SIMG-716
Linear Imaging Mathematics I
01 - Motivation

1 Mathematical Descriptions of Imaging Systems

$$\text{Input to Imaging System: } f[x,y,z,\lambda,t] \text{ or } f[x,y,z,\nu,t]$$

$$\text{Output of Imaging System: } g[x',y',z',\lambda,t] \rightarrow g[x',y']$$

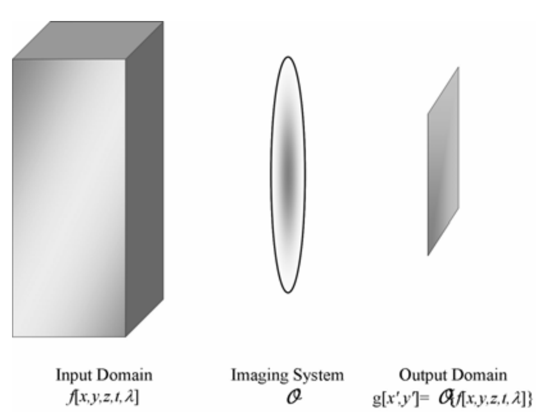
$$\text{Action of system specified by operator: } \mathcal{O}\{f[x,y,z,t,\lambda\} = g[x',y']$$

1. Representations of input f and output g are mathematical functions
   - Real-Valued Function: rule(s) that assign single number (the *amplitude*) to each location in multidimensional domain
   - Complex-Valued function: rule(s) that assign a complex amplitude to each location in multidimensional domain
     - complex amplitude may be written as sum of two real amplitudes in the form $f[x,y] = a[x,y] + i\ b[x,y]$ $(i \equiv \sqrt{-1})$
   - f and g generally defined over different domains
   - Often, domain of output g has fewer dimensions that domain of f, e.g.,

     $$\mathcal{O}\{f[x,y,z,\lambda,t]\} = g[x',y']$$

     - reduction in dimensionality affects ability to solve the inverse and system analysis imaging "tasks" (next section)

2. Mathematical description of system represented by operator $\mathcal{O}$ that relates the two functional representations of input and output – f and g
   - Action of system may relate input and output at same location (point operation): e.g., $g_1[x,y] = -2 \cdot f[x,y]$, $g_2[x,y] = f[x,y] + 2, g_3[x,y] = f[x-2,y+4]$
   - Action of $\mathcal{O}$ at one output location may combine input amplitudes over a local region of f (*neighborhood* operation, e.g., *convolution*)
   - $\mathcal{O}$ may combine input amplitudes over the entire input domain (*global* operation, e.g., image *transformation: Fourier, Radon*)



Schematic of imaging system that acts on a time-varying input with three spatial dimensions and color, $f[x,y,z,t,\lambda]$ to produce 2-D monochrome (gray scale) image $g[x',y']$.

2 Three Imaging "Tasks"

1. *Forward* or *direct* problem:

   find mathematical expression for image $g[x', \cdots]$ given complete knowledge of the input object $f[x, \cdots]$ and the system $\mathcal{O}$;

   $$\mathcal{O}\{f[x, \cdots]\} = g[x', \cdots]$$

2. *Inverse problem* (the most important and generally most difficult)

   expression for input $f[x, \cdots]$ is evaluated from measured image $g[x', \cdots]$ and known action of the system $\mathcal{O}$; goal is to find an *inverse operator* $\mathcal{O}^{-1}$:

   $$\mathcal{O}^{-1}\{g[x', \cdots]\} = f[x, \cdots]$$

3. *System analysis* problem:

   action of operator $\mathcal{O}$ must be determined from input $f[x, \cdots]$ and image $g[x', \cdots]$.

   - (similar in form to the inverse problem in many cases)

Variants:

1. knowledge of some or all of f, g, and $\mathcal{O}$ may be incomplete
2. Output image may be contaminated by random noise in inverse or system analysis task.

- If output domain has fewer dimensions than input domain, e.g., "depth" dimension is lost when mapping 3-D volume to 2-D plane via:

  $$\mathcal{O}\{f[x, y, z]\} = g[x, y]$$

  then the inverse imaging task is "ill posed" and the inverse operator does not exist – you cannot recover the "depth" information without additional data

  1. must modify the imaging system (and therefore the imaging operator **O**)
  2. must gather more measurements – a third dimension – to recover the depth
  3. Example: computed tomography

3 Examples of Imaging Tasks:

Consider solution of the imaging tasks for a few systems

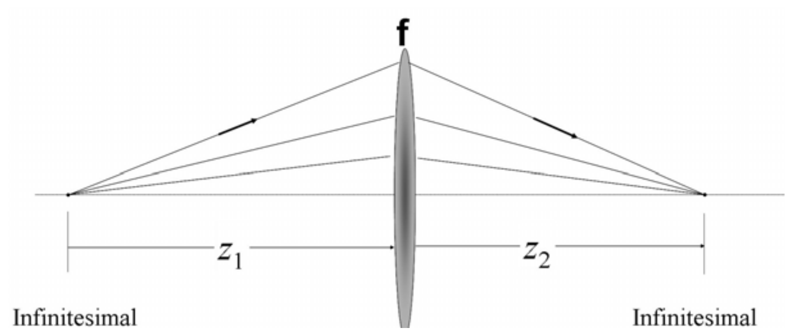
1. Optical imaging with different models of light
   (a) Ray Optics (geometrical optics)
   (b) Wave Optics (physical optics)
2. Optical imaging tasks
   (a) System Analysis of Hubble Space Telescope
   (b) Recovering information that is "lost" due to atmospheric effects
3. Medical Imaging
   (a) Gamma-ray imaging ("nuclear medicine")
      i. Pinhole Camera
      ii. Multiple Apertures
      iii. Coded Apertures
   (b) Radiography (X-ray imaging)
      i. loss of depth information by imaging system
      ii. Multiple-angle radiography (computed tomography = CT)

3.1 Optical Models of Imaging

Two models for light propagation:

1. as “rays”
   - simple model ($\lambda \to 0$)
   - obeys Snell’s law at interface
2. as “waves”
   - more complicated mathematics (but not “that” hard)
   - waves for different colors have different wavelengths/temporal frequencies

3.1.1 Imaging Tasks in Ray Optics Model:



Perfect optical imaging system in “ray optics” model with no aberrations. All rays from a single source point converge to “image” point.

1. Point source of energy emits geometrical “rays” of light that propagate in straight lines to infinity in all directions.
2. Imaging “system” is an optical “element” that interacts with any intercepted ray.
3. Physical process of interaction (usually refraction or reflection, sometimes diffraction) “diverts” the ray from its original propagation direction.
4. Rays are collected by a sensor
5. Spatial “density” of rays at sensor determines “strength” of measured signal (radiometry)

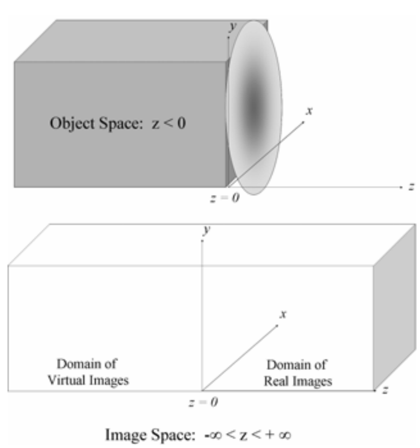
Mathematical Model of Ray-Optics Imaging: Very simple model of imaging with three parameters: one each for system, input, and output:

1. *System* is specified by "focal length" $\mathbf{f}$
2. *Input* specified by "object distance" z_1 (from object to optic)
3. *Output* specified by "image distance" z_2 (from optic to image)
4. *Imaging Equation* relates these three quantities

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{\mathbf{f}}$$

("focal length is the reciprocal of the sum of the reciprocals of the object and image distances")

- Note that z_2 increases as z_1 decreases until $z_1 < \mathbf{f}$, for which $z_2 < 0$ ("virtual" image)
- Imaging equation "pairs up" object and image planes located at distances z_1 and z_2 for a "fixed system" with focal length $\mathbf{f}$.
- We rarely seem to think about action of lens for:
  - 3-D objects, so that several object distances exist that are mapped to different image distances
  - "out-of-focus" planar objects located at a distance z_1 that does not satisfy the imaging equation for a fixed focal length $\mathbf{f}$ and image distance z_2, even though there are many obvious situations where we might want to calculate the appearance of such an image.
- Example:
  - 1. lens with focal length $\mathbf{f} > 0$ placed in plane $[x, y, 0]$
    2. object is to left of lens, so that domain of object space $[x, y, z < 0]$
    3. domain of "image space" is set of all locations where image is formed:
       (a) $z_1 > \mathbf{f} > 0 \implies z_2 > 0 \implies$ image is in the volume $[z, y, z > 0]$
       (b) $z_1 = \mathbf{f} > 0 \implies z_2 = \infty \implies$ no "image" is formed
       (c) $\mathbf{f} > z_1 > 0 \implies z_2 < 0 \implies$ image is formed "to right of lens" or "in front of lens" $\implies$ "virtual" image
  - domain of object space spans the "half" volume $[x, y, z < 0]$
  - domain of image space spans the full volume $[x, y, -\infty \leq z \leq +\infty]$



(top) Schematic of locations of "real" objects (i.e., "object space") and (bottom) corresponding image space for lens with focal length $\mathbf{f} > 0$ in ray optics model.

Solutions to Imaging Tasks:

1. Direct Task: Given z_1 and $\mathbf{f}$, find z_2:

$$\left(\frac{1}{\mathbf{f}} - \frac{1}{z_1}\right)^{-1} = z_2$$

2. Inverse Task: Given z_2 and $\mathbf{f}$, find z_1:

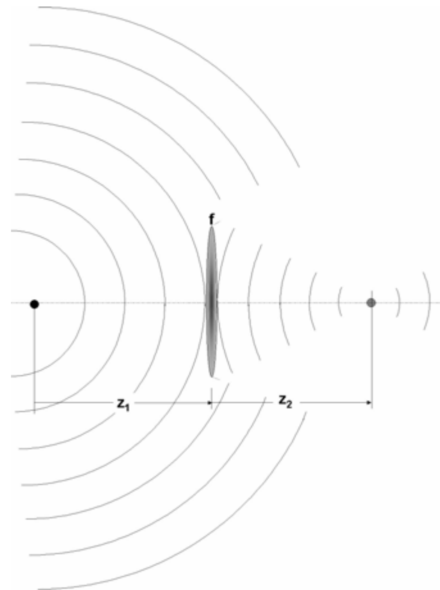
$$\left(\frac{1}{\mathbf{f}} - \frac{1}{z_2}\right)^{-1} = z_1$$

3. System Analysis Task: Given z_1 and z_2, find $\mathbf{f}$:

$$\left(\frac{1}{z_1} + \frac{1}{z_2}\right)^{-1} = \mathbf{f}$$

3.1.2 Wave Optics Model:

Essential to model unavoidable effects of *diffraction*



Wave model of light propagation in imaging system: several spherical waves shown were emitted by point source at times separated by identical intervals. Lens intercepts portion of each wavefront and alters curvatures. Waves propagate until converging to a (more or less) finite-sized image point.

- Each source point emits spherical waves of e-m radiation
  - propagates outward from the source at velocity of light.
  - radiation at all points on one wave surface emitted at same instant of time in a direction perpendicular to wave surface, thus corresponding to the ray in the simpler model.
- Rays may be modeled as "normal" to wavefronts
- Propagation from point source creates spherical wavefronts
  - all points on same wavefront were emitted by source at same instant of time ($\Longrightarrow$ all points on one wavefront have same *phase angle*)
- Function that describes spherical wave is valid everywhere, not than just along individual ray.
- Optical element (lens or mirror) intercepts portion of each spherical wavefront and changes its radius of curvature
- Model suggests another interpretation of the action of optical system
  - System attempts to produce "replica" of source point at new location (the "image")
  - Success of reproduction is determined by diameter of image of point source
    - diameter *decreases* as fidelity improves.
  - Quality of image produced by "flawless" optical system (without aberrations) *improves* if system intercepts larger portion of outgoing wave.
    - size of diffracted "point image" decreases as size of optic increases (assuming no aberrations).

Concept of "Resolution"

- Overlapping "blurred" images may be difficult or impossible to distinguish.
- Details of image are "averaged" together by diffraction
- Ultimate limit on capability of imaging system to resolve fine detail

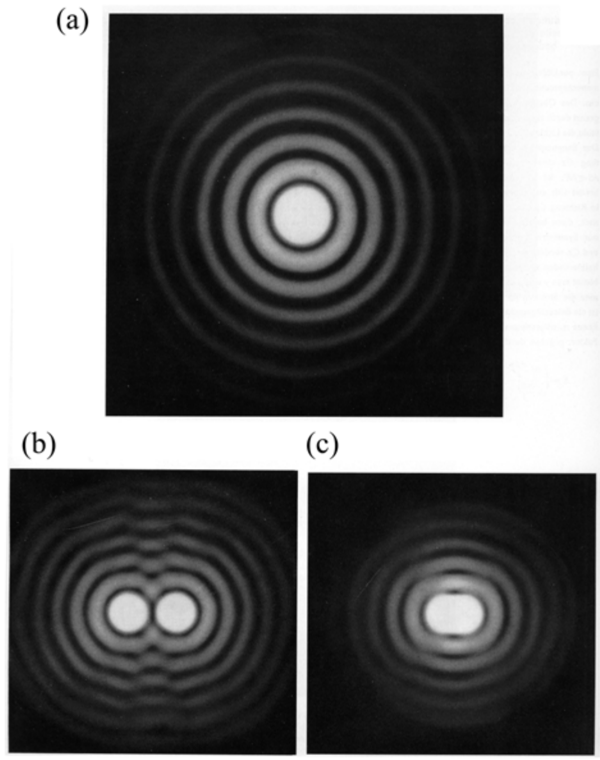
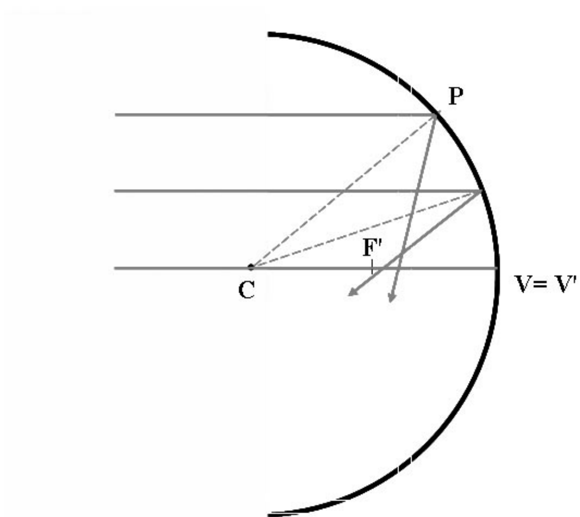


Illustration of resolution limit due to diffraction: (a) contrast-enhanced appearance of image of single point object (an "impulse"), showing diffractive "spreading;" (b) image of two such point objects showing overlap of the diffraction patterns; (c) image of same two point objects positioned closer together, showing difficulty of task to distinguish the pair.

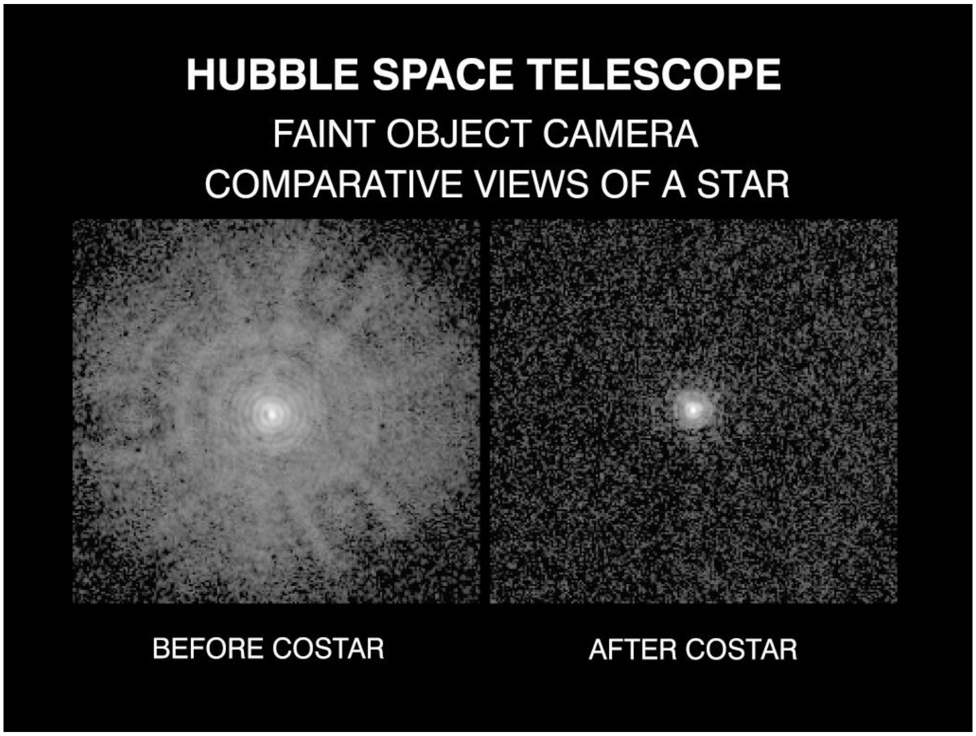
3.1.3 Analysis of Hubble Space Telescope optics

- System Analysis Task: Given knowledge of input $f[x,y]$ and output $g[x,y]$, find the mathematical equation for the action of the system $\mathcal{O}$.
- HST optics suffered from *spherical aberration* due to manufacturing defects
  - light incident on different parts of mirror focused at different distances from system vertex **V** (intersection of mirror with axis of symmetry)



Spherical aberration: parallel rays from object at infinity at different hights from axis are reflected by spherical mirror and cross axis at different distances, creates a "fuzzy" image.

- Necessary to determine formula(e) for corrector optics (*Corrective Optics Space Telescope Axial Replacement* = "*COSTAR*")
  - examine images of "known" objects
  - infer action of system that created those images
  - construct compensating optics
  - compensating optics now "built into" replacement instruments; COSTAR now removed and exhibited at National Air and Space Museum



Images of star from HST before and after COSTAR. (a) exhibits spherical aberration of primary mirror.

HST Images before (left) and after installation of COSTAR corrector

3.1.4 Imaging by Ground-Based Telescopes through Atmosphere:

1. Optical propagation disrupted by atmospheric effects, local variations of air temperature

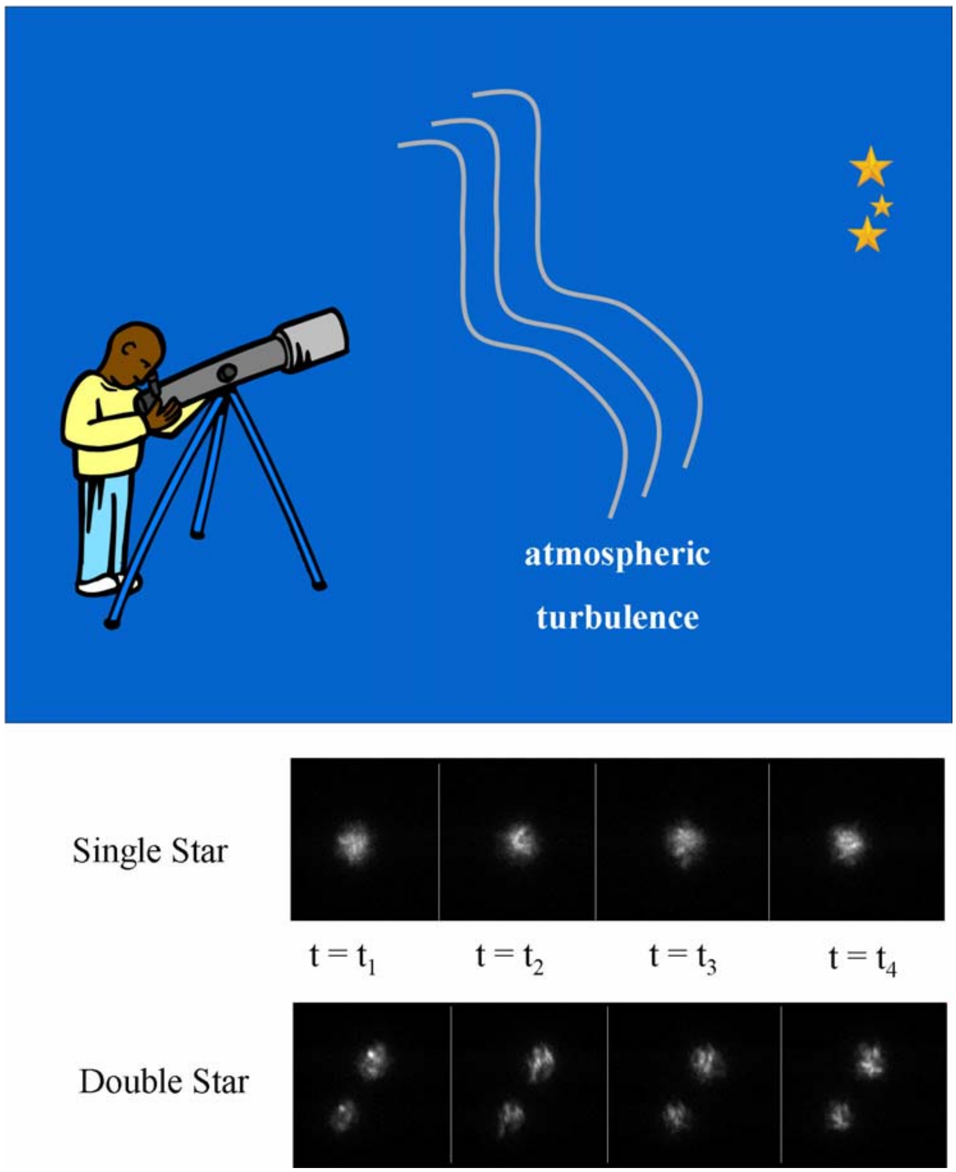
   (a) Resolution of telescope is limited by atmosphere, not by optics

      i. Localized "tilts" and "focus" changes are applied to incoming plane wave from distant object

      ii. due to local variations in index of refraction n

      iii. as air temperature $T \uparrow$, air density $\rho \downarrow \Longrightarrow$ refractive index $n \downarrow$

      iv. refractive index varies with local temperature that varies with time $\Longrightarrow$ optics of system produce different images over time



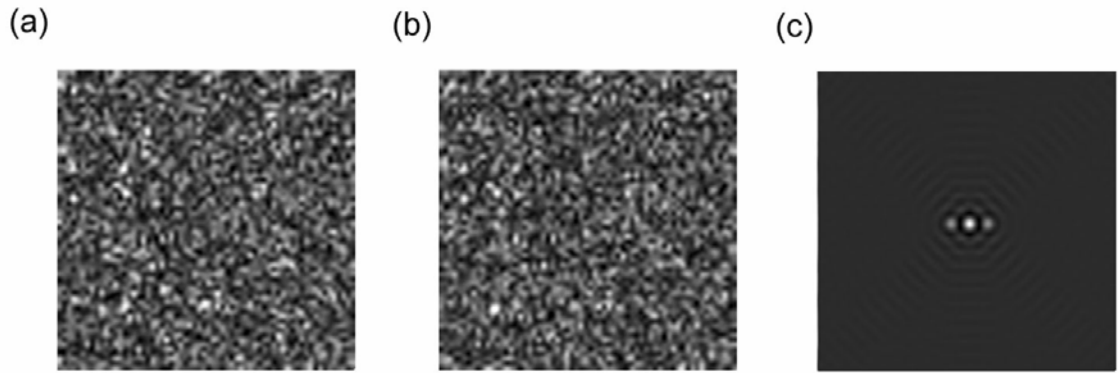
Atmospheric effects on star imagery: temporal variations in local temperature cause variations in refractive index; images of single and pair of stars taken through WIYN telescope are shown at four different times. Note that the two images of the double star have nearly identical forms at the different times.

2. Modify imaging system to deduce additional information about object from image or sequences: “Stellar speckle interferometry” by Labeyrie, 1970; very short exposures $\Longrightarrow$ “freeze” atmospheric effects in each, but few photons

   (a) Simulated short-exposure image of one star through “frozen” turbulent atmosphere generates pattern of “speckles”

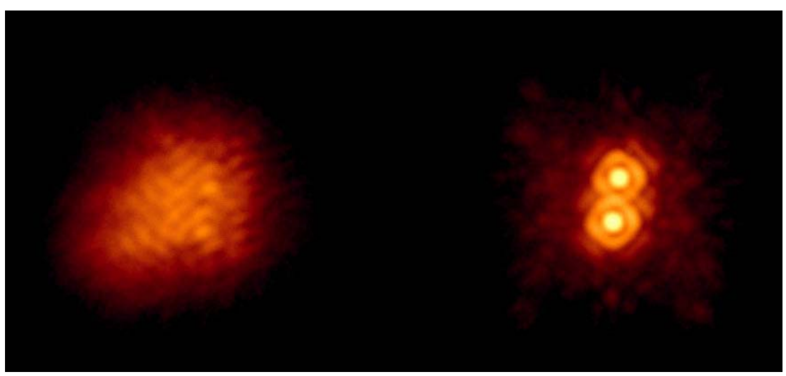
   (b) Each star in unresolved pair imaged through turbulent atmosphere generates identical patterns of “speckles” that appear "doubled"

   (c) Each image in set of short-exposure images of pair is processed to evaluate “autocorrelation;” averaged to form image (c) that conveys information about separation and angle of pair of stars

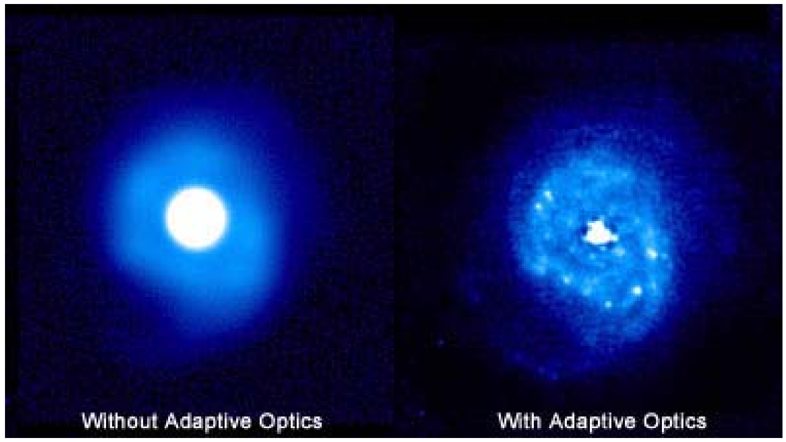


Stellar speckle interferometry: (a) image obtained from single star through turbulent atmosphere, showing “speckles” from refractive effects ; (b) corresponding image of unresolved double star – though difficult to discern, the “speckles” are paired; (c) result of Labeyrie algorithm applied to sequence of images of double-star speckles as in (b) to yield an estimated “autocorrelation” that reveals the separation of the two stars.

3. Now possible to measure effect of atmosphere on image in real time and compensate with optical element (“rubber mirror” = “adaptive optics”); examples shown of an unresolved double star using the Hale 200− in telescope on Palomar Mountain (www.astro.caltech.edu/palomar/AO/), the galaxy NGC 7469 (http://cfao.ucolick.org/ao/why.php), and Titan (largest moon of Saturn) through a ground-based conventional telescope, the Hubble Space telescope, and the Keck Telescope with adaptive optics (http://cfao.ucolick.org/ao/why.php).

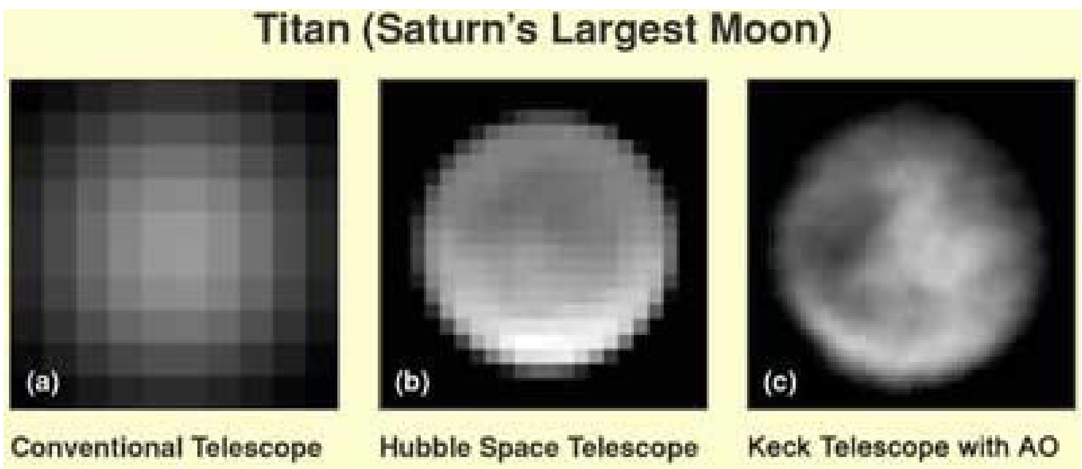


Images of double star *IW Taurii* with separation of 0.3 arcseconds taken with Hale telescope on Palomar Mountain ($d_0 = 200\,\text{in}$). The first image is taken through a turbulent atmosphere and the second image shows the image after turning on the adaptive optical compensation. The two stars are resolved by the adaptive optical system.



Images of galaxy NGC 7469 without and with adaptive optics.

Adaptive optics applied to one of the two Keck telescopes with effective aperture diameter $d_0 = 10\,\mathrm{m}$ has 20 times the light-gathering power as the Hubble Space Telescope and a theoretical resolution that is 4-5 times better if the atmospheric effects are compensated.



Comparison of images of Titan (largest moon of Saturn): (a) simulated image through ground-based conventional telescope; (b) image using Hubble Space Telescope ($d_0 = 2.4\,\mathrm{m} \cong 94.5\,\mathrm{in}$, $\Delta\theta \cong 0.05\,\mathrm{sec}$); (c) image using ground-based Keck telescope ($d_0 = 10\,\mathrm{m}$, $\Delta\theta \cong 0.01\,\mathrm{sec}$) and adaptive optics

3.2 Medical Imaging Applications

3.2.1 Gamma-Ray Imaging

Image gamma rays (high-energy photons) emitted by object of interest
e.g., kinetic energy of gamma rays emitted by technetium is ≃140 keV

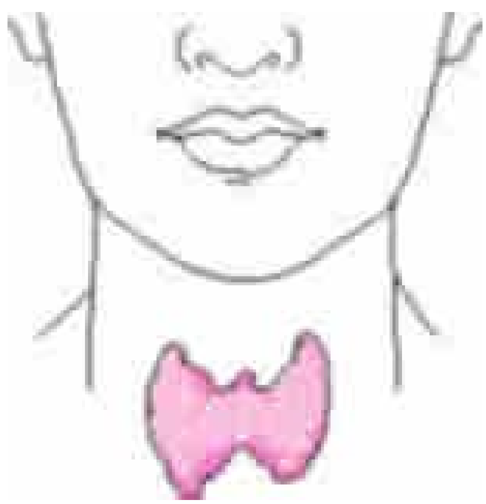
- $E \simeq 0.14\,\text{MeV} \simeq 2.24 \times 10^{-7}\,\text{erg}$ per photon
- Gamma rays have very short wavelength:

$$\lambda_\gamma = \frac{hc}{E} \simeq 9 \times 10^{-12}\,\text{m} \ll \lambda_{\text{visible}} \simeq 5 \times 10^{-7}\,\text{m}$$

- Gamma-ray photons pass through optical refractors virtually without deviation, and reflect only at very shallow "grazing" angles of incidence (as used by optics in *Chandra* Orbiting X-Ray Observatory)

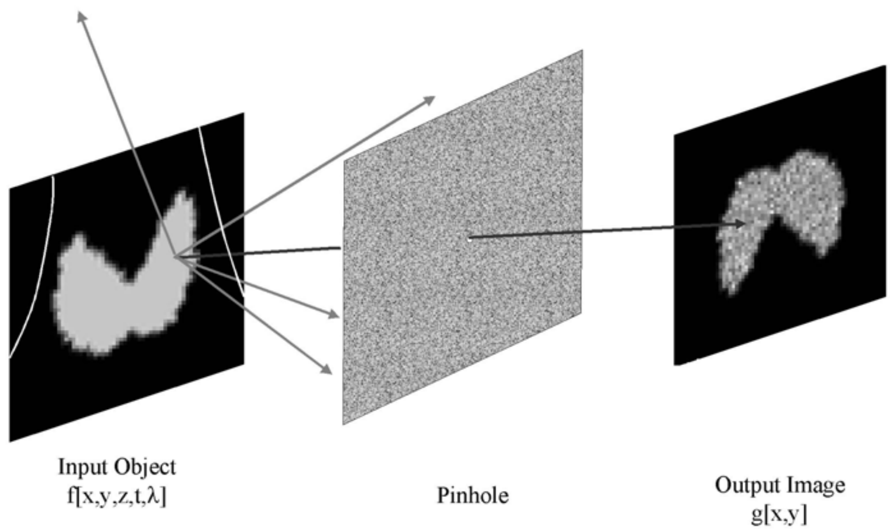
Grazing incidence optics used to focus X rays in Chandra orbiting X-ray observatory

- Gamma ray imaging of thyroid gland:



Location of thyroid gland in chest just below neck

- SImplest gamma-ray imaging system “selects” gamma rays rather than “redirects” them to form image
  - “pinhole” in absorptive sheet of led
  - only gamma rays that travel through pinhole can reach sensor
  - most gamma rays are discarded $\Longrightarrow$ “inefficient” system



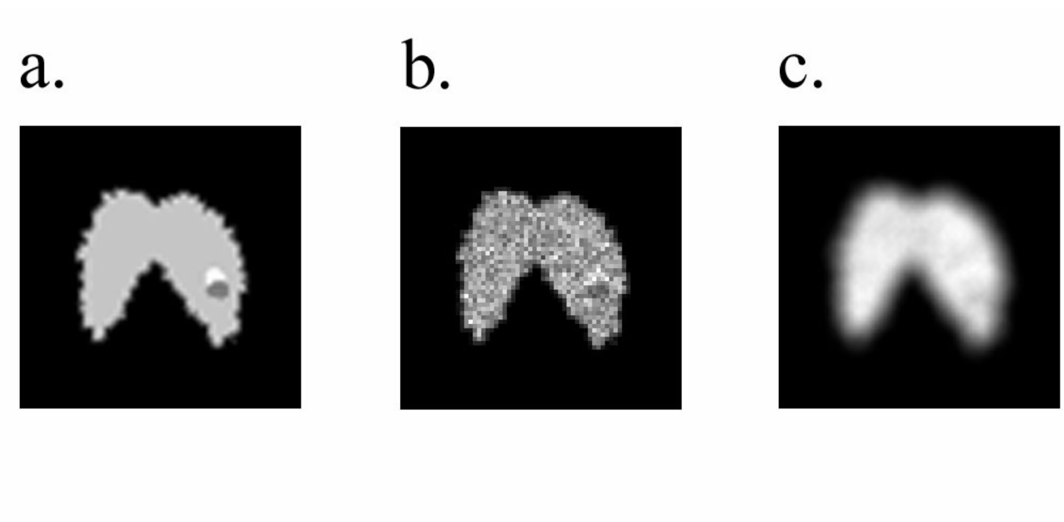
Schematic of gamma-ray pinhole imaging of thyroid gland. Iodine tagged with radioactive tracer is selectively absorbed by the thyroid. Energetic gamma-ray photons are emitted in all directions. A small percentage travel along paths that pass through pinhole in lead plate to expose sensor and form planar image $g[x, y]$.

Resolution of Gamma-Ray Imaging System

1. - Spatial resolution depends on d_0, diameter of pinhole
     - smaller hole $\Longrightarrow$ better spatial resolution
   - "Brightness" resolution (ability to measure differences in number of photons counted) depends on number of photons counted
     - Poisson noise in number N of counted photons
       * mean number of counted photons is N
       * variation in number of counted photons (the *standard deviation*) $\sigma = \sqrt{N}$
       * "signal-to-noise ratio" is

$$SNR = \frac{N}{\sigma} = \frac{N}{\sqrt{N}} = \sqrt{N}$$

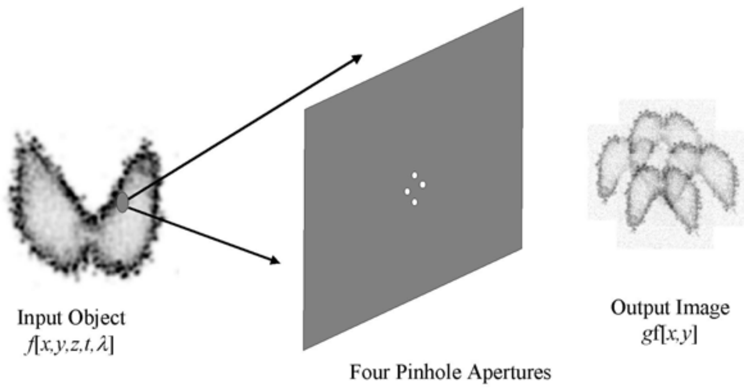
       * want to COUNT MORE PHOTONS to improve "brightness resolution"
     - Need to count more photons $\Longrightarrow$ better "brightness" resolution $\Longrightarrow$ either:
       (a) emit more photons $\Longrightarrow$ larger dose of radioactivity to patient
       (b) larger hole $\Longrightarrow$ worse spatial resolution
   - A QUANDARY!
2. Need to modify imaging system to address problem!



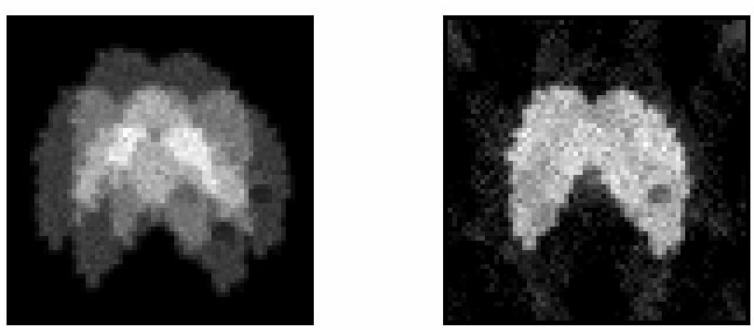
Simulation of effect of pinhole diameter d_0 on spatial resolution. (a) $f[x,y]$ is simulated thyroid with "hot" & "cold" spots; (b) simulated image obtained using small pinhole w/ added noise due to small number of counted photons. (c) Simulated image with large d_0, showing reduction in noise by averaging but "blur" due to exposure of different points on sensor by photons from same object point.

Strategy #1: Multiple Pinhole Imaging of Gamma Rays

1. Collect images through several small pinholes at once
2. Images may overlap
3. Postprocess to reconstruct image of object with improved signal-to-noise ratio (SNR)



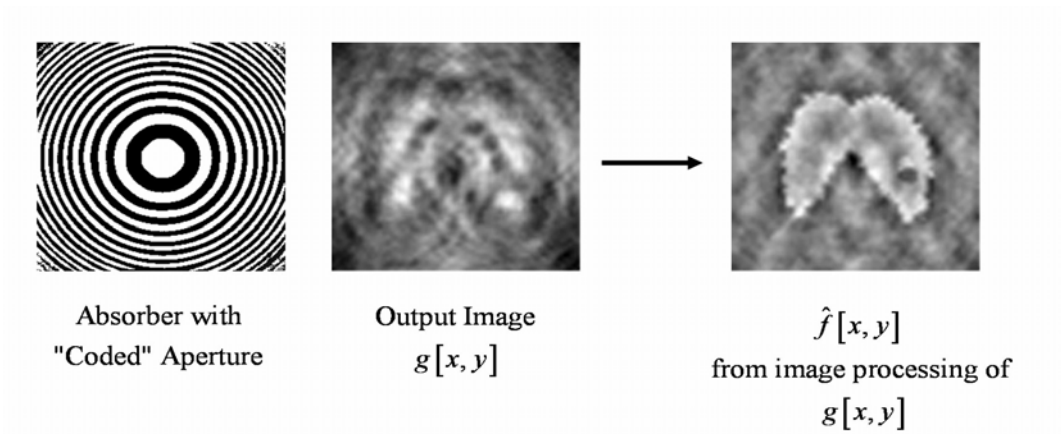
Gamma-ray camera with four pinholes, producing four overlapping images.



Simulation of "raw" output and subsequently processed gamma-ray images from camera with four pinholes (a) Raw image is the sum of four overlapping images. (b) The result after processing the image to "merge" the four overlapping images.

Strategy #2: "Coded Aperture Imaging" of Gamma Rays

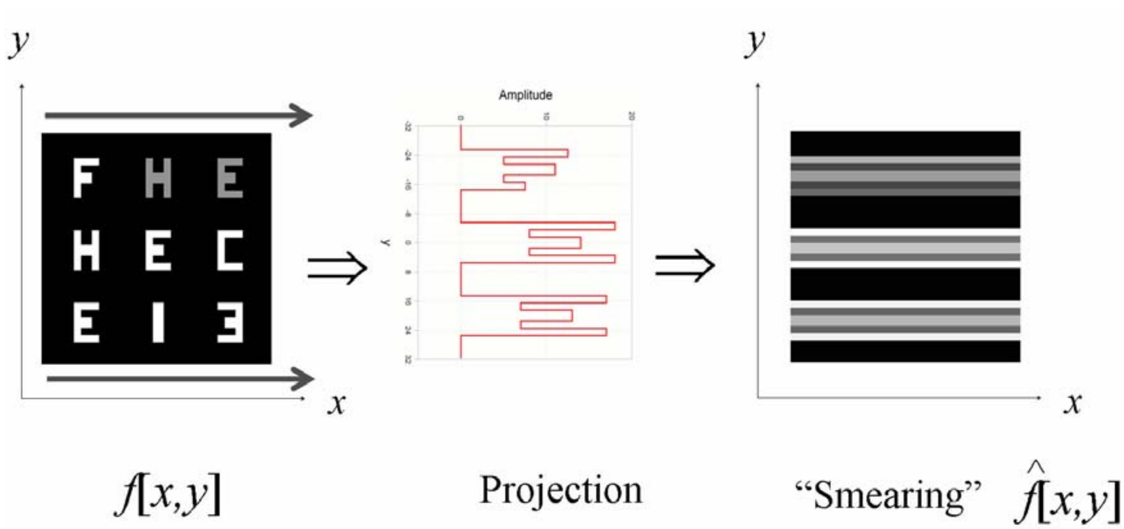
1. Drill out lead plate to open more "pinholes"
2. Pinholes merge together to form regions of "open space"
   (a) 50% of lead removed in this example
   (b) $\Longrightarrow$ MANY more photons transmitted to detector
   (c) Images from "pinholes" overap.
3. Pattern of detected photons is processed by a mathematical algorithm based on the pattern of pinholes
4. Postprocess to "reconstruct" (approximation of) original object with improved SNR.



Simulation of imaging through an "infinite" number of pinholes laid out in configuration in (a) ("coded aperture"); (b) output "image" obtained through this aperture; (c) "reconstructed" image obtained by postprocessing of (b) using knowledge of (a).

3.2.2 Radiography (X-Ray Imaging)

1. energetic X rays propagate in straight lines (as "rays")
2. Absorptions due to interactions with object reduce flux of X rays
3. Transmitted X rays measured by detector, fewer measured $\Longrightarrow$ more were absorbed
   (a) Reduction in dimensionality: 3-D object $f[x, y, z]$ measured as 2-D image, e.g., $g[x, y]$ $\Longrightarrow$ information about "depth" has been lost
   (b) Example shows a 2-D object $f[x, y]$ that is imaged as the 1-D function $g[x]$ by X rays traveling from top to bottom
4. All "depth" information about the absorption is lost; absorption could have occurred anywhere along the horizontal path as shown.



Reconstructed image $\hat{f}[x, y]$ from one projection.

5. So, how to recover depth information?

3.2.3 Computed Tomographic Radiography (CT)

1. Modify X-ray imaging system to measure X-ray absorption of object over "complete" set of azimuth angles.

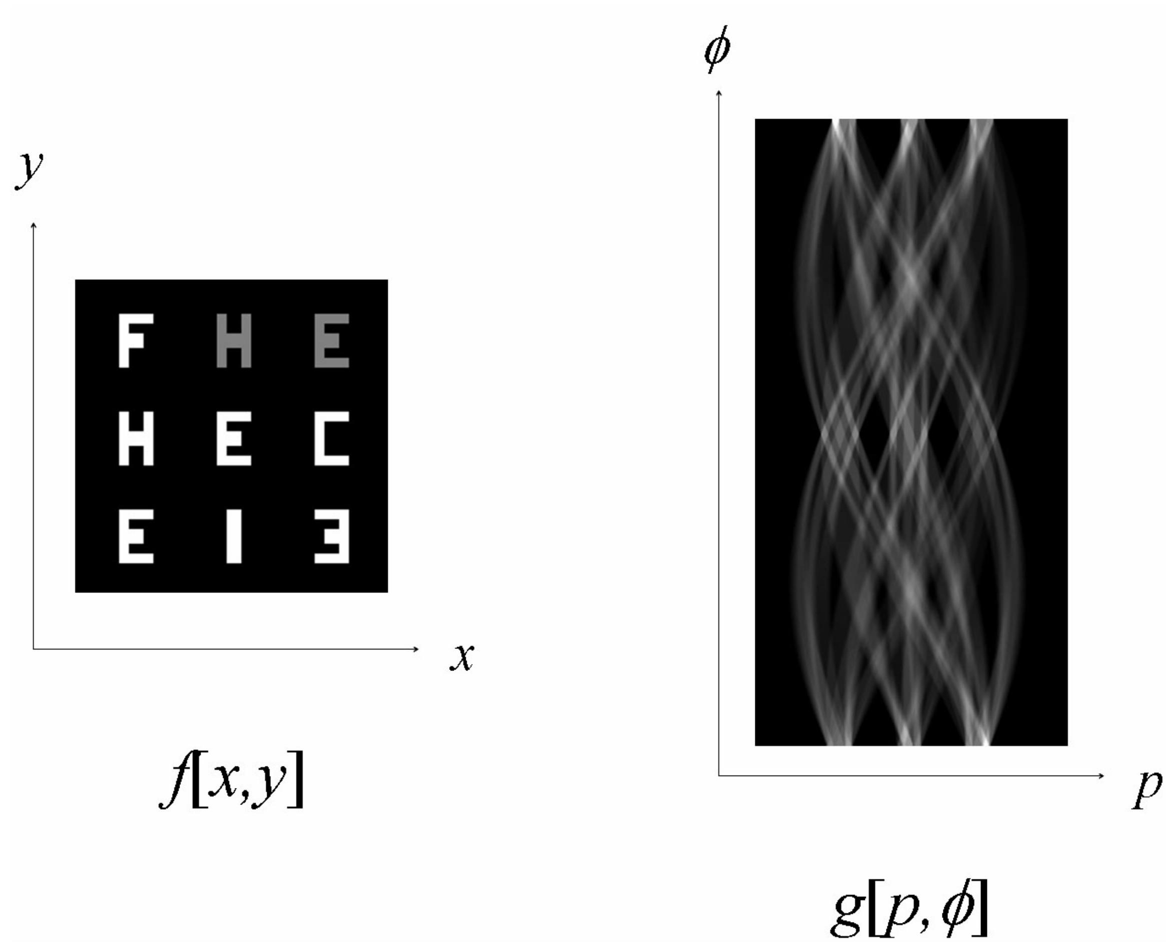
   (a) Example shown for 2-D object $f[x, y]$

   (b) At each azimuth angle ϕ: 1-D image $g[x]$ is the "sum" of the X-ray absorption along parallel lines (the "projections" of $f[x, y]$)

   (c) Ensemble of projections for set of azimuth angles produces 2-D "hybrid" image $g(x, \phi)$ (with both spatial and angle dimensions)

   (d) Plot of each location $[x, y]$ is a piece of sinusoid in (p, ϕ); representation is a "sinogram"

   (e) Reconstruction requires figuring out "lightnesses" of all points in objec that produce observed mix of sinusoids in $g(p, \phi)$

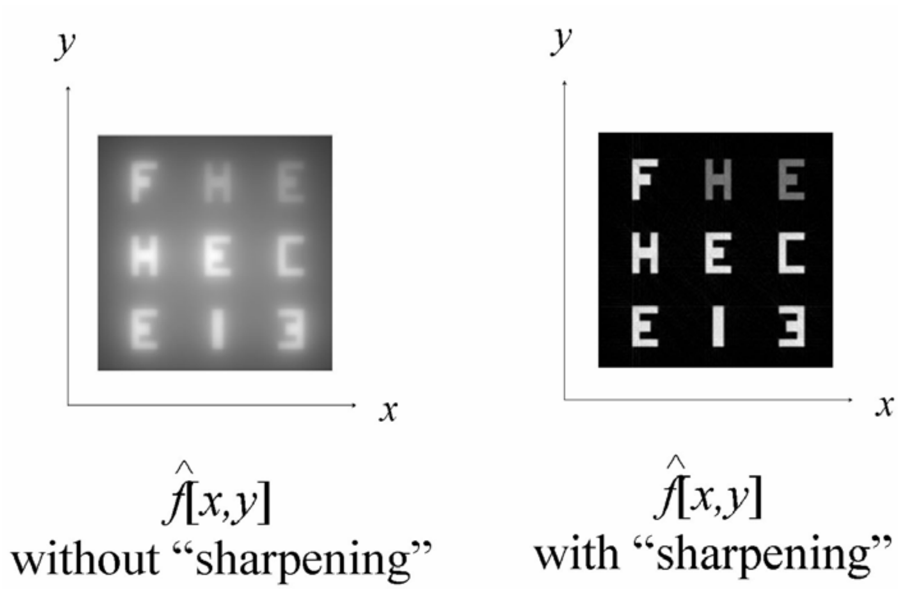


Simulation of data "projection" in X-ray computed tomography. X-ray transmission measured at each angle ϕ to compute "projection image" $g(p; \phi)$ with one spatial and one angle coordinate. Each point in original object becomes sinusoidal wave in projection image, hence name of representation as "sinogram."

2. Since each point in $[x, y]$ becomes a sinusoid in (p, ϕ), the reconstruction process may be described as figuring out the "lightness" of the input point that would generate that sinusoid. Recovery of $f[x, y]$ from $g(x, \phi)$ is the *inverse Radon transform* (Johan Radon, 1917), several algorithms exist, one method listed

   (a) "Smear" line of image data for each ϕ "back" into the 2-D space to produce "blurry" reconstruction

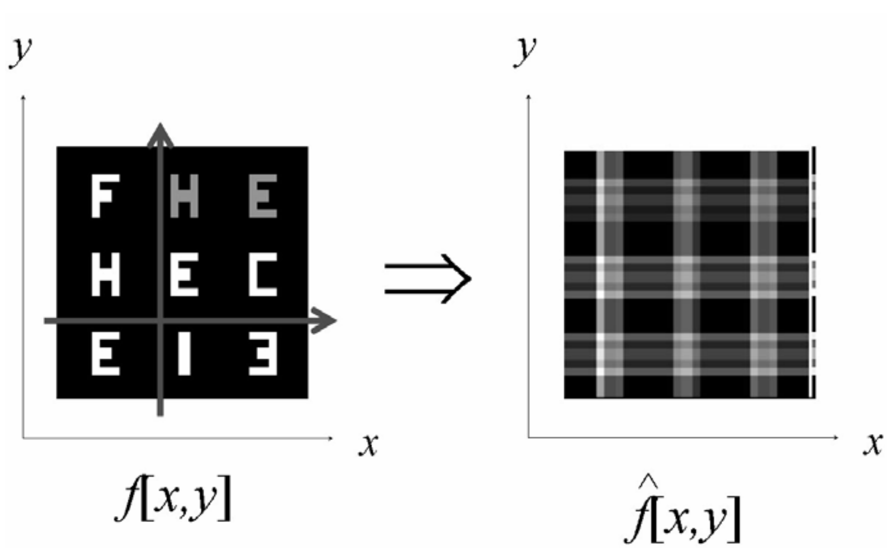
   (b) Apply "filter" to the reconstruction to "remove the blur" (i.e., to "sharpen" the image); the result is an "estimate" of the original object (denoted by the "carat" $\hat{f}[x, y]$)



Simulation of result of X-ray computed tomographic reconstruction. The "raw" reconstruction on left is "blurry." Image "sharpening" recovers the "sharp" image on the right.

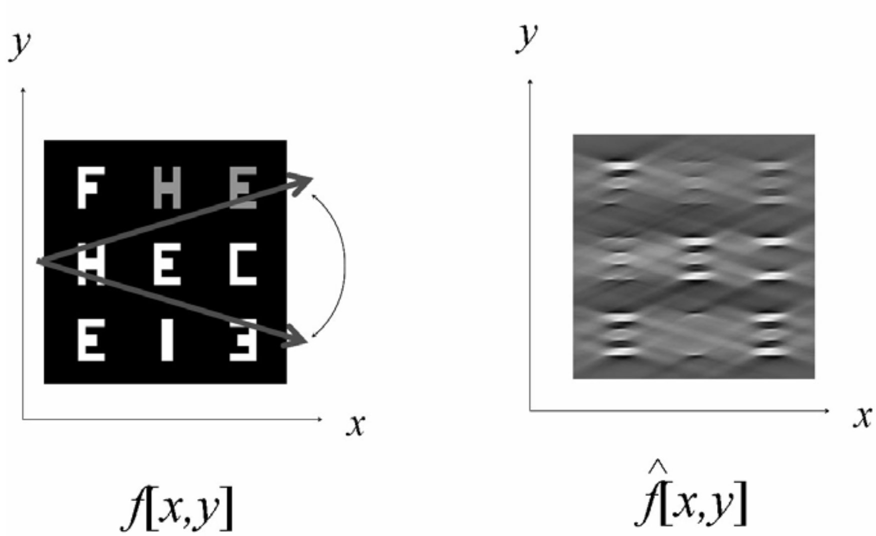
3. Reconstructions from data collected over subsets of azimuth angles

   (a) Two orthogonal angles, shows some information about the locations of the individual characters.



Original image and reconstruction from data collected at two angles separated by 90°.

(b) Limited range of angles ($-15° \leq \theta \leq +15°$)



Original image and reconstruction from data collected over limited range $-15° \leq \theta \leq +15°$, showing accurate renderings of horizontal lines

(simulations were generated using three different tools: custom software that I wrote, ImageJ with the Radon transform plugin, and Matlab with the Image Processing Toolbox)

4 Tools: Forward and Inverse Fourier transforms

- Alternative representation of spatial sinusoidal function with amplitude A_0, period X_0, and "initial phase" angle ϕ_0:
$$f[x] = A_0 \cos\left[2\pi \frac{x}{X_0} + \phi_0\right]$$
where ϕ_0 is a dimensionless quantity ("measured" in radians) and X_0 is a length (e.g., measured in mm)

- Recast spatial period as its reciprocal ξ_0:
$$\xi_0 = X_0^{-1}$$
measured in "reciprocal length (e.g., mm^{-1}, such as "cycles per millimeter"); ξ_0 is a "spatial frequency"
$$f[x] = A_0 \cos[2\pi\xi_0 x + \phi_0]$$

- Concept: the function $f[x]$ with infinite support is competely specified by the three quantities A_0, ξ_0 (or X_0), and ϕ_0 :

- (Almost) any function may be written as ("decomposed" into) unique sum of sinusoidal components with amplitude A and initial phase ϕ specified for each spatial frequency ξ
$$f[x] \iff (A[\xi], \phi[\xi])$$
  - Original function $f[x]$ is representation "in space domain" (because variable x has dimensions of length)
  - Alternate equivalent representation $(A[\xi], \phi[\xi])$ is "in frequency domain" because variable ξ has dimensions of reciprocal length ("spatial frequency" measured in "cycles per unit length)

- Common notation for the frequency-domain representation uses the same character as the space-domain representation, but in upper case, e.g.
$$(A[\xi], \phi[\xi]) \equiv F[\xi]$$

- The two representations are *equivalent*:
$$f[x] \iff F[\xi]$$

- The process of evaluating frequency-domain representation from space-domain representation is the *Fourier transformation* (or "Fourier transform" or "forward Fourier transform")

- We denote operator for Fourier transform by calligraphic letter "$\mathcal{F}$":
$$F[\xi] = \mathcal{F}\{f[x]\}$$

- "Fourier transformation decomposes $f[x]$ into its frequency components;" also may be called *Fourier analysis.*

- Because $F[\xi]$ represents function $f[x]$ in terms of its frequency components, is often called the *spectrum* of $f[x]$

- Process of evaluating space-domain function $f[x]$ from spectrum $F[\xi]$ is the *inverse Fourier transformation* (or just "inverse Fourier transform), denoted:
$$f[x] = \mathcal{F}^{-1}\{F[\xi]\}$$
which may be called *Fourier synthesis.*

- Much of the first half of this course will be spent deriving the mathematical expressions for the forwand and inverse Fourier transforms, but we can state them here:

$$\boxed{F[\xi] \equiv \int_{x=-\infty}^{x=+\infty} f[x] \cdot (\exp[+i \cdot 2\pi\xi x])^* \ dx}$$

where $i \equiv \sqrt{-1}$ and "$*$" denotes the complex conjugate operation.and:

$$\begin{aligned} \exp[+i \cdot 2\pi\xi x] &= \cos[2\pi\xi x] + i \cdot \sin[2\pi\xi x] \\ &\Longrightarrow (\exp[+i \cdot 2\pi\xi x])^* = \cos[2\pi\xi x] - i \cdot \sin[2\pi\xi x] \end{aligned}$$

which demonstrates that the process of *Fourier analysis* may be written as:

$$\boxed{F[\xi] \equiv \int_{x=-\infty}^{x=+\infty} f[x] \cdot \exp[-i \cdot 2\pi\xi x] \ dx = \int_{x=-\infty}^{x=+\infty} f[x] \cdot (\cos[2\pi\xi x] - i \cdot \sin[2\pi\xi x]) \ dx}$$

- *Fourier synthesis* has form:

$$\boxed{f[x] = \int_{\xi=-\infty}^{\xi=+\infty} F[\xi] \cdot (\exp[+i \cdot 2\pi\xi x]) \ d\xi = \int_{\xi=-\infty}^{\xi=+\infty} F[\xi] \cdot (\cos[2\pi\xi x] + i \cdot \sin[2\pi\xi x]) \ d\xi}$$

which just adds up the sinusoidal functions $\exp[+i \cdot 2\pi\xi x] = \cos[2\pi\xi x] + i \cdot \sin[2\pi\xi x]$ weighted by complex-valued amplitudes $F[\xi]$ over all spatial frequencies ξ

- These tools are very helpful, even essential, for solving the three imaging tasks if the imaging system satisfies certain properties or "constraints:"

  1. the system must be "linear," which means (reduced to its simplest form) that the action of the imaging system to the sum of two inputs is equal to the sum of the two individual outputs.
  2. the action of the system on an input function does not depend on the location of the input function in the field of view; this is called "shift invariance."

5 Goals of Course:

1. Develop an intuitive grasp of mathematical methods for describing action of general linear system on signals of one or more spatial dimensions.
    - "develop intuitive images" of the mathematics
2. Develop consistent mathematical formalism for characterizing linear imaging systems; requires derivation of equations used to describe:
    - action of imaging system
    - its effect on *quality* of output image
3. Derive representations of images that are defined over:
    - *continuous domains*, convenient for describing:
        - realistic objects
        - realistic imaging systems
        - resulting images
    - *discrete domains*, essential for computer modeling of
        - objects
        - systems
        - images
    - continuous range of amplitude
    - discrete range of amplitude

- Representations in discrete coordinates (*sampled* functions) are essential for computer modeling general objects, images, and systems.
- Discrete images and systems are represented most conveniently as vectors and matrices