2}$$
$$\Im\left\{\frac{z_1}{z_2}\right\} = \frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}.$$

2.5.1 Note:

Special care must be exercised when applying some familiar rules of algebra when imaginary or complex numbers are used. Nahin points out some examples of such relationships that fail, such as:

$$\sqrt{ab} = \sqrt{a}\ \sqrt{b}$$

which yields an incorrect result when both a and b are negative:

$$\begin{aligned}
\sqrt{(-|a|)\cdot(-|b|)} &= \sqrt{|a|\cdot|b|} = \sqrt{|a|}\cdot\sqrt{|b|} \\
\sqrt{a} &= \sqrt{-|a|} = i\cdot\sqrt{|a|} \text{ if } a<0 \\
\sqrt{b} &= \sqrt{-|b|} = i\cdot\sqrt{|b|} \text{ if } b<0 \\
\sqrt{a}\sqrt{b} &= \left(i\cdot\sqrt{|a|}\right)\left(i\cdot\sqrt{|b|}\right) \\
&= -1\cdot\sqrt{|a|}\sqrt{|b|} \neq \sqrt{|a|}\cdot\sqrt{|b|} = \sqrt{ab} \text{ if } a,b<0
\end{aligned}$$

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3 Graphical Representation of Complex Numbers

- Expression for sum of two complex numbers has same form as sum of two 2-D vectors
- Arithmetic of complex numbers is analogous to that of 2-D vectors with real-valued components.
- $z = a + ib$ is equivalent to ordered pair of real numbers $[a, b]$
- Domain of individual complex numbers is equivalent to 2-D domain of real numbers
  - set of individual complex numbers (a "one-dimensional" set) does not exhibit the property of *ordered size* that exists for the 1-D array of real numbers.
  - Consider two real numbers a and b
    - If both $a > 0$ and $b > 0$, then $ab > 0$.
    - Establishes a metric for relative sizes of the real numbers.
- Corresponding relationship does not exist for the set of "1-D" complex numbers $a + ib$
- Complex numbers may be ordered in size only by using a true 1-D metric.
- "Length" of the complex number $z = a + ib$ is equivalent to the length of the equivalent 2-D vector $[a, b]$. Mathematicians typically call this quantity the "modulus" or "absolute value" of the complex number

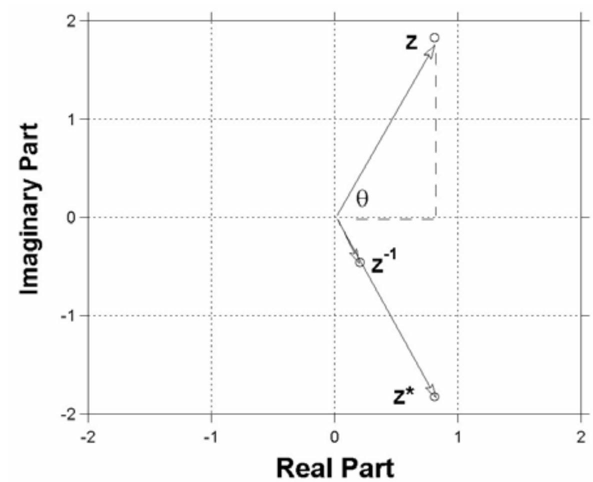
$$|z| = \sqrt{z\cdot z^*} = \sqrt{(a+ib)\ \ (a-ib)} = \sqrt{a^2+b^2}$$

- Magnitude of z is an appropriate metric of ordered size for complex numbers.
- Analogy between complex number and an ordered pair ensures that z may be depicted graphically with imaginary part on y-axis in a 2-D plane.

– *Argand* diagram of the complex number (*phasor*): $z = (|z|, \phi)$

  * magnitude $|z|$
  * polar , azimuth, or phase angle ϕ

$$\phi = \tan^{-1}\left[\frac{b}{a}\right] = \tan^{-1}\left[\frac{\Im\{z\}}{\Re\{z\}}\right]$$



Argand Diagrams of z, z^{-1}, and z^*. If the phase angle of z is ϕ_0, then the phase angles of z^{-1} and z^* are identically $-\phi_0$

- Subtle (but very IMPORTANT) problem with definition of phase angle ϕ

  – range of valid phase angles is $-\infty < \phi < +\infty$

  – arctangent function is multiply valued over any contiguous range exceeding π radians

    * calculation of arctangent of ratio of two lengths returns angle in interval $-\frac{\pi}{2} \leq \phi < +\frac{\pi}{2}$

  – One interval of 2π radians is selected as the *principal value* of the phase

    * "symmetric" interval $-\pi \leq \phi < +\pi$ (our convention)
    * "one-sided" interval $0 \leq \phi < 2\pi$ (common for sparkies)

  – Some computer languages compute arctangent of ratio of imaginary and real parts.

    * Computes only angle in interval $-\frac{\pi}{2} \leq \phi < +\frac{\pi}{2}$
    * Additional calculations must be performed based on the algebraic signs of the real and imaginary parts to select the appropriate quadrant and assign the correct angle

  – IDL has a two-argument inverse tangent function

  – need to know the algebraic signs of real and imaginary parts to locate phase angle in proper quadrant

3.1 Real and Imaginary Parts of Complex Number in Polar Form

$$\Re\{z\} = \Re\{a + ib\} = a = r\ \cos[\phi]$$
$$\Im\{z\} = \Im\{a + ib\} = a = r\ \sin[\phi]$$
$$z = \Re\{z\} + i\ \Im\{z\} = r\ \cos[\phi] + r\ (i\ \sin[\phi]) = r\ (\cos[\phi] + i\ \sin[\phi])$$

4 Euler Relation

Apply Taylor-series representations for cosine, sine, and e^u

$$\cos[\phi] = \frac{\phi^0}{0!} - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots = \sum_{n=0}^{+\infty} (-1)^n \frac{\phi^{2n}}{(2n)!}, \text{ (even powers only)}$$

$$\sin[\phi] = \frac{\phi^1}{1!} - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \cdots = \sum_{m=0}^{+\infty} (-1)^m \frac{\phi^{2m+1}}{(2m+1)!}, \text{ (odd powers only)}$$

$$e^u = \frac{u^0}{0!} + \frac{u^1}{1!} + \frac{u^2}{2!} + \cdots = \sum_{n=0}^{+\infty} \frac{u^n}{n!}$$

Substitute i^2 for -1, i^3 for $-i$, i^4 for +1, *etc.*, to obtain the *Euler Relation*:

$$\begin{aligned}
\cos[\phi] + i\ \sin[\phi] &= \left(\frac{\phi^0}{0!} - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots\right) + i\left(\frac{\phi^1}{1!} - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \cdots\right) \\
&= \frac{\phi^1}{1!} + i\frac{\phi^1}{1!} + i^2\frac{\phi^2}{2!} + i^3\frac{\phi^3}{3!} + i^4\frac{\phi^4}{4!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(i\phi)^n}{n!} = e^{+i\phi}
\end{aligned}$$

5 Equivalent Expressions for z

$$z = \Re\{z\} + i\ \Im\{z\} = |z|\, e^{i\phi} = |z|\ (\cos[\phi] + i\ \sin[\phi])$$

Euler relation for product, reciprocal, and ratio of complex numbers

$$\begin{aligned}
z_1\ z_2 &= |z_1|\ e^{i\Phi\{z_1\}}\ |z_2|\ e^{i\Phi\{z_2\}} = |z_1|\ |z_2| e^{i(\Phi\{z_1\}+\Phi\{z_2\})} \\
\frac{1}{z_2} &= \frac{1}{|z_2|\, e^{+i\Phi\{z_2\}}} = \frac{1}{|z_2|} e^{-i\Phi\{z_2\}} \\
\frac{z_1}{z_2} &= \frac{|z_1|}{|z_2|} e^{i(\Phi\{z_1\}-\Phi\{z_2\})}
\end{aligned}$$

- Magnitude of ratio is ratio of magnitudes
- Phase of ratio is difference of phases.

6 Complex-Valued Functions

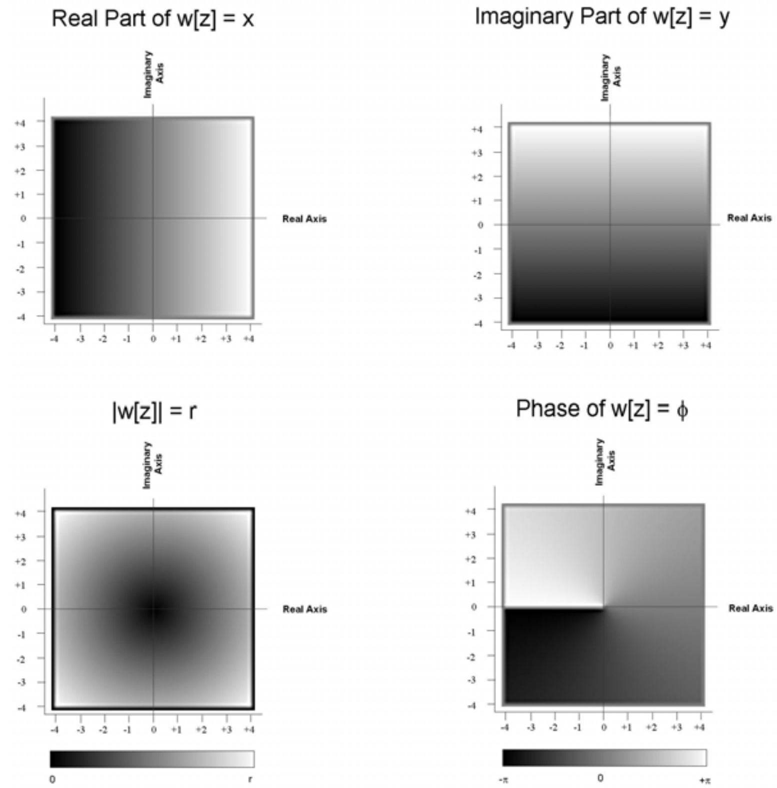
- The most common "complex functions" in imaging applications have a real-valued domain and a complex-valued range.
  - More restrictive definition than used in mathematical analysis: both domain and range are complex valued:

$$w[z] = w[x+iy]$$

$$w[z] = \Re\{w[x+iy]\} + i\ \Im\{w[x+iy]\} = |w[z]|\ e^{i\Phi\{w[z]\}}$$

  - Both $|w[z]|$ and $\Phi\{w[z]\}$ are real-valued functions evaluated for each location z in the complex plane.

– Both real and imaginary parts of $w[z]$ may be represented pairs of 2-D "images" as "gray values" for each coordinate $[x, y]$:



The complex function $w[z] = z = x + iy$, represented as real part, imaginary part, magnitude, and phase. Note the discontinuity in the phase angle at $\phi = \pm\pi$

- Analysis and manipulation of $w[z]$ is VERY useful in linear systems.
  - contour integration of $w[z]$ in the 1-D complex domain (equivalent to the 2-D real plane) is very useful when evaluating properties of some real-valued special functions, such as $SINC[x]$
- Concerned with more restrictive definition of complex functions with real-valued domains
  - Denoted by same symbols that have been used for functions with real-valued ranges, $f[x]$
  - f is a complex number (unless otherwise noted).

$$f[x] = \Re\{f[x]\} + i\,\Im\{f[x]\} \equiv f_R[x] + i\ f_I[x]$$

$$\begin{aligned}|f[x]| &= \sqrt{(f_R[x])^2 + (f_I[x])^2}\\ \Phi\{f[x]\} &= \tan^{-1}\left(\frac{f_I[x]}{f_R[x]}\right)\\ f_R[x] &= |f[x]|\ \cos[\Phi\{f[x]\}]\\ f_I[x] &= |f[x]|\ \sin[\Phi\{f[x]\}]\end{aligned}$$

6.1 Phase of a Complex-Valued Function:

- $\Phi\{f[x]\} = \tan^{-1}\left(\frac{f_I[x]}{f_R[x]}\right)$ may be evaluated for ANY complex-valued function
- $\Phi\left[\cos\left[2\pi\frac{x}{X}\right]\right] = 2\pi\frac{x}{X}$ applies only to sinusoids

6.2 Hermitian Function:

- Real part is even and imaginary part is odd
- Complex conjugate of a Hermitian function is equal to "reversed" function:

$$f^*[x] = f[-x] \Longrightarrow f^*[-x] = f[x]$$

6.3 Power of Complex Function:

- Power of complex function is 1-D real-valued function obtained by squaring the (real-valued) magnitude:

$$|f[x]|^2 = f[x] \times f^*[x] = (f_R[x])^2 + (f_I[x])^2$$

and obviously is called the *squared magnitude* of $f[x]$.

7 Complex Sinusoid

- - *Complex sinusoid* or *complex linear-phase exponential*:

$$f[x] = e^{+2\pi i\xi_0 x}$$

  - Real and imaginary parts of $f[x]$ are obtained from Euler relation at each value of the coordinate x:

$$\begin{aligned}e^{+i\theta} &= \cos[\theta] + i\ \sin[\theta]\\ \Longrightarrow e^{+2\pi i\xi_0 x} &= \cos[2\pi\xi_0 x] +\ i\sin[2\pi\xi_0 x]\end{aligned}$$

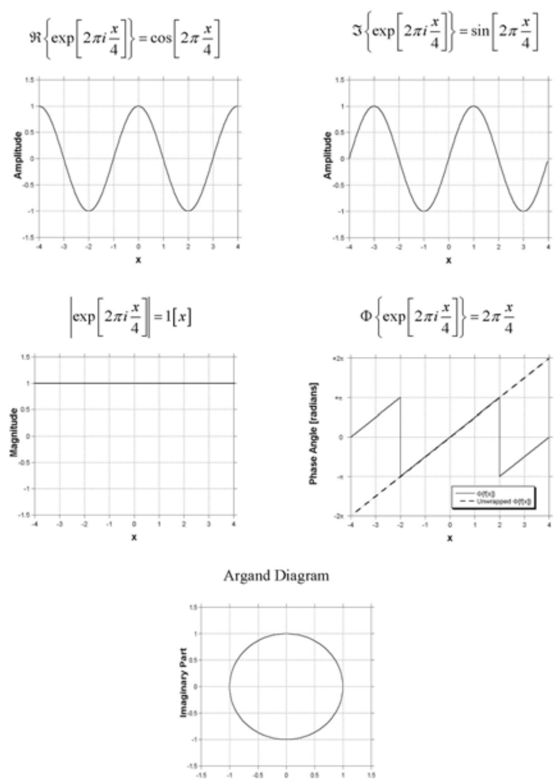
– Real and imaginary part of $e^{+2\pi i\xi_0 x}$ are identical to even and odd parts, respectively.

– Magnitude of the complex sinusoid is unity for all x

$$\left|e^{+2\pi i\xi_0 x}\right| = \sqrt{\cos^2\left[2\pi\xi_0 x\right] + \sin^2\left[2\pi\xi_0 x\right]} = 1$$

– Phase angle is linear function of x:

$$\begin{aligned}\Phi\left\{e^{+2\pi i\xi_0 x}\right\} &= \tan^{-1}\left[\frac{+\sin\left[+2\pi\xi_0 x\right]}{\cos\left[+2\pi\xi_0 x\right]}\right] \\ &= \tan^{-1}\left[\tan\left[+2\pi\xi_0 x\right]\right] = +2\pi\xi_0 x\end{aligned}$$



Representations of the complex sinusoidal function $f[x] = e^{+2\pi i\xi_0 x} = COS\left[2\pi\xi_0 x\right] + i\ SIN\left[2\pi\xi_0 x\right]$ as (a) real part; (b) imaginary part; (c) magnitude; (d) phase; and (e) Argand diagram.

- Real and imaginary parts of $f[x]$ are smoothly varying functions of x
- Phase is proportional to x
- Inverse tangent function is single-valued and continuous only over the range of the so-called "principal value", assumed to be $[-\pi, +\pi)$.
  – "Wrapped" phase computed in the principal interval; the phase exhibits a discontinuity from $+\pi$ to $-\pi$ at intervals of $(2\xi_0)^{-1}$is the "solid line" in *d*.
  – "Unwrapped" phase is necessary in some applications (e.g., complex logarithm)

- Phase unwrapping algorithm assumes that the derivative of the phase is constrained to some finite range.

- Complex conjugate of complex sinusoid is:

$$\begin{aligned}\left(e^{+2\pi i\xi_0 x}\right)^* &= e^{+2\pi(-i)\xi_0 x} = e^{-2\pi i\xi_0 x}\\ &= (\cos[2\pi\xi_0 x] + i\ \sin[2\pi\xi_0 x])^* = \cos[2\pi\xi_0 x] - i\ \sin[2\pi\xi_0 x]\\ &= \cos[-2\pi\xi_0 x] + i\ \sin[-2\pi\xi_0 x]\end{aligned}$$

  where the respective evenness and oddness of the cosine and sine functions have been used.

- Corresponding expressions for magnitude and phase of complex conjugate of the linear-phase complex exponential are:

$$|e^{-2\pi i\xi_0 x}| = \sqrt{\cos^2[-2\pi\xi_0 x] + \sin^2[-2\pi\xi_0 x]} = 1$$

$$\begin{aligned}\Phi\{e^{-2\pi i\xi_0 x}\} &= \tan^{-1}\left[\frac{+\sin[-2\pi\xi_0 x]}{\cos[-2\pi\xi_0 x]}\right]\\ &= \tan^{-1}[\tan[-2\pi\xi_0 x]] = -2\pi\xi_0 x\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\left(e^{2\pi i\xi_0 x} + e^{-2\pi i\xi_0 x}\right) &= \cos[2\pi\xi_0 x]\\ \frac{1}{2i}\left(e^{2\pi i\xi_0 x} - e^{-2\pi i\xi_0 x}\right) &= \sin[2\pi\xi_0 x]\,.\end{aligned}$$

8 DeMoivre's Theorem:

Generalize product of complex numbers to compute the n^{th} power of z :

$$\begin{aligned}z^n &= \left(|z|\, e^{i\theta}\right)^n = |z|^n\left[e^{i\theta}\right]^n = |z|^n[\cos[\theta] + i\ \sin[\theta]]^n\\ &= |z|^n\left[e^{in\theta}\right] = |z|^n[\cos[n\theta] + i\ \sin[n\theta]]\end{aligned}$$

- Equate the two expressions in θ to obtain *DeMoivre's Theorem*:

$$\left(e^{i\theta}\right)^n = \cos[n\theta] + i\ \sin[n\theta]$$

- Complex number raised to a numerical power n by raising magnitude to that power and multiply phase angle by same number.

- Essential for finding complex-valued roots of equations

$$e^{+i\theta}\ e^{\pm i\phi} = e^{i(\theta\pm\phi)} = \cos[\theta\pm\phi] + i\ \sin[\theta\pm\phi]$$

$$\begin{aligned}e^{+i\theta}\ e^{\pm i\phi} &= (\cos[\theta] + i\ \sin[\theta])(\cos[\phi] \pm i\ \sin[\phi])\\ &= \cos[\theta]\ \cos[\phi] \pm i^2\ \sin[\theta]\ \sin[\phi] + i\ \sin[\theta]\ \cos[\phi] \pm i\ \cos[\theta]\ \sin[\phi]\\ &= \cos[\theta]\ \cos[\phi] \mp \sin[\theta]\ \sin[\phi] + i\ (\sin[\theta]\ \cos[\phi] \pm \cos[\theta]\ \sin[\phi])\end{aligned}$$

Equating real and imaginary parts of right-hand sides to derive two familiar and useful trigonometric identities

$$\begin{aligned}\cos[\theta\pm\phi] &= \cos[\theta]\ \cos[\phi] \mp \sin[\theta]\ \sin[\phi]\\ \sin[\theta\pm\phi] &= \sin[\theta]\ \cos[\phi] \pm \cos[\theta]\ \sin[\phi]\end{aligned}$$

9 Representation of Complex Sinusoids:

- Specified completely by three real-valued quantities:
  - spatial frequency ξ_0
  - maximum amplitude A_0
  - phase angle at the origin ϕ_0(the *initial phase*)

$$(A_0)_R = A_0 \cos[\phi_0]$$
$$(A_0)_I = A_0 \sin[\phi_0]$$
$$A_0 = \sqrt{(A_0)_R^2 + (A_0)_I^2}$$
$$\phi_0 = \tan^{-1}\left[\frac{(A_0)_I}{(A_0)_R}\right]$$

10 Generalized Spatial Frequency – Negative Frequencies

$$\Phi\{f[x]\} = \tan^{-1}\left[\frac{f_I[x]}{f_R[x]}\right].$$

- Generalized concept of spatial frequency of a complex-valued function is obtained by defining spatial frequency as proportional to the "rate of change" of phase:

$$\xi[x] = \frac{1}{2\pi}\frac{\partial\Phi}{\partial x}$$

  (the higher the frequency, the faster the change in phase angle)

- Spatial frequency of complex unit-magnitude, linear-phase exponential is:

$$f[x] = 1[x] \;\; e^{+i\Phi\{f[x]\}} = e^{+2\pi i\xi_0 x} = \cos[2\pi\xi_0 x] + i\cdot\sin[2\pi\xi_0 x]$$
$$\Phi\{f[x]\} = \tan^{-1}\left[\frac{\sin[2\pi\xi_0 x]}{\cos[2\pi\xi_0 x]}\right] = \tan^{-1}[\tan[2\pi\xi_0 x]] = 2\pi\xi_0 x$$
$$\Longrightarrow \xi[x] = \frac{1}{2\pi}\frac{\partial\Phi\{f[x]\}}{\partial x} = \xi_0$$

- Spatial frequency of a complex function is constant if phase includes at most the sum of linear and constant functions of x.
- Phase of a 1-D complex-valued function may include higher-order functions of x
  - spatial frequency will vary with coordinate x.
  - ξ evaluated at specific coordinate x_0 is the *instantaneous* spatial frequency.
- Because spatial frequency is defined by a derivative, it may take on any real value in the infinite range $(-\infty, +\infty)$
- NEGATIVE spatial frequencies are possible
  - Concept may be disconcerting at first glance; the meaning of a spatial oscillation with a frequency of -1 sinusoidal cycle per mm may not be obvious.
  - Negative spatial frequency is analogous to readily accepted and visualized concept of negative velocity.
  - $f[x]$ is negative in all regions of the domain where the phase angle $\Phi\{f[x]\}$ *decreases* as x increases.

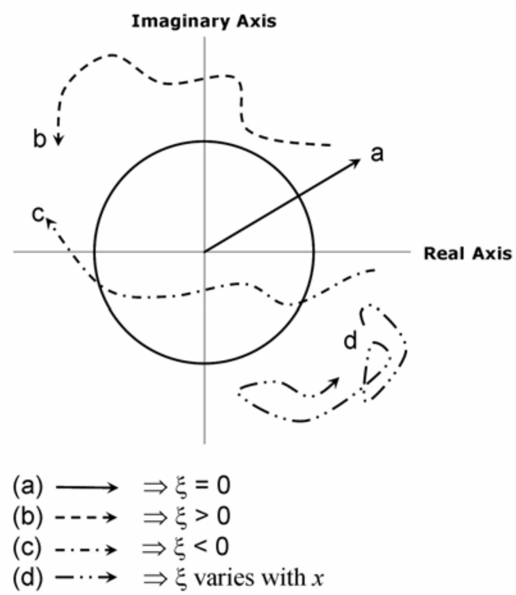


Figure 1: Argand diagrams of complex-valued 1-D functions of x with spatial frequencies that are null, positive, negative, and vary with x.

11 Argand Diagrams of Complex-Valued Functions

- Follow the "tip" of the phasor around the complex plane
- Analogous to simultaneous display of two functions on oscilloscope that is called a *Lissajous figure*