IMGS-616-20142 Solutions to Final Exam

Since we had almost done exactly the first five problems in class (#4 more than once), I expected most of you to do those 5 and then select one other. I was quite wrong.
My primary comment is that many of you seem to have problems with the basic principles of calculus, (and even algebra and trigonometry).

1. What property (or "properties") of an input function $f[x]$ that makes $f[x]$ useful for testing the frequency response of an imaging system? Put another way, what are the qualities of the input function $f[x]$ that make it useful for solving the "system analysis" task? Explain the reasons and list at least three examples of inputs that satisfy this property. Discuss or describe any practical issues resulting from the three cases.

   Solution: *The implication is that the system is linear and shift invariant, and so may be described by a convolution. The transfer function* $H[\xi] = \mathcal{F}\{h[x]\}$ *determines the "frequency response" of the system. To test the frequency response, apply a function* $f[x]$ *(or more than one function) and measure the frequency response of the output* $g[x]$*. The action of the system in the space domain is described by :*

$$f[x] * h[x] = g[x]$$

and the spectra of the input and output ($F[\xi]$ *and* $H[\xi]$*) are related to the transfer function* $H[\xi] = \mathcal{F}\{h[x]\}$ *via:*

$$F[\xi] \cdot H[\xi] = G[\xi]$$

which means that the magnitude of the transfer function is the ratio of the magnitude spectra of the output and input:

$$\Longrightarrow \quad H[\xi] = \frac{G[\xi]}{F[\xi]} = \frac{|G[\xi]|}{[F[\xi]]} \cdot \exp[+i \cdot (\Phi_G[\xi] - \Phi_F[\xi])]$$
$$\Longrightarrow \quad |H[\xi]| = \frac{|G[\xi]|}{[F[\xi]]}$$

and the phase spectrum is the difference of the phase spectra of the output and input:

$$\Longrightarrow \Phi_H[\xi] = \Phi_G[\xi] - \Phi_F[\xi]$$

The condition on the magnitude shows that the magnitude spectrum of the input cannot be zero to solve for the frequency response: $|F[\xi]| \neq 0$ *at all* ξ.

The best input functions that satisfy the condition are those with constant magnitude spectrum:

$$|F[\xi]| = 1[\xi] \Longrightarrow f[x] = \mathcal{F}^{-1}\{1[\xi] \cdot \exp[+i \cdot \Phi[\xi]]\}$$

Among the obvious functions that satisfy this condition are:

$$(1)\ f_1[x] = \delta[x] \Longrightarrow F_1[\xi] = 1[\xi] \cdot \exp[+i \cdot 0[\xi]]$$
$$(2)\ f_2[x] = \delta[x - x_0] \Longrightarrow F_2[\xi] = 1[\xi] \cdot \exp[+i \cdot 2\pi \cdot \xi \cdot (-x_0)]$$
$$(3)\ f_3[x] = \exp\left[+i \cdot \pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right] \Longrightarrow F_3[\xi] = |\alpha_0| \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi\left(\frac{\xi}{\left(\frac{1}{\alpha_0}\right)}\right)^2\right]$$

Another function that "works" is the constant magnitude spectrum with random phase:

$$(4)\ F_4[\xi] = 1[\xi] \cdot \exp[+i \cdot \pi \cdot N[\xi]]$$

where $N[\xi]$ *is a real-valued function such that:*

$$-1 \leq N[\xi] < +1$$

The space-domain representation consists of Gaussian-distributed random noise.

The practical issue arising if using $f_1[x] = \delta[x]$ *or* $f_2[x] = \delta[x - x_0]$ *is that the "dynamic range" of amplitude is infinite; it's impossible to create a realistic function with amplitude that jumps from "off" to infinite amplitude over an infinitesimal length (or time, if a temporal signal). The practical issue for the chirp function is its infinite support in the space domain, so the frequency response measurement is affected by the finite size of the aperture. The practical issue for the random phase is recording* $F[\xi]$ *to allow comparison to* $G[\xi]$.

2. Prove **TWO** of these three theorems

(a) *Need to specify some definitions first:*

$$\begin{aligned}\mathcal{F}\{f[x]\} &= F[\xi] = \int_{x=-\infty}^{x=+\infty} f[x]\cdot(\exp[+i\cdot 2\pi\cdot\xi x])^* \; dx \\ &= \int_{-\infty}^{+\infty} f[x]\cdot\exp[-i\cdot 2\pi\cdot\xi x]\; dx\end{aligned}$$

$$\mathcal{F}^{-1}\{F[\xi]\} = f[x] = \int_{\xi=-\infty}^{\xi=+\infty} F[\xi]\cdot\exp[+i\cdot 2\pi\cdot\xi x]\; d\xi$$

NOTE: you cannot apply the "forward" Fourier transform ("synthesis") to a function in the frequency domain, as several of you tried to do, i.e., you cannot do this:

$$\int_{x=-\infty}^{+\infty} F[\xi]\cdot\exp[-i\cdot 2\pi\cdot\xi x]\; dx!!$$

(b) "transform-of-a-transform" theorem:

$$\begin{aligned}\mathcal{F}\{F[x]\} &= \int_{-\infty}^{+\infty} F[x]\cdot\exp[-i\cdot 2\pi\cdot\xi x]\; dx \\ &= \int_{-\infty}^{+\infty} F[u]\cdot\exp[-i\cdot 2\pi\cdot\xi u]\; du \\ &= \int_{-\infty}^{+\infty} F[u]\cdot\exp[+i\cdot 2\pi\cdot(-\xi)\cdot u]\; du \\ &= f[-\xi]\end{aligned}$$

(c) filter theorem:

$$\begin{aligned}\mathcal{F}\{f[x]*h[x]\} &= \int_{x=-\infty}^{+\infty}\left(\int_{\alpha=-\infty}^{+\infty} f[\alpha]\cdot h[x-\alpha]\; d\alpha\right)\exp[-i\cdot 2\pi\cdot\xi x]\; dx \\ &= \int_{\alpha=-\infty}^{+\infty} f[\alpha]\cdot\left(\int_{x=-\infty}^{+\infty} h[x-\alpha]\exp[-i\cdot 2\pi\cdot\xi x]\; dx\right) d\alpha \\ &= \int_{\alpha=-\infty}^{+\infty} f[\alpha]\cdot\left(\int_{x=-\infty}^{+\infty} h[u]\exp[-i\cdot 2\pi\cdot\xi(u+\alpha)]\; du\right) d\alpha \\ &= \int_{\alpha=-\infty}^{+\infty} f[\alpha]\cdot\left(\int_{x=-\infty}^{+\infty} h[u]\exp[-i\cdot 2\pi\cdot\xi u]\exp[-i\cdot 2\pi\cdot\xi a]\; du\right) d\alpha \\ &= \int_{\alpha=-\infty}^{+\infty} f[\alpha]\cdot\exp[-i\cdot 2\pi\cdot\xi a]\left(\int_{x=-\infty}^{+\infty} h[u]\exp[-i\cdot 2\pi\cdot\xi u]\; du\right) d\alpha \\ &= \left(\int_{\alpha=-\infty}^{+\infty} f[\alpha]\cdot\exp[-i\cdot 2\pi\cdot\xi a]\; d\alpha\right)\cdot\left(\int_{x=-\infty}^{+\infty} h[u]\exp[-i\cdot 2\pi\cdot\xi u]\; du\right) \\ &= F[\xi]\cdot H[\xi]\end{aligned}$$

(d) derivative theorem:

$$\begin{aligned}
\frac{df}{dx} &= \frac{d}{dx}\left\{\mathcal{F}^{-1}\left\{F\left[\xi\right]\right\}\right\} \\
&= \frac{d}{dx}\int_{-\infty}^{+\infty} F\left[\xi\right]\cdot\exp\left[+i\cdot 2\pi\cdot\xi x\right]\, d\xi \\
&= \int_{-\infty}^{+\infty} \frac{d}{dx}\left(F\left[\xi\right]\cdot\exp\left[+i\cdot 2\pi\cdot\xi x\right]\right)\, d\xi \\
&= \int_{-\infty}^{+\infty} F\left[\xi\right]\cdot\frac{d}{dx}\left(\exp\left[+i\cdot 2\pi\cdot\xi x\right]\right)\, d\xi \\
&= \int_{-\infty}^{+\infty} F\left[\xi\right]\cdot\left(+i\cdot 2\pi\cdot\xi\right)\left(\exp\left[+i\cdot 2\pi\cdot\xi x\right]\right)\, d\xi \\
&= \int_{-\infty}^{+\infty} \left(+i\cdot 2\pi\cdot\xi\cdot F\left[\xi\right]\right)\cdot\left(\exp\left[+i\cdot 2\pi\cdot\xi x\right]\right)\, d\xi \\
&= \mathcal{F}^{-1}\left\{+i\cdot 2\pi\cdot\xi\cdot F\left[\xi\right]\right\} \\
&\Longrightarrow \mathcal{F}\left\{\frac{df}{dx}\right\} = \mathcal{F}\left\{\mathcal{F}^{-1}\left\{+i\cdot 2\pi\cdot\xi\cdot F\left[\xi\right]\right\}\right\} \\
&\Longrightarrow \mathcal{F}\left\{\frac{df}{dx}\right\} = +i\cdot 2\pi\cdot\xi\cdot F\left[\xi\right]
\end{aligned}$$

Some people tried to say the following (ACK!)

$$\begin{aligned}
\mathcal{F}\left\{\frac{df}{dx}\right\} &= \int_{-\infty}^{+\infty} \frac{d}{dx}\left(f\left[x\right]\right)\cdot\exp\left[-i\cdot 2\pi\cdot\xi x\right]\, dx \\
&= \int_{-\infty}^{+\infty} f\left[x\right]\cdot\frac{d}{dx}\left\{\exp\left[-i\cdot 2\pi\cdot\xi x\right]\right\}\, dx
\end{aligned}$$

which violates the principles of basic differential calculus – even a very cursory knowledge of calculus should have shown that this is clearly incorrect

3. For the matrix $\underline{\mathbf{A}}$ shown:

$$\underline{\mathbf{A}} = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ -1 & 0 & 1 & -2 \end{bmatrix}$$

(a) find the eigenvalues and use to determine if the matrix is invertible:

$$\underline{\mathbf{A}}\underline{\mathbf{x}}_n = \lambda_n \underline{\mathbf{x}}_n$$

The matrix is circulant, so we already know the normalized eigenvectors:

$$\hat{\underline{\mathbf{x}}}_0 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \hat{\underline{\mathbf{x}}}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ +i \\ -1 \\ -i \end{bmatrix}, \hat{\underline{\mathbf{x}}}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \hat{\underline{\mathbf{x}}}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -i \\ -1 \\ +i \end{bmatrix}$$

$$\begin{aligned} \underline{\mathbf{A}}\underline{\mathbf{x}}_0 &= \lambda_0 \underline{\mathbf{x}}_0 \\ \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ -1 & 0 & 1 & -2 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = -2 \cdot \left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) \\ &\Longrightarrow \boxed{\lambda_0 = -2} \end{aligned}$$

which is the same as the sum of the elements in each row.

$$\begin{aligned} \underline{\mathbf{A}}\underline{\mathbf{x}}_1 &= \lambda_1 \underline{\mathbf{x}}_1 \\ \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ -1 & 0 & 1 & -2 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 \\ +i \\ -1 \\ -i \end{bmatrix} &= \begin{bmatrix} -1-i \\ 1-i \\ 1+i \\ -1+i \end{bmatrix} = 2 \cdot (-1-i) \cdot \left(\frac{1}{2}\begin{bmatrix} 1 \\ +i \\ -1 \\ -i \end{bmatrix} \right) \\ &\Longrightarrow \boxed{\lambda_0 = -2 \cdot (1+i)} \end{aligned}$$

$$\begin{aligned} \underline{\mathbf{A}}\underline{\mathbf{x}}_2 &= \lambda_0 \underline{\mathbf{x}}_2 \\ \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ -1 & 0 & 1 & -2 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} &= \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = -2 \cdot \left(\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right) \\ &\Longrightarrow \boxed{\lambda_2 = -2} \end{aligned}$$

$$\begin{aligned} \underline{\mathbf{A}}\underline{\mathbf{x}}_1 &= \lambda_1 \underline{\mathbf{x}}_1 \\ \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ -1 & 0 & 1 & -2 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 \\ -i \\ -1 \\ +i \end{bmatrix} &= \begin{bmatrix} -1+i \\ 1+i \\ 1-i \\ -1-i \end{bmatrix} = 2 \cdot (-1+i) \cdot \left(\frac{1}{2}\begin{bmatrix} 1 \\ -i \\ -1 \\ +i \end{bmatrix} \right) \\ &\Longrightarrow \boxed{\lambda_3 = 2 \cdot (-1+i)} \end{aligned}$$

(b) evaluate the inverse matrix $\underline{\mathbf{A}}^{-1}$ or the pseudoinverse matrix $\underline{\mathbf{A}}^{\dagger}$, as appropriate.

$$\underline{\mathbf{\Lambda}} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2\cdot(1+i) & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2\cdot(1-i) \end{bmatrix} = -2\cdot\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-i \end{bmatrix}$$

$$\begin{aligned}
\underline{\mathbf{\Lambda}}^{-1} &= \frac{1}{2}\cdot\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-i \end{bmatrix}^{-1} \\
&= \frac{1}{2}\cdot\begin{bmatrix} \frac{1}{1} & 0 & 0 & 0 \\ 0 & \frac{1}{1+i} & 0 & 0 \\ 0 & 0 & \frac{1}{1} & 0 \\ 0 & 0 & 0 & \frac{1}{1-i} \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1-i & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1+i \end{bmatrix}
\end{aligned}$$

$$\underline{\mathbf{A}}^{-1} = -\frac{1}{8}\begin{bmatrix} 3 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & 1 & 3 & -1 \\ -1 & 1 & 1 & 3 \end{bmatrix}$$

(c) Evaluate the product of $\underline{\mathbf{A}\mathbf{A}}^{-1}$ or $\underline{\mathbf{A}\mathbf{A}}^{\dagger}$, as appropriate.

duh!

$$\begin{aligned}
\underline{\mathbf{A}\mathbf{A}}^{-1} &= \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ -1 & 0 & 1 & -2 \end{bmatrix}\left(-\frac{1}{8}\begin{bmatrix} 3 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & 1 & 3 & -1 \\ -1 & 1 & 1 & 3 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

4. Evaluate and sketch $|g[x]|^2 = |f[x] * h[x]|^2$ for this object $f[x]$ and impulse response $h[x]$ (You must simplify the result):

$$\begin{aligned} f[x] &= \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \\ h[x] &= \delta[x+b_0] + \delta[x-b_0] \end{aligned}$$

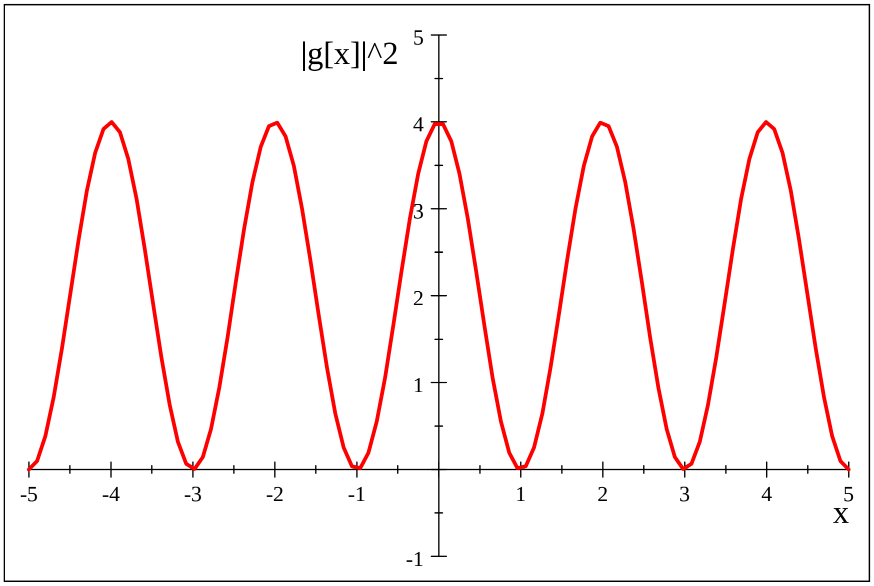
SOLUTION: *(you've done this at least twice already!)*

$$\begin{aligned} g[x] &= f[x] * h[x] \\ &= \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] * (\delta[x+b_0] + \delta[x-b_0]) \\ &= \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] * \delta[x+b_0] + \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] * \delta[x-b_0] \\ &= \exp\left[+i\pi\left(\frac{x+b_0}{\alpha_0}\right)^2\right] + \exp\left[+i\pi\left(\frac{x-b_0}{\alpha_0}\right)^2\right] \\ &= \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \cdot \exp\left[+i\pi\left(\frac{b_0}{\alpha_0}\right)^2\right] \cdot \exp\left[+i\pi\frac{2xb_0}{\alpha_0^2}\right] \\ &\quad + \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \cdot \exp\left[+i\pi\left(\frac{b_0}{\alpha_0}\right)^2\right] \cdot \exp\left[-i\pi\frac{2xb_0}{\alpha_0^2}\right] \end{aligned}$$

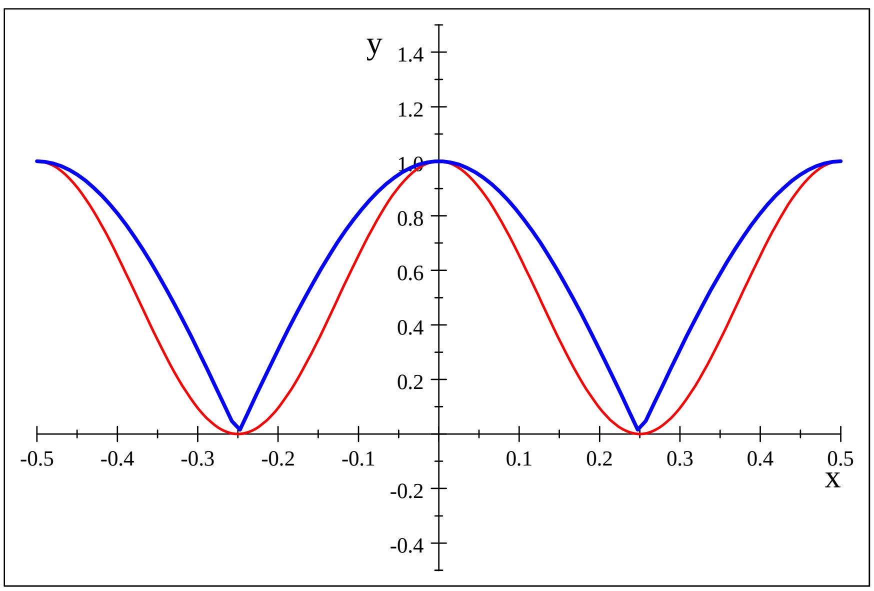
$$\begin{aligned} g[x] &= \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \cdot \exp\left[+i\pi\left(\frac{b_0}{\alpha_0}\right)^2\right] \cdot \left(\exp\left[+i\pi\frac{2xb_0}{\alpha_0^2}\right] + \exp\left[-i\pi\frac{2xb_0}{\alpha_0^2}\right]\right) \\ g[x] &= \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \cdot \exp\left[+i\pi\left(\frac{b_0}{\alpha_0}\right)^2\right] \cdot 2\cos\left[2\pi\frac{xb_0}{\alpha_0^2}\right] \\ |g[x]|^2 &= \left|\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \cdot \exp\left[+i\pi\left(\frac{b_0}{\alpha_0}\right)^2\right] \cdot 2\cos\left[2\pi\frac{xb_0}{\alpha_0^2}\right]\right|^2 \\ &= \left|\exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right]\right|^2 \cdot \left|\exp\left[+i\pi\left(\frac{b_0}{\alpha_0}\right)^2\right]\right|^2 \cdot \left|2\cos\left[2\pi\frac{xb_0}{\alpha_0^2}\right]\right|^2 \\ &= 1\cdot 1\cdot 4\cdot\left|\cos\left[2\pi x\left(\frac{b_0}{\alpha_0^2}\right)\right]\right|^2 \\ &= 4\cdot\cos^2\left[2\pi x\left(\frac{b_0}{\alpha_0^2}\right)\right] = 4\cdot\left(\frac{1}{2}+\frac{1}{2}\cos\left[2\pi x\left(\frac{2b_0}{\alpha_0^2}\right)\right]\right) \\ &: \quad \textit{(since } \cos^2[\theta] = \frac{1}{2}+\frac{1}{2}\cos[2\theta]\textit{)} \\ &= 2+2\cos\left[2\pi x\left(\frac{2b_0}{\alpha_0^2}\right)\right] = 2+2\cdot\cos\left[2\pi\frac{x}{\left(\frac{\alpha_0^2}{2b_0}\right)}\right] \end{aligned}$$

For $\alpha_0 = 2$ *and* $b_0 = 1$*:*

$$|g[x]|^2 = 2 + 2 \cdot \cos\left[2\pi \frac{x}{\left(\frac{2^2}{2\cdot 1}\right)}\right] = 2 + 2 \cdot \cos\left[2\pi \frac{x}{2}\right]$$



Several of you sketched $\cos^2[\theta]$ *with "cusps" at the zero crossings similar to* $|\cos[\theta]|$



$\cos^2[2\pi x]$ *in red and* $|\cos[2\pi x]|$ *in blue (with cusps)*

5. Consider the 1-D function $f[x] = \exp[+i\pi x]$

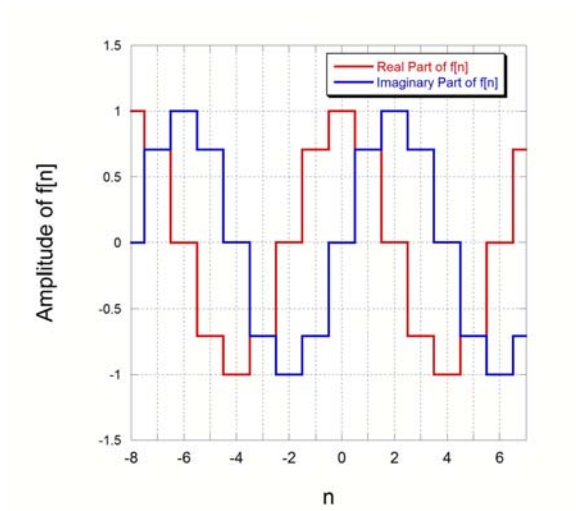
Note the "i" in the exponent! This is NOT $\exp[+\pi x]$*!*

(a) Evaluate and sketch the discrete Fourier transform (DFT) of $f[n]$ derived from $f[x]$ by sampling at spacing $\Delta x = 0.25$ units with $N = 32$.

Most of you immediately started by writing down sampling equations, rather than graphing the function (sigh!). The picture helps considerably. The sampled function is:

$$f[n] = \exp\left[+i \cdot 2\pi \cdot \frac{x}{2}\right] \cdot \frac{1}{0.25} COMB\left[\frac{x}{0.25}\right] \implies \frac{2}{0.25} = 8 \textit{ samples per period}$$

The function is a complex-valued sinusoid with period of 2 units sampled at a spacing of $\Delta x = 0.25$ *units, so there each cycle will include 8 samples; there is no aliasing. For* $N = 32$ *and a period of 8 samples, there are EXACTLY 4 cycles of the sinusoid in 32 samples, and therefore no "leakage".*



real and imaginary parts of $f[n]$ *for* $\Delta x = \frac{1}{4}$ *and* $N = 32$

Since there is no aliasing and will be no leakage, the spectrum will resemble that of the continuous function. We can evaluate the continuous Fourier transform and then determine the sample of the corresponding frequency in the DFT.

$$f[x] = \exp\left[+i \cdot 2\pi \cdot \frac{x}{2}\right] \implies \xi_0 = +\frac{1}{2}\frac{\text{cycle}}{\text{unit length}}$$

$$\implies F[\xi] = \delta\left[\xi - \frac{1}{2}\right]$$

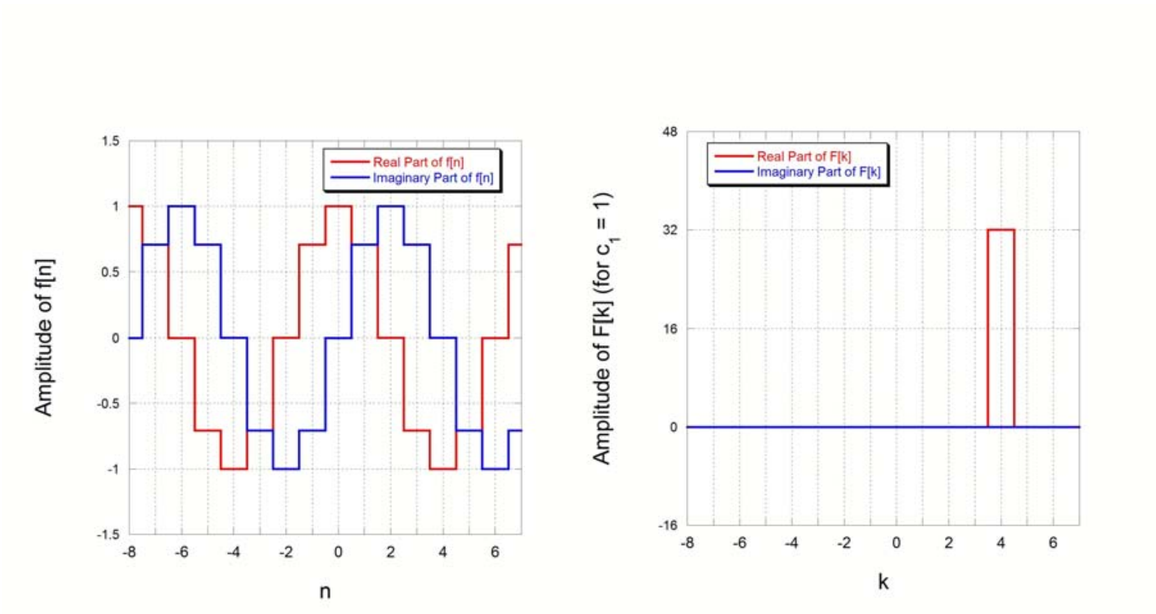
From the given values for N *and* Δx*, it is easy to find the interval* $\Delta\xi$ *in the frequency domain:*

$$\Delta x = 0.25 \text{ units}, N = 32 \implies \Delta\xi = \frac{1}{N \cdot \Delta x} = \frac{1}{32 \cdot \frac{1}{4}} = \frac{1}{8}\frac{\text{cycle}}{\text{sample}}$$

The only remaining step is to find the sample index k_0 that corresponds to that spatial frequency:

$$\begin{aligned}\xi_0 &= +\frac{1}{2}\frac{\text{cycle}}{\text{unit length}} = k_0 \cdot \Delta\xi \\ &= k_0 \cdot \frac{1}{8} \Longrightarrow \boxed{k_0 = +4} \\ &\boxed{F[k] = \begin{cases} N \cdot c_1 = 32 \cdot c_1 & \text{if} \quad k = 4 \\ 0 & \text{if} \quad k \neq 4 \end{cases}}\end{aligned}$$

The DFT evaluated for $N = 32$ will exhibit a single real-valued impulse located at $k = +4$ and zero elsewhere. The "height" of the impulse depends on the normalization constant used (typically $c_1 = 1, \frac{1}{N} = \frac{1}{32}$, or $\frac{1}{\sqrt{32}}$, your choice)



(a) real and imaginary parts of $f[n]$ for $\Delta x = \frac{1}{4}$ and $N = 32$; (b) DFT $F[k]$ for $c_1 = 1$, $c_2 = \frac{1}{32}$.

(b) Evaluate and sketch the DFT of $f[n]$ sampled at the same spacing with $N = 128$.

The spatial frequency is still $\xi_0 = +\frac{1}{2}\frac{\text{cycle}}{\text{unit length}}$. *Now find* $\Delta\xi$ *for* $N = 128$ *and* $\Delta x = \frac{1}{4}$

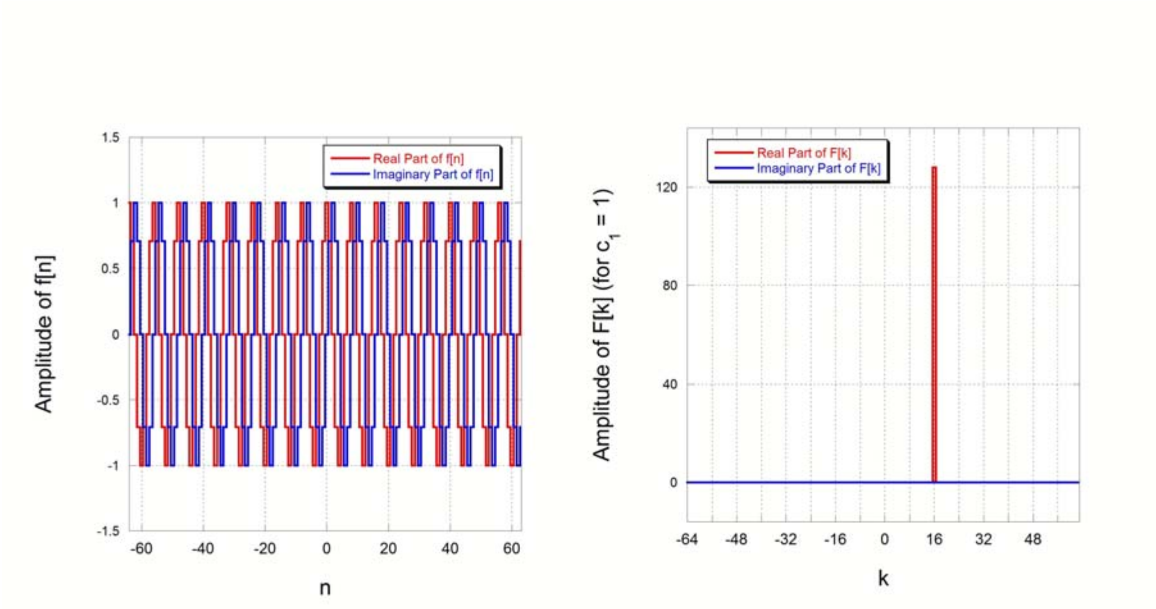
$$\Delta\xi = \frac{1}{N \cdot \Delta x} = \frac{1}{128 \cdot \frac{1}{4}} = \frac{1}{32}$$

Now find the index k *in the frequency domain for this frequency* ξ_0:

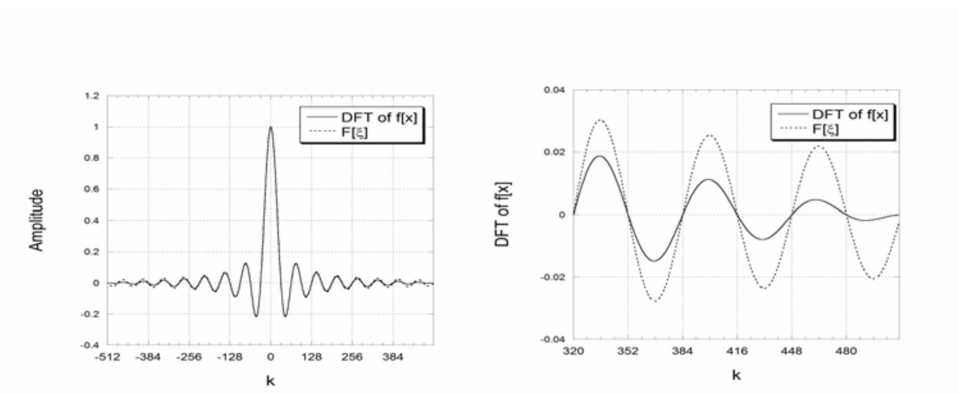
$$\begin{aligned} \xi_0 &= \frac{1}{2} = k_0 \cdot \Delta\xi = k_0 \cdot \frac{1}{32} \\ &\Longrightarrow k_0 = 16 \end{aligned}$$

$$\boxed{F[k] = \begin{cases} N \cdot c_1 = 128 \cdot c_1 & \text{if} \quad k = 16 \\ 0 & \text{if} \quad k \neq 16 \end{cases}}$$

You can also surmise this by recognizing that there will be 16 cycles of the sinusoid in $f[n]$, *so the DFT is composed of a single impulse at* $k = +16$ *with amplitude either* $1, N$, *or* $\sqrt{N}$ *for* $c_1 = N^{-1}, 1$, *or* $\sqrt{N}^{-1}$, *respectively.*



6. A colleague uses a discrete Fourier transform program to evaluate the spectrum $F[k]$ of $f[n]$, which is a sampled rectangle function with width parameter $b_0 = 32$ samples for $N = 1024$ and $\Delta x = \frac{1}{32}$ mm. The graph of the sampled spectrum is shown in the solid line. Samples of the spectrum of the continuous rectangle function are shown as the dashed line. The second graph is a magnified view of the details at the right-hand side of the first graph.



(a) Determine the size of the interval between samples of the DFT (i.e., in the frequency domain) for the stated value of N.

$$\begin{aligned} N &= 1024 \\ \Delta x &= \frac{1}{32}\,\text{mm} \\ N \cdot \Delta x \cdot \Delta\xi &= 1 \implies \Delta\xi = \frac{1}{1024 \cdot \frac{1}{32}\,\text{mm}} = \frac{1}{32}\,\text{mm}^{-1} = \frac{1}{32}\frac{\text{cycle}}{\text{mm}} \end{aligned}$$

$$\boxed{\Delta\xi = \frac{1}{32}\,\text{mm}^{-1} = \frac{1}{32}\frac{\text{cycle}}{\text{mm}}}$$

(b) Determine the maximum of the absolute value of the frequency in the DFT in "cycles per millimeter."

$$\xi_{\text{max}} = \frac{1}{2 \cdot \Delta x} = \frac{1}{2 \cdot \frac{1}{32}\,\text{mm}} = 16\,\text{mm}^{-1} = 16\frac{\text{cycle}}{\text{mm}}$$

which also could be obtained from:

$$\xi_{\text{max}} = \frac{N}{2} \cdot \Delta\xi = \frac{1024}{2} \cdot \frac{1}{32}\frac{\text{cycle}}{\text{mm}} = 16\frac{\text{cycle}}{\text{mm}}$$

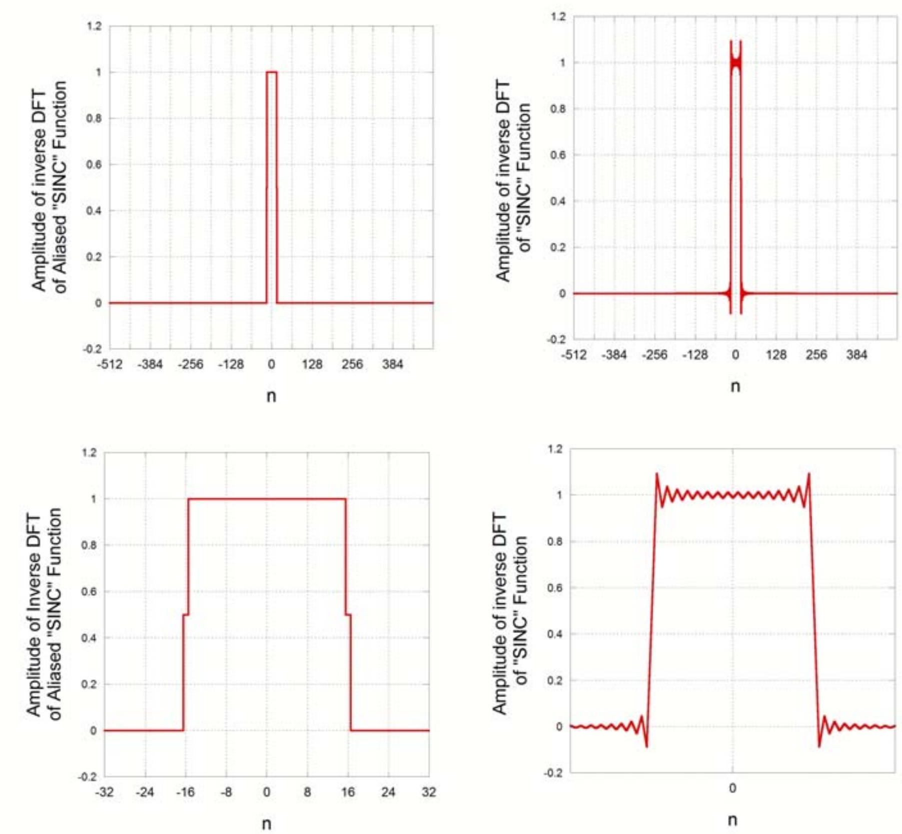
$$\boxed{\xi_{\text{max}} = 16\frac{\text{cycle}}{\text{mm}}}$$

(c) In words, explain the discrepancy between the two plotted lines in the graph; the detail plot on the right may be helpful to visualize the differences. Your explanation should be as detailed as you can make it; one-word "explanations" are not sufficient. You might start by specifying the differences between the two graphs.

Solution: *We know that the spectrum of the continuous rectangle function is a SINC function with infinite support, which means that the sampled rectangle MUST be aliased. The discrepancy between the values of the SINC function and the DFT of the rectangle function is due to the contributions from the infinite number of periods of the displaced SINC function summed together.*

(d) Sketch the space-domain functions that would be obtained by evaluating the inverse DFT over 1024 samples for both of the lines in the graph; be sure to label your axes.

Solution: *We know that the inverse DFT of the aliased SINC function must yield the samples of the rectangle. The inverse DFT of the "actual" windowed SINC function is the inverse DFT of the product of the SINC with a wide rectangle, so the result is the convolution of the continuous rectangle and the SINC, which "overshoots" at the transition points:*



(top row) full-scale plots of inverse DFT of the two cases ; (bottom row) magnified plots over interval $-32 \leq n \leq +32$*, showing that the inverse DFT of the aliased SINC function yields the samples of the rectangle, while those of the actual values of the windowed SINC yields the convolution of the rectangle and the SINC function due to the rectangular window of width* $b_0 = N \cdot \Delta x$

7. Consider three cases of complex-valued noise $n[x]$ that is "bipolar" (positive and negative values): (a) the power spectrum $|N_a[\xi]|^2$ is constant over all frequencies; (b) the power spectrum $|N_b[\xi]|^2$ is large at low frequencies and goes to zero in the limit $|\xi| \to \infty$; (c) the power spectrum $|N_c[x]|^2$ is zero at $\xi = 0$ and greater than zero at nonzero frequencies.

Several seemed to confuse the "autocorrelation" with the "impulse response of a system." The autocorrelation is a way to characterize some properties of a stochastic function, specifically to determine if its numerical values at different locations are "related" or if they are "random."It does NOT represent an effect on another signal. We did exactly this problem in class, so you have seen it. It might have been useful to write down the expression for the autocorrelation of a function:

$$f[x] \bigstar f[x] = f[x] * f^*[-x] = \int_{-\infty}^{+\infty} f[\alpha] \cdot f^*[\alpha - x] \, d\alpha$$

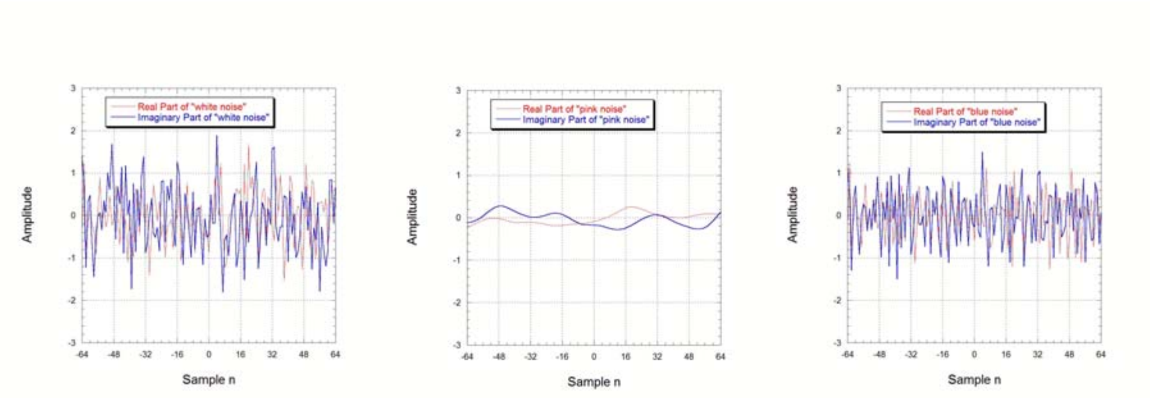
Several made the pretty obvious mistake of assuming that the fact that $|N[\xi]|^2 = 1$ *means that* $N[\xi] = 1$*, which clearly is NOT true! You need the Wiener-Khinchin theorem, which states that the Fourier transform of the autocorrelation is the squared magnitude of the signal spectrum:*

$$\begin{aligned} \mathcal{F}\{f[x] \bigstar f[x]\} &= |F[\xi]|^2 \geq 0 \\ &\Longrightarrow f[x] \bigstar f[x] = \mathcal{F}^{-1}\left\{|F[\xi]|^2\right\} \end{aligned}$$

(a) Describe any significant "features" of the space-domain representations of the noise, put another way, what are the expected "qualities" of the three instances of the noise: $n_a[x] = \mathcal{F}_1^{-1}\{N_1[\xi]\}$, ...

Solution: *The three noise functions are respectively "white," "pink," and "blue." The gray values of "white noise" are completely uncorrelated, which means that values at adjacent coordinates are unrelated* $\Longrightarrow$ *any noise sample may take on any numerical value regardless of values of neighbors. The dominance of low spatial frequencies in "pink noise" means that adjacent values ARE correlated: adjacent samples of noise tend to have the same algebraic sign. The dominance of large spatial frequencies in "blue noise" means that adjacent values tend to have opposite signs.*

Examples of noise distributions for the three classes:



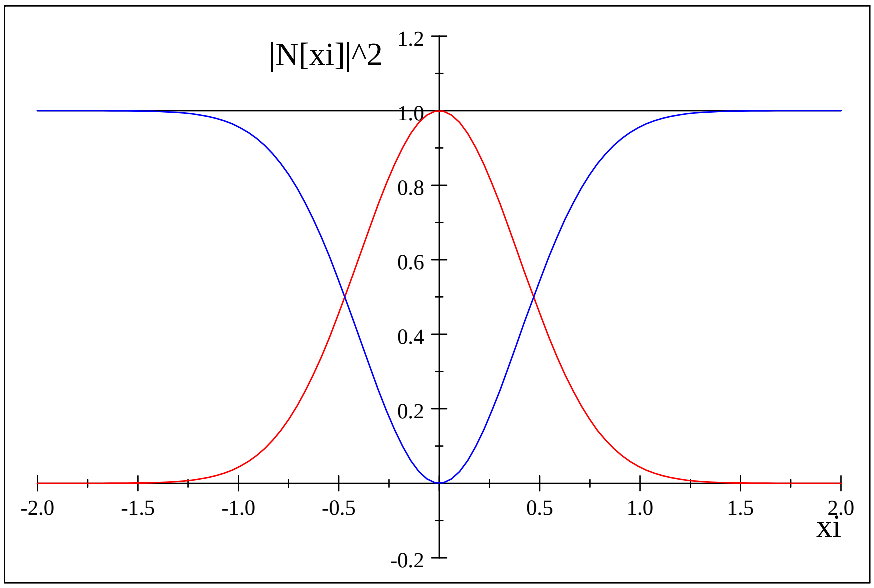
(b) Sketch the expected "shapes" of the autocorrelations of the three cases (you may do this for specific power spectra of your choice that satisfy the stated conditions, but this is not required);

Solution: *I'll use three functions to illustrate the result:*

$$|N_{\text{white}}[\xi]|^2 = 1[\xi]$$

$$|N_{\text{pink}}[\xi]|^2 = GAUS\left[\frac{\xi}{b_0}\right] = \exp\left[-\pi\left(\frac{\xi}{b_0}\right)^2\right]$$

$$|N_{\text{blue}}[\xi]|^2 = 1 - GAUS\left[\frac{\xi}{b_0}\right] = 1 - \exp\left[-\pi\left(\frac{\xi}{b_0}\right)^2\right]$$



Graphs of three examples of noise power spectra: (black) "white noise;" (red) "pink noise" with $b_0 = 1$; (blue) "blue noise" with $b_0 = 1$.

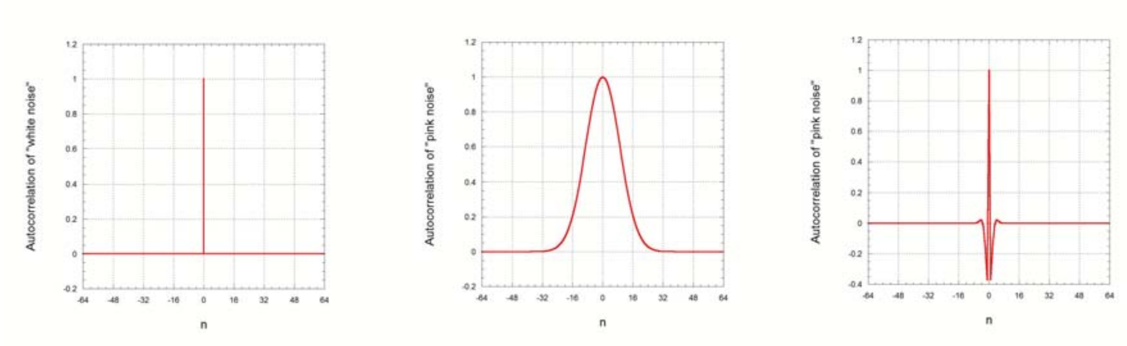
Some people used $|N_c[\xi]|^2 = |N_{\text{blue}}[\xi]|^2 \propto \xi^2$*, which results in a signal with very large amplitudes at high frequencies and that does not have an easily derived autocorrelation. When you have the option of choosing a function, why not choose a function that has an easily derived transform (or inverse transform, in this case)?*

Now use the Wiener-Khintchin theorem to evaluate the autocorrelations:

$$\begin{aligned}
\mathcal{F}\{n[x]\bigstar n[x]\} &= |N[\xi]|^2 \implies n[x]\bigstar n[x] = \mathcal{F}^{-1}\left\{|N[\xi]|^2\right\} \\
\textit{white noise example} &: \quad n[x]\bigstar n[x] \propto \mathcal{F}^{-1}\{1[\xi]\} = \delta[x] \\
\textit{pink noise example} &: \quad n[x]\bigstar n[x] \propto \mathcal{F}^{-1}\left\{\exp\left[-\pi\left(\frac{\xi}{b_0}\right)^2\right]\right\} \\
&= |b_0|\cdot\exp\left[-\pi\left(\frac{x}{\left(\frac{1}{b_0}\right)}\right)^2\right] \\
\textit{blue noise example} &: \quad n[x]\bigstar n[x] \propto \mathcal{F}^{-1}\left\{1-\exp\left[-\pi\left(\frac{\xi}{b_0}\right)^2\right]\right\} \\
&= \delta[x] - |b_0|\cdot\exp\left[-\pi\left(\frac{x}{\left(\frac{1}{b_0}\right)}\right)^2\right]
\end{aligned}$$

(c) Describe the features that are visible in the three autocorrelation functions and give reasons for their existence.

Solution: *The autocorrelations of these examples of white and pink noise are both nonnegative, while that of blue noise is positive for $x = 0$ (zero shift) and negative for adjacent values of x.*



The autocorrelation of the white noise is a Dirac delta function, which indicates that the noise is "perfectly correlated" for zero translation, so that

$$\int_{-\infty}^{+\infty} f[\alpha] \cdot f^*[\alpha - 0] \; d\alpha = \int_{-\infty}^{+\infty} |f[\alpha]|^2 \; d\alpha > 0$$

but that the autocorrelation at other translations is zero:

$$\int_{-\infty}^{+\infty} f[\alpha] \cdot f^*[\alpha - x] \; d\alpha = 0 \text{ for } x \neq 0$$

In words, white noise is completely uncorrelated to itself at any translation except zero shift.

"Pink" noise consists primarily of low-frequency sinusoidal functions, so the summation tends to be correlated over distances other than zero. This means that adjacent amplitudes of the noise tend to be similar – nearby noise amplitudes are "positively correlated."

"Blue" noise consists of high-frequency sinusoidal functions, so the sinusoids (and thus the noise) change sign over short distances, so adjacent values tend to have opposite sign – they are "anticorrelated."

8. You are familiar with the notations for the sum and product of several terms:

$$\sum_{n=0}^{\infty} x^n = x^0 + x^1 + x^2 + \cdots$$
$$\prod_{n=0}^{\infty} = a_n = a_0 \cdot a_1 \cdot a_2 \cdot \cdots$$

We now define a notation "$\bigodot$" for the convolution of $N+1$ examples of functions:

$$\bigodot_{n=0}^{N} f_n[x] \equiv f_0[x] * f_1[x] * \cdots * f_N[x]$$

Evaluate this function for two cases for the general value of N

(a) $f_n[x] = SINC\left[\frac{x}{n+1}\right]$

$$\begin{aligned}
g[x] &= \bigodot_{n=0}^{N} SINC\left[\frac{x}{n+1}\right] = SINC\left[\frac{x}{1}\right] * SINC\left[\frac{x}{2}\right] * \cdots * SINC\left[\frac{x}{N+1}\right] \\
G[\xi] &= (1 \cdot RECT[\xi]) \cdot (2 \cdot RECT[2\xi]) \cdot \cdots \cdot ((N+1) \cdot RECT[(N+1)\xi]) \\
&= (1 \cdot 2 \cdot 3 \cdot \cdots \cdot (N+1)) \cdot RECT[(N+1)\xi] = (N+1)! \cdot RECT[(N+1)\xi] \\
g[x] &= \frac{(N+1)!}{N+1} SINC\left[\frac{x}{N+1}\right] = \boxed{g[x] = N! \cdot SINC\left[\frac{x}{N+1}\right]}
\end{aligned}$$

In words, the convolution of a bunch of SINC functions of different width is a scaled replica of the widest SINC.

(b) $f_n[x] = \exp[-\pi x^2]$

$$\begin{aligned}
g[x] &= \bigodot_{n=0}^{N} \exp\left[-\pi x^2\right] \equiv \left(\exp\left[-\pi x^2\right]\right)_0 * \left(\exp\left[-\pi x^2\right]\right)_1 * \cdots * \left(\exp\left[-\pi x^2\right]\right)_N \\
G[\xi] &= \left(\exp\left[-\pi \xi^2\right]\right)_0 \cdot \left(\exp\left[-\pi \xi^2\right]\right)_1 \cdot \cdots \cdot \left(\exp\left[-\pi \xi^2\right]\right)_N \\
&= \exp\left[-\pi \cdot (N+1) \cdot \xi^2\right] = \exp\left[-\pi\left(\sqrt{N+1} \cdot \xi\right)^2\right] = GAUS\left[\sqrt{N+1} \cdot \xi\right] \\
g[x] &= \mathcal{F}^{-1}\left\{GAUS\left[\sqrt{N+1} \cdot \xi\right]\right\} = \frac{1}{\sqrt{N+1}} \cdot GAUS\left[\frac{x}{\sqrt{N+1}}\right] \\
&= \frac{1}{\sqrt{N+1}} \exp\left[-\pi\left(\frac{x}{\sqrt{N+1}}\right)^2\right] \\
&\boxed{g[x] = \frac{1}{\sqrt{N+1}} \cdot GAUS\left[\frac{x}{\sqrt{N+1}}\right] = \frac{1}{\sqrt{N+1}} \exp\left[-\pi \cdot \frac{x^2}{N+1}\right]}
\end{aligned}$$

The convolution of a set of identical Gaussian functions is a WIDER Gaussian function!

(c) $f_n[x] = \exp\left[(-1)^n \cdot i \cdot \pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right]$

$$g[x] = \bigodot_{n=0}^{N} \exp\left[(-1)^n \cdot i \cdot \pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right] = \exp\left[+i\pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right] * \exp\left[-i\pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right] * \cdots$$

Note that there is an even number of functions if N is odd and an odd number of functions if N is even. The convolution of one pair of adjacent functions with opposite signs is:

$$\begin{aligned}
&\exp\left[+i\pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right] * \exp\left[-i\pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right] \\
&= \mathcal{F}^{-1}\left\{\left(|\alpha_0| \cdot \exp\left[+i\frac{\pi}{4}\right] \cdot \exp\left[-i\pi\xi^2\right]\right) \cdot \left(|\alpha_0| \cdot \exp\left[-i\frac{\pi}{4}\right] \cdot \exp\left[+i\pi\xi^2\right]\right)\right\} \\
&= \mathcal{F}^{-1}\left\{|\alpha_0|^2 \cdot 1[\xi]\right\} \\
&= |\alpha_0|^2 \cdot \delta[x]
\end{aligned}$$

so each pair of functions with positive and negative sign convolves to form a Dirac delta function with area $|\alpha_0|^2$. The convolution of N functions evaluates to a Dirac delta function if there are an even number of functions (so that N is ODD). If N is even, the previous N/2 pairs of functions evaluated to a scaled Dirac delta function, so the additional term yields a scaled quadratic-phase function:

$$\begin{aligned}
\left(|\alpha_0|^2\right)^{\frac{N+1}{2}} \cdot \delta[x] &= |\alpha_0|^{N+1} \cdot \delta[x] \text{ if } N \text{ is odd} \\
|\alpha_0|^N \cdot \delta[x] * \exp\left[+i\pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right] &= |\alpha_0|^N \cdot \exp\left[+i\pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right] \text{ if } N \text{ is even}
\end{aligned}$$

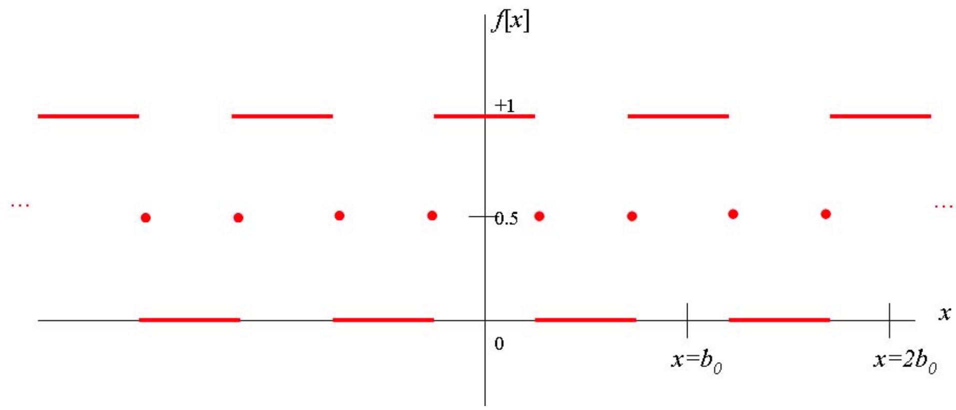
$$\boxed{g[x] = \begin{cases} |\alpha_0|^{N+1} \cdot \delta[x] & \text{if } N \text{ is odd} \\ |\alpha_0|^N \cdot \exp\left[+i\pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right] & \text{if } N \text{ is even} \end{cases}}$$

9. The space-domain function $f[x]$ is a "50% square wave," which means that the amplitude "switches" from a value of unity ("on") to a value of zero ("off") at uniformly spaced intervals. Assume that the period of the square wave is b_0

   (a) Write down an equation for this function and sketch it.

   The function is written based on the period of the elements in the COMB function

$$\begin{aligned} f[x] &= RECT\left[\frac{x}{\left(\frac{b_0}{2}\right)}\right] * \frac{1}{|b_0|} COMB\left[\frac{x}{b_0}\right] \\ &= \sum_{n=-\infty}^{+\infty} RECT\left[\frac{x - n \cdot b_0}{\left(\frac{b_0}{2}\right)}\right] \end{aligned}$$

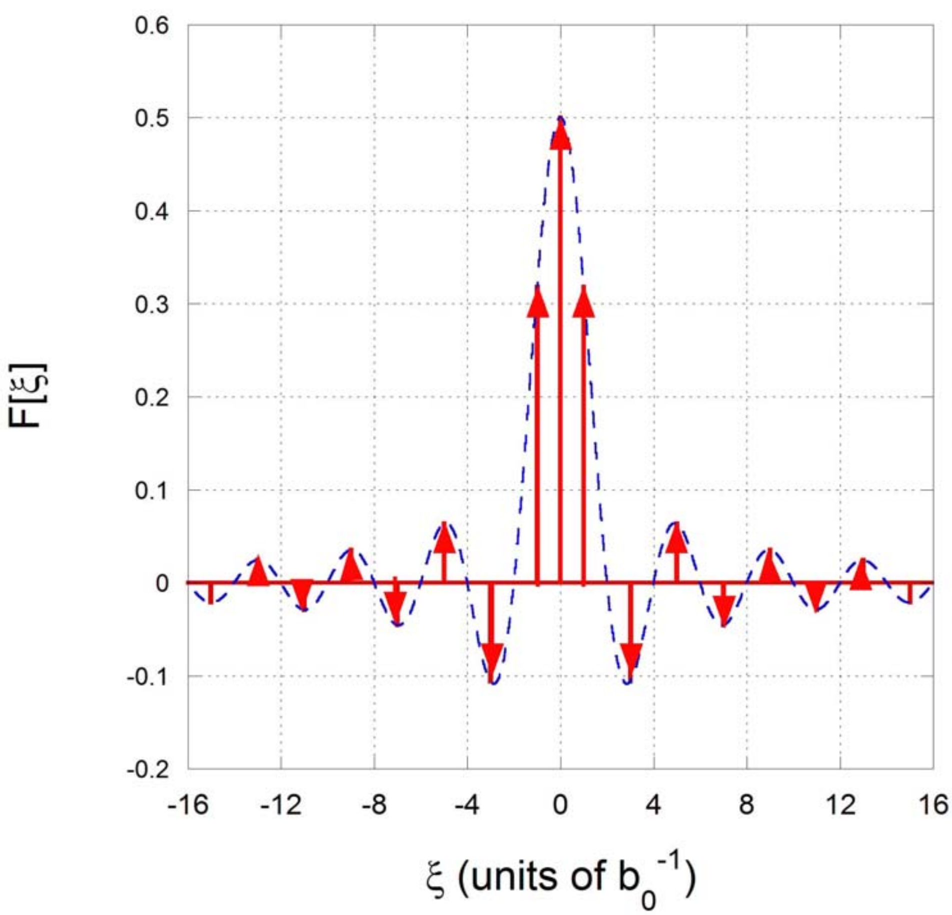


(b) Evaluate the formula for and sketch its spectrum

$$
\begin{aligned}
F[\xi] &= \mathcal{F}\left\{RECT\left[\frac{x}{\left(\frac{b_0}{2}\right)}\right] * \frac{1}{|b_0|}COMB\left[\frac{x}{b_0}\right]\right\} \\
&= \left|\frac{b_0}{2}\right| \cdot SINC\left[\frac{b_0\xi}{2}\right] \cdot COMB[b_0\xi] \\
&= \left|\frac{b_0}{2}\right| \cdot SINC\left[\frac{b_0\xi}{2}\right] \cdot \sum_{k-\infty}^{+\infty} \delta[b_0 \cdot \xi - k] \\
&= \left|\frac{b_0}{2}\right| \cdot SINC\left[\frac{b_0\xi}{2}\right] \cdot \sum_{k-\infty}^{+\infty} \delta\left[b_0 \cdot \left(\xi - \frac{k}{b_0}\right)\right] \\
&= \left|\frac{b_0}{2}\right| \cdot SINC\left[\frac{b_0\xi}{2}\right] \cdot \frac{1}{|b_0|} \cdot \sum_{k-\infty}^{+\infty} \delta\left[\xi - \frac{k}{b_0}\right] \\
&= \frac{1}{2} \cdot SINC\left[\frac{\xi}{\left(\frac{2}{b_0}\right)}\right] \cdot \sum_{k-\infty}^{+\infty} \delta\left[\xi - \frac{k}{b_0}\right] \\
&= \frac{1}{2} \cdot \sum_{k-\infty}^{+\infty} SINC\left[\frac{\xi}{\left(\frac{2}{b_0}\right)}\right] \cdot \delta\left[\xi - \frac{k}{b_0}\right] \\
&= \frac{1}{2} \cdot \sum_{k-\infty}^{+\infty} SINC\left[\frac{\frac{k}{b_0}}{\left(\frac{2}{b_0}\right)}\right] \cdot \delta\left[\xi - \frac{k}{b_0}\right] \\
&= \frac{1}{2} \cdot \sum_{k-\infty}^{+\infty} SINC\left[\frac{k}{2}\right] \cdot \delta\left[\xi - \frac{k}{b_0}\right]
\end{aligned}
$$

$$
\begin{aligned}
F[\xi] &= \frac{1}{2} \cdot \delta[\xi] + \frac{1}{2} \cdot SINC\left[\frac{1}{2}\right] \\
&\quad \cdot \left(\delta\left[\xi + \frac{1}{b_0}\right] + \delta\left[\xi - \frac{1}{b_0}\right]\right) + \frac{1}{2} \cdot SINC\left[\frac{3}{2}\right] \cdot \left(\delta\left[\xi + \frac{3}{b_0}\right] + \delta\left[\xi - \frac{3}{b_0}\right]\right) + \cdots \\
&= \frac{1}{2} \cdot \delta[\xi] + \frac{1}{2} \cdot \frac{2}{\pi} \cdot \left(\delta\left[\xi + \frac{1}{b_0}\right] + \delta\left[\xi - \frac{1}{b_0}\right]\right) \\
&\quad + \frac{1}{2} \cdot \left(-\frac{2}{3\pi}\right) \cdot \left(\delta\left[\xi + \frac{3}{b_0}\right] + \delta\left[\xi - \frac{3}{b_0}\right]\right) + \cdots \\
&= \frac{1}{2} \cdot \delta[\xi] + \frac{1}{\pi} \cdot \left(\delta\left[\xi + \frac{1}{b_0}\right] + \delta\left[\xi - \frac{1}{b_0}\right]\right) \\
&\quad - \frac{1}{3\pi} \cdot \left(\delta\left[\xi + \frac{3}{b_0}\right] + \delta\left[\xi - \frac{3}{b_0}\right]\right) + \frac{1}{5\pi} \cdot \left(\delta\left[\xi + \frac{5}{b_0}\right] + \delta\left[\xi - \frac{5}{b_0}\right]\right) - \cdots
\end{aligned}
$$

so the frequencies in the square wave are $\xi = 0, \pm b_0^{-1}, \pm 3 \cdot b_0^{-1}, \cdots, \pm(2\ell+1) \cdot b_0^{-1}$. *This is a SINC function sampled at a spacing equal to half the reciprocal of the width parameter. This is the same pattern as that of the amplitude generated* *from a 50% square-wave diffraction grating observed in the Fraunhofer diffraction region.*



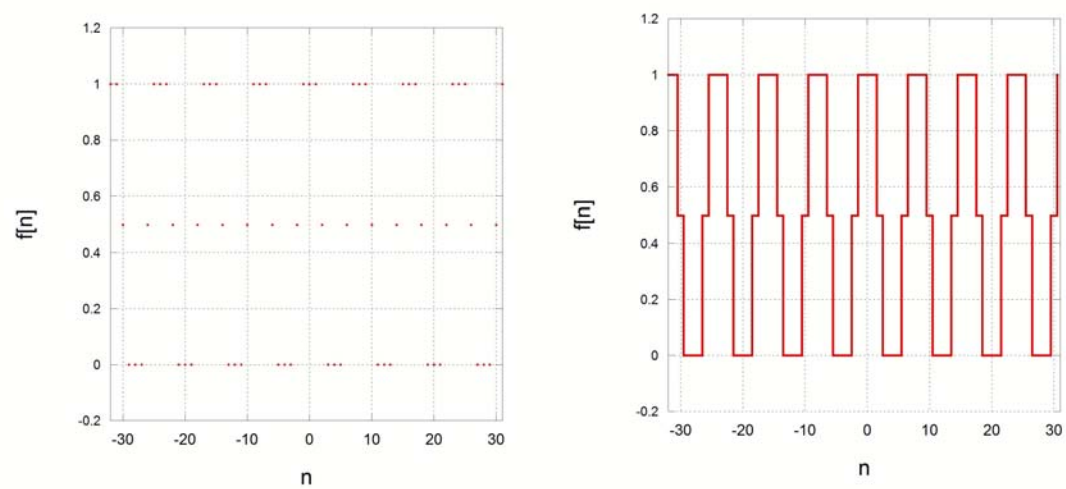
Since the period is 8 units, and since you must sample more than twice per period of the most rapidly oscillating term, we can see that a period of the fifth-order term is $\frac{8}{5} < 2$*, so this component will not be sampled sufficiently to avoid aliasing. We'll see this again in part (d)*

(c) Write down the expression for a sampled version of $f[x]$ evaluated at $N = 64$; the function is sampled with Δx selected so that there are 8 periods of $f[x]$ in the sampled function.

$$\frac{b_0}{\Delta x} = 8 \implies \Delta x = \frac{b_0}{8}$$

$$f[x] = \left(RECT\left[\frac{x}{4 \cdot \Delta x}\right] * \frac{1}{|16 \cdot \Delta x|} COMB\left[\frac{x}{16 \cdot \Delta x}\right] \right) \cdot \frac{1}{|\Delta x|} COMB\left[\frac{x}{\Delta x}\right]$$

Here's what the 64 samples look like, both as samples and with the samples connected by lines using the nearest neighbor interpolator. You can see that a period of 8 samples includes in sequence: 3 samples at value "1," 1 at value "$\frac{1}{2}$," 3 at value "0," and another at value "$\frac{1}{2}$."



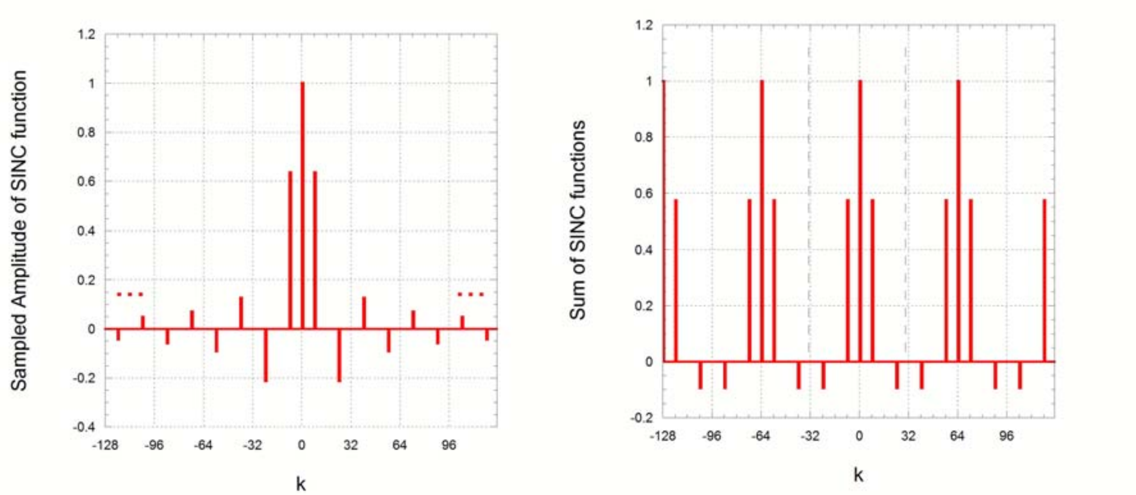
Samples of square wave with $N = 64$ and $b_0 = 8 \cdot \Delta x$: (a) without interpolation between samples; (b) with interpolation between samples.

(d) Evaluate and sketch the DFT of the sampled function evaluated at $N = 64$; determine the smallest TWO **positive**-valued frequencies of $f[x]$ (i.e., $\xi > 0$) that are aliased by the sampling process and locate them on the graph of the DFT.

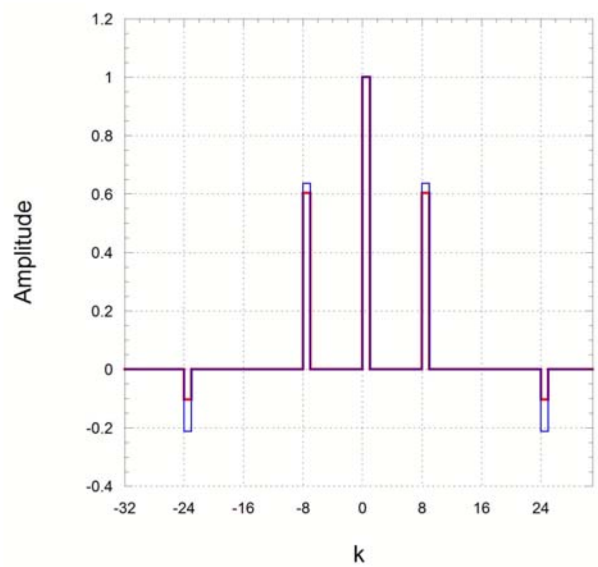
Since the sampled function includes an integer number of periods, there is no problem with leakage. However, the square wave clearly is aliased, which means that frequency components that "should be" at large spatial frequencies appear at smaller frequencies. The "continuous" Fourier transform is evaluated via the filter and modulation theorems:

$$
\begin{aligned}
F[\xi] &= \left(\frac{1}{2}\cdot\sum_{k-\infty}^{+\infty} SINC\left[\frac{k}{2}\right]\cdot\delta\left[\xi-\frac{k}{b_0}\right]\right) * COMB[\Delta x\cdot\xi] \\
&= \left(\frac{1}{2}\cdot\sum_{k-\infty}^{+\infty} SINC\left[\frac{k}{2}\right]\cdot\delta\left[\xi-\frac{k}{b_0}\right]\right) * \sum_{\ell=-\infty}^{+\infty}\delta[\Delta x\cdot\xi-\ell] \\
&= \left(\frac{1}{2}\cdot\sum_{k-\infty}^{+\infty} SINC\left[\frac{k}{2}\right]\cdot\delta\left[\xi-\frac{k}{b_0}\right]\right) * \sum_{\ell=-\infty}^{+\infty}\delta\left[\Delta x\cdot\left(\xi-\frac{\ell}{\Delta x}\right)\right] \\
&= \frac{1}{2\cdot\Delta x}\cdot\left(\sum_{k-\infty}^{+\infty} SINC\left[\frac{k}{2}\right]\cdot\delta\left[\xi-\frac{k}{b_0}\right]\right) * \sum_{\ell=-\infty}^{+\infty}\delta\left[\xi-\frac{\ell}{\Delta x}\right] \\
&= \frac{1}{2\cdot\Delta x}\cdot\sum_{\ell=-\infty}^{+\infty}\left(\sum_{k-\infty}^{+\infty} SINC\left[\frac{k}{2}\right]\cdot\delta\left[\xi-\frac{k}{b_0}\right]\right) * \delta\left[\xi-\frac{\ell}{\Delta x}\right]
\end{aligned}
$$

So the DFT is composed of the sum of an infinite number of replicas of the same sampled SINC function:



(a) sampled SINC spectrum resulting from Fourier transform of continuous square wave; (b) sum of replicas of sampled SINC function, showing that the overlapping terms change the amplitude at the overlapping frequencies.



Samples of SINC function in blue, DFT evaluated at $N = 64$ showing reduced amplitude due to aliasing.

From the graph, you can see that the first two orders have frequencies sufficiently small as to avoid aliasing, so the smallest two aliased frequencies are:

$$\begin{aligned}\xi_5 &= 5 \cdot \Delta\xi = \frac{5}{b_0} = \frac{5}{N \cdot \Delta x} = \frac{5}{64} \cdot \frac{1}{\Delta x} \\ \xi_7 &= 7 \cdot \Delta\xi = \frac{7}{N \cdot \Delta x} = \frac{7}{64} \cdot \frac{1}{\Delta x}\end{aligned}$$

Since $\Delta x = \frac{1}{64}$*samples, then:*

$$\begin{aligned}k_1 &= 8 \implies \xi_1 = 8 \cdot \Delta\xi = \frac{8}{N \cdot \Delta x} = \frac{8}{64}\frac{\text{cycles}}{\text{sample}} = \frac{1}{8}\frac{\text{cycles}}{\text{sample}} < \frac{1}{2}\frac{\text{cycles}}{\text{sample}} \\ k &= 8 + 16 \implies \xi_3 = 24 \cdot \Delta\xi = \frac{24}{N \cdot \Delta x} = \frac{24}{64}\frac{\text{cycles}}{\text{sample}} = \frac{3}{8}\frac{\text{cycles}}{\text{sample}} < \frac{1}{2}\frac{\text{cycles}}{\text{sample}} \\ k &= 24 + 16 \implies \xi_3 = 40 \cdot \Delta\xi = \frac{40}{N \cdot \Delta x} = \frac{40}{64}\frac{\text{cycles}}{\text{sample}} = \frac{5}{8}\frac{\text{cycles}}{\text{sample}} > \frac{1}{2}\frac{\text{cycles}}{\text{sample}} \\ k &= 40 + 16 \implies \xi_3 = 56 \cdot \Delta\xi = \frac{56}{N \cdot \Delta x} = \frac{56}{64}\frac{\text{cycles}}{\text{sample}} = \frac{7}{8}\frac{\text{cycles}}{\text{sample}} > \frac{1}{2}\frac{\text{cycles}}{\text{sample}}\end{aligned}$$

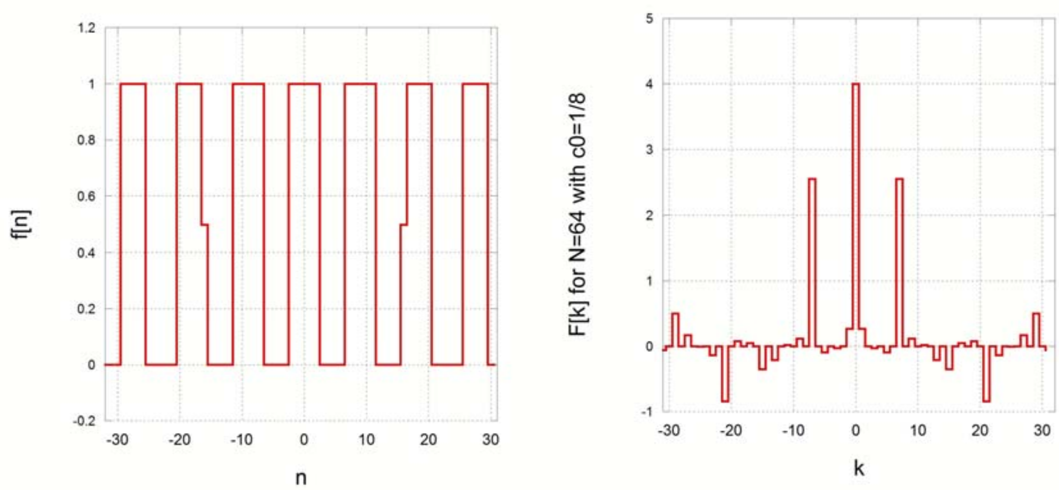
$$\begin{aligned}\xi_5 &= \frac{5}{8}\frac{\text{cycles}}{\text{sample}} \\ \xi_7 &= \frac{7}{8}\frac{\text{cycles}}{\text{sample}}\end{aligned}$$

10. **Extension of previous problem, which you must do before doing this one**

Consider the same space domain function $f[x]$ that is now sampled at $N = 64$ points with Δx selected such that there are 7 periods of $f[x]$ in the sampled function.

(a) Evaluate and sketch the DFT of the sampled function evaluated at $N = 64$ samples.

If there are 7 periods in 64 samples, then the length of each period is $64/7 = 9\frac{1}{7}$ *pixels. In this case, the aliased frequencies in the spectrum do not "overlap" the "unaliased" frequencies, as they did in the case of 8 cycles in 64 samples. The spectrum in the case of 7 cycles in 64 samples shows many more terms.*



(b) Determine the smallest TWO **positive**-valued frequencies of $f[x]$ (i.e., $\xi > 0$) that are aliased by the sampling process and locate them on the graph of the DFT.

$$\begin{aligned}
k_1 &= 7 \\
k_2 &= 7 + 14 = 21 \\
k_3 &= 21 + 14 = 35 \implies \xi_3 = 35 \cdot \Delta\xi = \frac{35}{N \cdot \Delta x} = \frac{35}{64} \cdot \frac{1}{\Delta x} = \frac{35}{64} \frac{\text{cycles}}{\text{sample}} > \frac{1}{2} \frac{\text{cycles}}{\text{sample}} \\
k_4 &= 35 + 14 = 49 \implies \xi_4 = 49 \cdot \Delta\xi = \frac{49}{N \cdot \Delta x} = \frac{49}{64} \cdot \frac{1}{\Delta x} = \frac{49}{64} \frac{\text{cycles}}{\text{sample}} > \frac{1}{2} \frac{\text{cycles}}{\text{sample}}
\end{aligned}$$

$$\xi = \frac{35}{64} \frac{\text{cycles}}{\text{sample}} \ \textit{and} \ \xi = \frac{49}{64} \frac{\text{cycles}}{\text{sample}} \ \textit{are aliased}$$