IMGS-616-20142 Final Exam 18 December 2014

3 HOURS, Do SIX (6) of the following (equal weight, though not equal difficulty)
Closed Book, No Notes, NO Calculators, No Phones, iPods, etc.; Put them away
WRITE ON ONE SIDE OF PAGE ONLY ("unprinted" side of scratch paper)
Submit ONLY the selected problems <u>IN NUMERICAL ORDER</u>
SHOW YOUR WORK! At minimum, list thought process leading to conclusions.
Note that you must finish #9 to do #10
STANDARD HINT: sketches help equations, LABEL AXES on graphs

1. What property (or "properties") of an input function $f[x]$ that makes $f[x]$ useful for testing the frequency response of an imaging system? Put another way, what are the qualities of the input function $f[x]$ that make it useful for solving the "system analysis" task? Explain the reasons and list at least three examples of inputs that satisfy this property. Discuss or describe any practical issues resulting from the three cases.

2. Prove **TWO** of these three theorems

   (a) "transform-of-a-transform" theorem

   (b) filter theorem

   (c) derivative theorem

3. For the matrix $\underline{\mathbf{A}}$ shown:

$$\underline{\mathbf{A}} = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ -1 & 0 & 1 & -2 \end{bmatrix}$$

   (a) find the eigenvalues and use to determine if the matrix is invertible; explain what the eigenvalues "mean" for each eigenvector.

   (b) evaluate the inverse matrix $\underline{\mathbf{A}}^{-1}$ or the pseudoinverse matrix $\underline{\mathbf{A}}^{\dagger}$, as appropriate.

   (c) Evaluate the product of $\underline{\mathbf{AA}}^{-1}$ or $\underline{\mathbf{AA}}^{\dagger}$, as appropriate.

4. For the listed object $f[x]$ and impulse response $h[x]$, evaluate $|g[x]|^2 = |f[x] * h[x]|^2$ and sketch for $\alpha_0 = 2$ and $b_0 = 1$.You will need to simplify the expression.

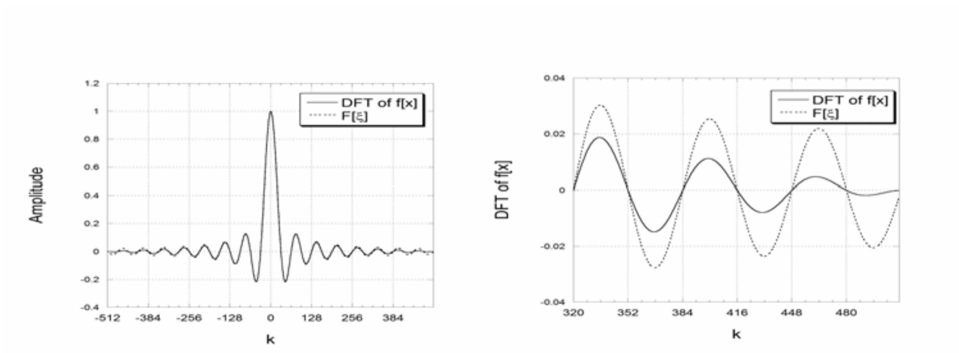
$$\begin{aligned} f[x] &= \exp\left[+i\pi\left(\frac{x}{\alpha_0}\right)^2\right] \\ h[x] &= \delta[x + b_0] + \delta[x - b_0] \end{aligned}$$

5. Consider the 1-D function $f[x] = \exp[+i\pi x]$

   (a) Evaluate and sketch the discrete Fourier transform (DFT) of $f[n]$ derived from $f[x]$ by sampling at spacing $\Delta x = 0.25$ units with $N = 32$.

   (b) Evaluate and sketch the DFT of $f[n]$ sampled at the same spacing with $N = 128$.

6. A colleague uses a discrete Fourier transform program to evaluate the spectrum $F[k]$ of $f[n]$, which is a sampled rectangle function with width parameter $b_0 = 32$ samples for $N = 1024$ and $\Delta x = \frac{1}{32}$ mm. The graph of the sampled spectrum is shown in the solid line. Samples of the spectrum of the continuous rectangle function are shown as the dashed line. The second graph is a magnified view of the details at the right-hand side of the first graph.



(a) Determine the size of the interval between samples of the DFT (i.e., in the frequency domain) for the stated value of N.

(b) Determine the maximum of the absolute value of the frequency in the DFT in "cycles per millimeter."

(c) In words, explain the discrepancy between the two plotted lines in the graph; the detail plot on the right may be helpful to visualize the differences. Your explanation should be as detailed as you can make it; one-word "explanations" are not sufficient. You might start by specifying the differences between the two graphs.

(d) Sketch the space-domain functions that would be obtained by evaluating the inverse DFT over 1024 samples for both of the lines in the graph; be sure to label your axes AND to explain any observed features.

7. Consider three cases of complex-valued noise $n[x]$ that is "bipolar" (positive and negative values): (a) the power spectrum $|N_a[\xi]|^2$ is constant over all frequencies; (b) the power spectrum $|N_b[\xi]|^2$ is large at low frequencies and goes to zero in the limit $|\xi| \to \infty$; (c) the power spectrum $|N_c[x]|^2$ is zero at $\xi = 0$ and greater than zero at nonzero frequencies.

(a) Describe any significant "features" of the space-domain representations of the noise, put another way, what are the expected "qualities" of the three instances of the noise: $n_a[x] = \mathcal{F}_1^{-1}\{N_1[\xi]\}$, ...

(b) Sketch the expected "shapes" of the autocorrelations of the three cases (you may do this for specific power spectra of your choice that satisfy the stated conditions, but this is not required);

(c) Describe the features that are visible in the three autocorrelation functions and give reasons for their existence..

8. You are familiar with the notations for the sum and product of several terms:

$$\sum_{n=0}^{\infty} x^n = x^0 + x^1 + x^2 + \cdots$$

$$\prod_{n=0}^{\infty} = a_n = a_0 \cdot a_1 \cdot a_2 \cdots$$

We now define a notation "$\bigodot$" for the convolution of $N+1$ examples of functions:

$$\bigodot_{n=0}^{N} f_n[x] \equiv f_0[x] * f_1[x] * \cdots * f_N[x]$$

Evaluate this function for an arbitrary value of N in these cases

(a) $f_n[x] = SINC\left[\frac{x}{n+1}\right]$

(b) $f_n[x] = \exp[-\pi x^2]$

(c) $f_n[x] = \exp\left[(-1)^n \cdot i \cdot \pi \cdot \left(\frac{x}{\alpha_0}\right)^2\right]$

9. The space-domain function $f[x]$ is a "50% square wave," which means that the amplitude "switches" from a value of unity ("on") to a value of zero ("off") at uniformly spaced intervals. Assume that the period of the square wave is b_0

(a) Write down an equation for this function and sketch it.

(b) Evaluate the formula for and sketch its spectrum

(c) Write down the expression for a sampled version of $f[x]$ evaluated at $N = 64$; the function is sampled with Δx selected so that there are 8 periods of $f[x]$ in the sampled function.

(d) Evaluate and sketch the DFT of the sampled function evaluated at $N = 64$; determine the smallest TWO **positive**-valued frequencies of $f[x]$ (i.e., $\xi > 0$) that are aliased by the sampling process and locate them on the graph of the DFT.

10. **Extension of previous problem, which you must finish to do this one**

Consider the same space domain function $f[x]$ that is now sampled at $N = 64$ points with Δx selected such that there are 7 periods of $f[x]$ in the sampled function.

(a) Evaluate and sketch the DFT of the sampled function evaluated at $N = 64$ samples.

(b) Determine the smallest TWO **positive**-valued frequencies of $f[x]$ (i.e., $\xi > 0$) that are aliased by the sampling process and locate them on the graph of the DFT.