$$\mathcal{F}\left\{ r\left[\frac{x-x_1}{b_0}\right] e^{+i\pi\left(\frac{x}{\alpha_0}\right)^2}\right\}$$

$$\cong e^{+i\frac{\pi}{4}}\, r\left[\frac{\xi - \frac{x_1}{\alpha_0^2}}{\frac{b_0}{\alpha_0^2}}\right] |\alpha_0|\, e^{-i\pi(\alpha_0\xi)^2}$$

"NEAR FIELD"

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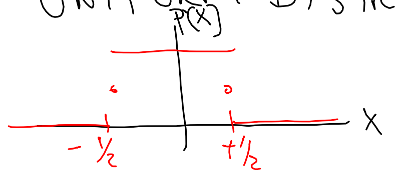
CENTRAL LIMIT THEOREM

$$\left(f(x)\right)_1 \otimes \left(f(x)\right)_2 \otimes \left(f(x)\right)_3 \otimes \cdots \otimes \left(f(x)\right)_N = \; ? \quad \cong \text{GAUSSIAN}$$

"STATISTICAL"

$n(x)$, e.g., UNIFORM DISTRIBUTION



$$n_1(x) + n_2(x) \Rightarrow P_{1+2}(n) = P_1(n) \otimes P_2(x)$$

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$$(f(x))_1 * (f(x))_2 * \cdots * (f(x))_N \cong \underbrace{A_0 \, e^{-\pi\left(\frac{x}{d_0}\right)^2}}_{A_0,\ d_0 \text{ TO BE DETERMINED}} = g(x)$$

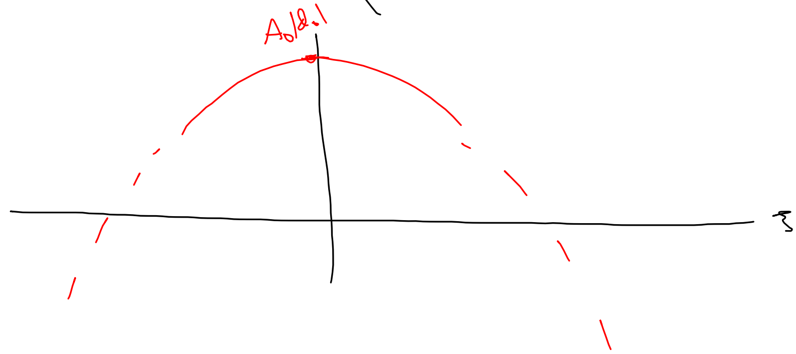
$\mathcal{F}_1$

$$(F(\xi))^N \cong A_0|d_0| \, e^{-\pi(d_0\xi)^2}$$

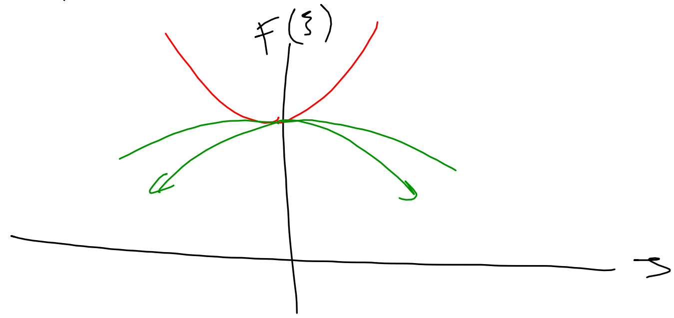
$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \cdots$$

$$e^{-\pi(d_0\xi)^2} = \sum_{n=0}^{\infty} \frac{\left[-\pi d_0^2 \xi^2\right]^n}{n!} = 1 - \pi d_0^2 \xi^2 + \frac{\pi^2 d_0^4 \xi^4}{2} - \cdots$$

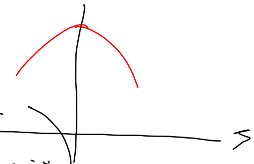
$$A_0|d_0| \, e^{-\pi(d_0\xi)^2} \cong A_0|d_0| \left(1 - \pi d_0^2 \xi^2\right) \quad \text{FOR } |\xi| \gtrsim 0$$



UNLESS $F(\xi)$ LOOKS LIKE

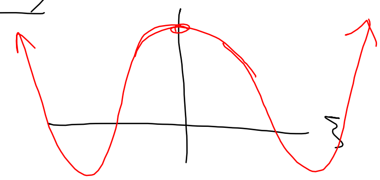


$$f(x) = \mathrm{Rect}(x) \Rightarrow F(\xi) = \mathrm{sinc}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}$$



$$\mathrm{sinc}(\xi) = \frac{\left(\frac{(\pi\xi)^1}{1!} - \frac{(\pi\xi)^3}{3!} + \frac{(\pi\xi)^5}{5!} - \cdots\right)}{\pi\xi}$$

$$= 1 - \frac{(\pi\xi)^2}{6} + \frac{(\pi\xi)^4}{120} - \cdots$$



$$\mathrm{sinc}(\xi) \cong 1 - \frac{(\pi\xi)^2}{6} \quad \text{FOR } |\xi| \gtrsim 0$$

$$\left(\mathrm{sinc}(\xi)\right)^N \cong \left(1 - \frac{(\pi\xi)^2}{6}\right)^N \quad \text{FOR } |\xi| \gtrsim 0$$

$$(1-u)^N = 1 - Nu + \frac{N(N-1)}{2!}u^2 + \frac{N(N-1)(N-2)}{3!}u^3 + \cdots$$

IF u IS SMALL!

$$\left(\mathrm{sinc}(\xi)\right)^N \approx 1 - N\cdot\frac{(\pi\xi)^2}{6} \cong A_0|d_0|\,e^{-\pi(d_0\xi)^2}$$

IF CENTRAL LIMIT Th. VALID

$$A_0|d_0|\,e^{-\pi(d_0\xi)^2} \simeq A_0|d_0|\left(1 - \frac{\pi d_0^2\xi^2}{1!} + \frac{(\pi d_0^2\xi^2)^2}{2!} - \cdots\right)$$

$$\left(\mathrm{sinc}(\xi)\right)^N \simeq 1 - N\frac{\pi^2\xi^2}{6} \simeq A_0|d_0|\left(1 - \pi d_0^2\xi^2\right)$$

$$A_0|d_0| = 1 \Rightarrow A_0 = \frac{1}{|d_0|}$$

$$N\frac{\pi^{\cancel{2}}\cancel{\xi^2}}{6} = \cancel{\pi} d_0^2 \cancel{\xi^2}$$

$$\frac{N\pi}{6} = d_0^2 \Rightarrow |d_0| = \sqrt{\frac{N\pi}{6}}$$

$$G(\xi) \cong A_0|d_0|\,e^{-\pi(d_0\xi)^2} = \boxed{1\cdot e^{-\pi\left(\sqrt{\frac{N\pi}{6}}\right)^2\xi^2}}$$

$$F(\xi) \cong \left(G(\xi)\right)^{\frac{1}{N}} = \left(e^{-\pi\left(\sqrt{\frac{N\pi}{6}}\right)^2\xi^2}\right)^{\frac{1}{N}}$$

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$$G(\xi) \cong e^{-\pi\left(\sqrt{\frac{N\pi}{6}}\right)^2 \xi^2}$$

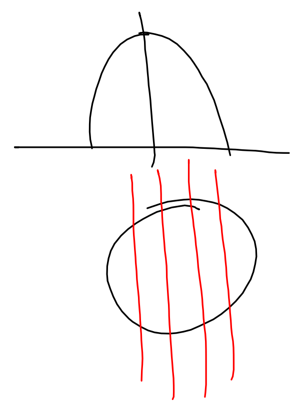
d_0

$$g(x) = \frac{1}{|d_0|} e^{-\pi\left(\frac{x}{d_0}\right)^2} = \boxed{\frac{1}{\sqrt{\frac{N\pi}{6}}} e^{-\pi\left(\frac{x}{\sqrt{\frac{N\pi}{6}}}\right)^2}}$$

$$= \frac{1}{\sqrt{\frac{N\pi}{6}}} \text{GAUS}\left(\frac{x}{\sqrt{\frac{N\pi}{6}}}\right)$$

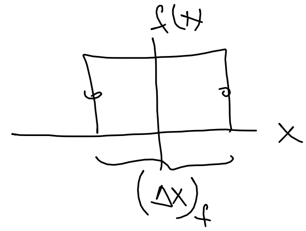
SHAPE OF $(\text{RECT}(x))_1 * (\text{RECT}(x))_2 * \cdots * (\text{RECT}(x))_N$

$$\left(\mathrm{SINC}(x)\right)_1 \ast \left(\mathrm{SINC}(x)\right)_2 \ast \cdots \left(\mathrm{SINC}(x)\right)_6 = \mathrm{SINC}(x) \neq \mathrm{GAUS}$$

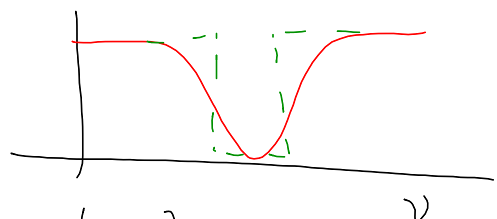


Width Metrics → Uncertainty Relation

$(\Delta x)_f \rightarrow$ "WIDTH" OF $f(x)$

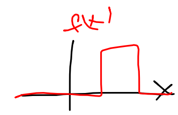


$(\Delta \xi)_F \rightarrow$ "WIDTH" OF $F(\xi)$



"EQUIVALENT WIDTH"

$$(\Delta x)_f = \frac{\int_{-\infty}^{+\infty} f(x)\,dx}{f(0)} = \frac{F(\xi=0)}{f(x=0)} = \frac{F(0)}{f(0)}$$



$$(\Delta \xi)_F = \frac{\int_{-\infty}^{+\infty} F(\xi)\,d\xi}{F(0)} = \frac{f(0)}{F(0)}$$

$$(\Delta x)_f \cdot (\Delta \xi)_F = \frac{F(0)}{f(0)} \cdot \frac{f(0)}{F(0)} = 1$$