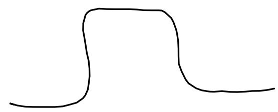
November 11, 2014

"APPROXIMATIONS" TO THE FOURIER TRANSFORM

(1) MOMENT THEOREM

$$F[\xi] = \sum_{n=0}^{\infty} a_n \xi^n ; \quad |\xi| \gtrsim 0$$

$$e^{-\pi|x|^n}$$



(2) STATIONARY-PHASE APPROXIMATION

$$f(x) = r[x]\, e^{+i\pi x^2} \qquad |\xi| >> 0$$

$$\longrightarrow r\left[\frac{x-x_0}{b_0}\right] e^{\pm i\pi\left(\frac{x}{\alpha_0}\right)^2}$$

(3) CENTRAL LIMIT THEOREM

(4) WIDTH METRICS

November 11, 2014

MOMENTS

$$m_\ell\{f(x)\} = \int_{-\infty}^{+\infty} f(x)\, x^\ell \, dx = \langle x^\ell \rangle_f$$

$\ell = 0 \Rightarrow \underline{1(x)}$

$\ell = 1 \Rightarrow x$

$\ell = 2 \Rightarrow x^2$

$\vdots$

$$F(\xi) = \int_{-\infty}^{+\infty} f(x)\, e^{-i2\pi\xi x} \, dx$$

$$\frac{d^\ell F}{d\xi^\ell} = \int_{-\infty}^{+\infty} f(x) \frac{d^\ell}{d\xi^\ell}\left(e^{-i2\pi\xi x}\right) dx$$

$$(-i2\pi x)^\ell e^{-i2\pi\xi x}$$

$$= (-i2\pi)^\ell \int_{-\infty}^{+\infty} \left(f(x)\, x^\ell\right) e^{-i2\pi\xi x} \, dx$$

$$\left(\frac{1}{-i2\pi}\right)^\ell F^{(\ell)}(\xi) = \mathcal{F}\left\{f(x)\, x^\ell\right\}$$

$$\left(-\frac{1}{i2\pi}\right)^\ell F^{(\ell)}[\xi = 0] = \int_{-\infty}^{+\infty} f(x)\, x^\ell \cdot \cancelto{1}{e^{-i2\pi \cdot 0 x}} \, dx = m_\ell$$

$$F^{(\ell)}[0] = (-i2\pi)^\ell \, m_\ell$$

$$f(x) = \text{RECT}(x) \Rightarrow F(\xi) = \text{SINC}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi} = \frac{\pi\xi - \frac{(\pi\xi)^3}{3!} + \frac{(\pi\xi)^5}{5!} - \cdots}{\pi\xi}$$

$$= 1 - \frac{(\pi\xi)^2}{6} + \frac{(\pi\xi)^4}{120} - \cdots$$

$$m_\ell\{\text{RECT}(x)\} = \int_{-\infty}^{+\infty} x^\ell \, \text{RECT}(x)\, dx = \int_{-1/2}^{1/2} x^\ell \, dx$$

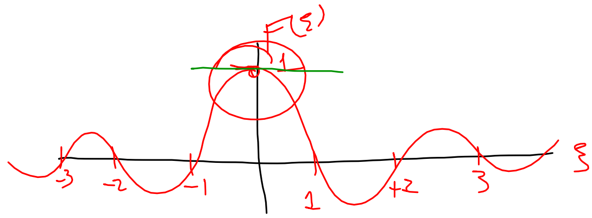
$$= \left.\frac{x^{\ell+1}}{\ell+1}\right|_{-1/2}^{+1/2}$$

$$m_0 = \int_{-\infty}^{+\infty} \text{RECT}(x) \cdot x^0 \, dx = \int_{-\infty}^{+\infty} \text{RECT}(x)\, dx = 1$$

$$m_1 = 0 = \int \text{RECT}(x) \cdot x \, dx = \int_{-1/2}^{+1/2} x \, dx = 0$$

$$m_2 = \left.\frac{x^3}{3}\right|_{-1/2}^{1/2} = \frac{1}{24} - \left(-\frac{1}{24}\right) = \frac{1}{12}$$

$$F(\xi) = \mathrm{sinc}(\xi) = 1 - \frac{(\pi\xi)^2}{6} + \frac{(\pi\xi)^4}{120} - \cdots$$



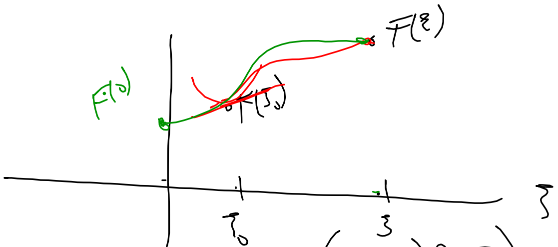
$m_0 = 1$

$m_1 = 0$

$m_2 = \frac{1}{12}$

$m_3 = 0$

$$F(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n F}{d\xi}\right|_{\xi=\xi_0} (\xi-\xi_0)^n$$



$$\rightarrow F(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(F^{(n)}[0]\right)(\xi)^n \;;\quad \xi_0 = 0$$

$$\boxed{F(\xi) = \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{n!}(-i2\pi)^n \underline{\underline{m_n}}\right]}_{a_n} \xi^n}$$

$$f(x) = \mathrm{RECT}(x) \longrightarrow F(\xi) = \mathrm{SINC}(\xi) = 1 - \frac{(\pi\xi)^2}{3!} + \frac{(\pi\xi)^4}{5!} - \cdots$$

$$m_0 = 1$$

$$m_1 = m_3 = 0$$

$$m_2 = \frac{1}{12}$$

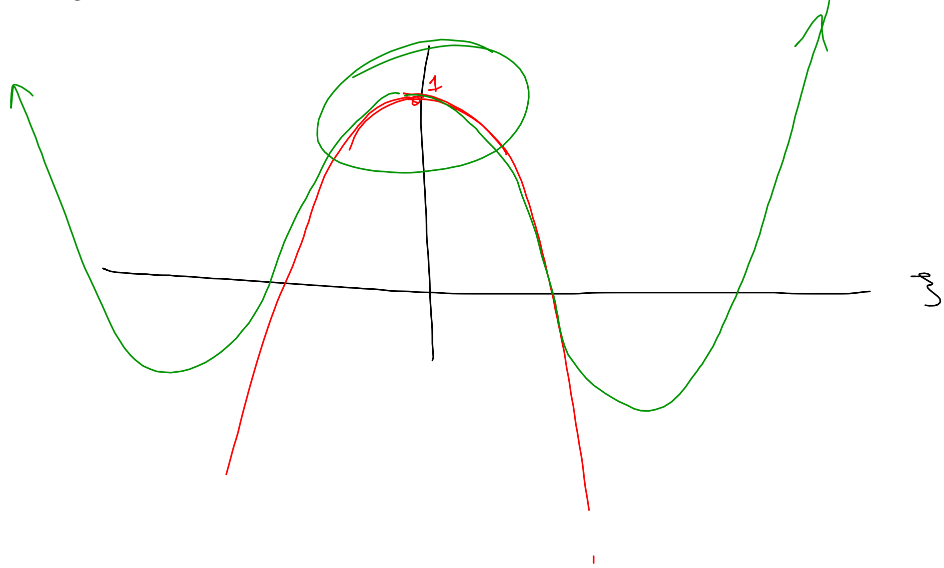
$$F(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} (-i2\pi)^n \, m_n \, \xi^n$$

$$= 1 \cdot \xi^0 + 0 + \frac{1}{2}(-i2\pi)^2 \frac{1}{12} \xi^2$$

$$1 \quad + \left(\frac{-1}{24}\right) \cdot 4 \cdot \pi^2 \xi^2$$

$$- \frac{1}{6}(\pi\xi)^2$$

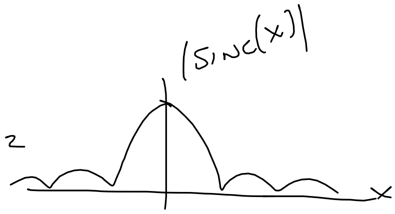
Sinc(ξ) for $|\xi| \gtrsim 0$ $\qquad F(\xi) = 1 - \frac{(\pi\xi)^2}{6} + \frac{(\pi\xi)^4}{120}$



$$
\begin{tikzcd}
 & f(x) \arrow[dl, "\int f(x) x^{\ell} dx"'] \arrow[dr, shift left] & \\
m_{\ell} \arrow[rr, shift left] & & F(\xi) \arrow[ul, shift left] \arrow[ll, shift left]
\end{tikzcd}
$$

$\ell = 0, 1, \cdots,$

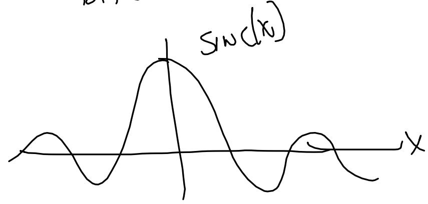
STATIONARY PHASE



$$f(x) = |\mathrm{sinc}((x-2))| \cdot e^{+i\pi x^2}$$

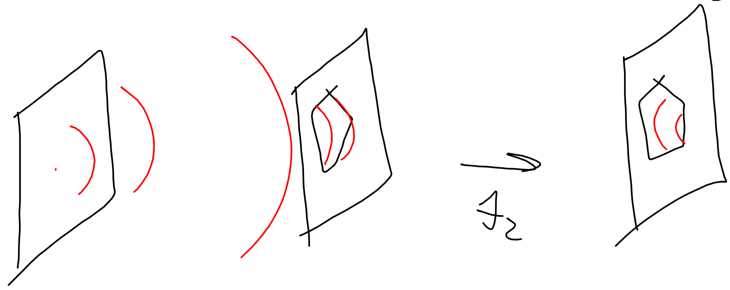
$$\mathcal{F}\{|\mathrm{sinc}(x)|\} * e^{+i\frac{\pi}{4}} e^{-i\pi\xi^2}$$

$$f_2(x) = \underbrace{\mathrm{sinc}(x)}_{\text{BIPOLAR}}\, e^{i\pi x^2}$$

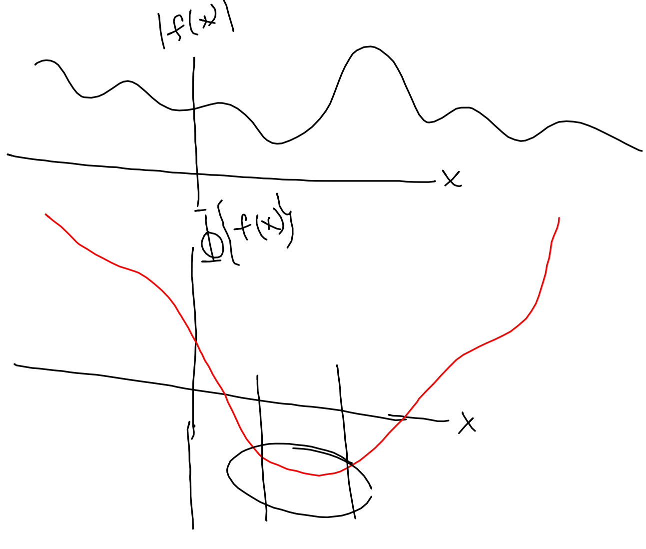


$$\mathrm{sinc}\left(\frac{x - x_0}{b_0}\right) e^{i\pi \frac{x^2}{\alpha_0}} \xrightarrow{\mathcal{F}} F(\xi) \simeq \mathrm{sinc}\left(\frac{\xi - \frac{x_0}{\alpha_0^2}}{\frac{b_0}{\alpha_0^2}}\right) e^{-i\pi(\alpha_0 \xi)^2} e^{+i\frac{\pi}{4}}$$

$\alpha_0 = 1$ $\qquad$ x_0, b_0



METHOD OF STATIONARY PHASE → M.O. STEEPEST DESCENTS



$$\int |f(x)| e^{i\Phi(x)} \, dx$$

$$\xi(x) = \frac{1}{2\pi} \frac{d\Phi}{dx}$$

November 11, 2014

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{+\infty} \underbrace{\left[f(x)\; e^{-i2\pi\xi x}\right]} dx$$

$$\int_{-\infty}^{+\infty} \left(|f(x)|\, e^{i\Phi(x)}\right) e^{-i2\pi\xi x}\, dx$$

$$= \int_{-\infty}^{+\infty} |f(x)|\; e^{i(\underbrace{\Phi(x)}_{\alpha x^2} - 2\pi\xi x)}\; dx$$

$$= \int_{-\infty}^{+\infty} \underbrace{|f(x)|}\; e^{i\xi \underbrace{\left(\frac{\Phi(x)}{\xi} - 2\pi x\right)}_{\mu(x)}}\; dx = F(\xi)$$

$$\Phi[x] = \pi\left(\frac{x}{\alpha_0}\right)^2$$

$$\int_{-\infty}^{+\infty} |f[x]| \, e^{i\xi\left(\frac{\pi\left(\frac{x}{\alpha_0}\right)^2}{\xi} - 2\pi x\right)} dx$$

$$e^{i\xi \cdot \underbrace{\mu[x]}}$$

"SMOOTH"

"WELL BEHAVED"

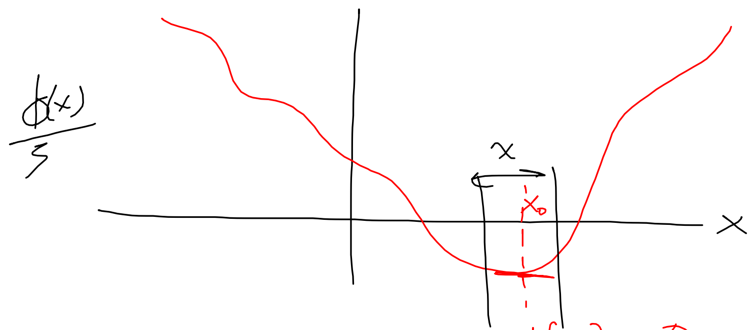
$$\mu[x] = \pi \frac{x^2}{\alpha_0^2 \xi} - 2\pi x$$

$$\mu[x] = \sum_{n=0}^{\infty} \frac{\mu^{(n)}(x_0)}{n!}(x-x_0)^n = \frac{\mu[x_0]}{0!}(x-x_0)^0 + \frac{\mu'(x_0)}{1!}(x-x_0)^1$$

TAYLOR SERIES

$$+ \frac{\mu''(x_0)}{2}(x-x_0)^2 + \frac{\mu'''(x_0)}{6}(x-x_0)^3$$

$$+ \cdots$$



$$\mu'(x_0) = 0$$

$$\mu(x) = \mu(x_0) + 0 + \frac{\mu''(x_0)}{2}(x-x_0)^2 + \cancel{\frac{\mu'''(x_0)}{6}(x-x_0)^3} + \dots$$

$$\cancel{\frac{\mu'''(x_0)}{6}(x-x_0)^3} \rightarrow 0$$

$$(x-x_0)^3 \ll (x-x_0)^2$$

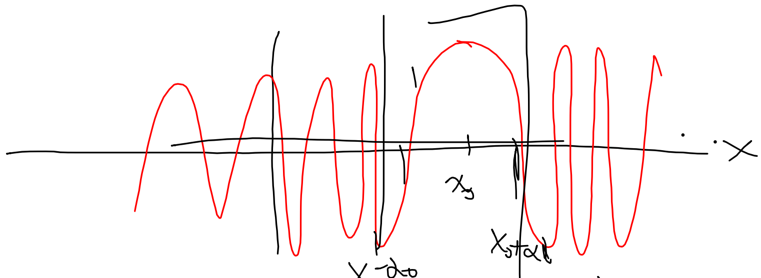
$$\int |f(x)| \, e^{i\xi \underbrace{\left(\frac{\phi(x)}{\xi} - 2\pi x\right)}_{\mu(x)}} dx = \int |f(x)| \, e^{i\xi\mu(x)} \, dx$$

$$\mu(x) \simeq \mu(x_0) + \underset{\mu'(x_0)=0}{\uparrow} \frac{\mu''(x_0)}{2}(x-x_0)^2 + 0$$

$$= \int_{\cancel{-\infty}\; x_0-\epsilon}^{\cancel{+\infty}\; x_0+\epsilon} |f(x_0)| \, e^{i\xi\mu(x_0)} \, e^{+i\left(\frac{\xi\mu''(x_0)}{2}\right)(x-x_0)^2} dx$$

$$F(\xi) \cong |f(x_0)| \, e^{i\xi\mu(x_0)} \int_{\cancel{x_0-\epsilon}\; -\infty}^{\cancel{x_0+\epsilon}\; +\infty} e^{i\left(\frac{\xi\mu''(x_0)}{2}\right)(x-x_0)^2} dx$$

$$= |f(x_0)| \, e^{i\xi\mu(x_0)} \int_{\cancel{x_0-\epsilon}\; -\infty}^{x_0+\epsilon} e^{i\pi \frac{(x-x_0)^2}{\left(\underbrace{\sqrt{\frac{2\pi}{\xi\mu''(x_0)}}}_{\alpha_0}\right)^2}} dx \qquad \left(\frac{x-x_0}{\alpha_0}\right)^2$$



$$\cos\left(\pi\frac{(x-x_0)^2}{\alpha_0^2}\right)$$

$$e^{+i\pi\frac{(x-x_0)^2}{\alpha_0^2}} \Longrightarrow |\alpha_0|\, e^{+i\frac{\pi}{4}}\, \cancelto{1}{e^{-i\pi(\alpha_0\xi)^2}}\, \cancelto{1}{e^{-i2\pi x_0\xi}} = F(\xi)$$

$$F(\xi) \cong |f(x_0)|\, e^{i\xi\mu(x_0)} \underbrace{\int_{-\infty}^{+\infty} e^{+i\pi \frac{(x-x_0)^2}{\left(\sqrt{\frac{2\pi}{\mu''(x_0)\xi}}\right)^2}}\, dx}_{\sqrt{\frac{2\pi}{\mu''(x_0)\xi}}\, e^{i\frac{\pi}{4}}} \quad \left[\frac{x-x}{\alpha}\right]^2$$

$$\boxed{\hat{F}(\xi) = |f(x_0)| \sqrt{\frac{2\pi}{\mu''(x_0)\cdot\xi}}\, e^{i\frac{\pi}{4}}\, e^{i\xi\mu(x_0)}}$$

$$\mu(x) \longrightarrow \mu'(x) = 0 \Rightarrow x_0 \Rightarrow \mu(x_0)$$

$$\mu''(x) \longrightarrow \mu''(x_0)$$

$$\int_{-\infty}^{+\infty} \left(\underbrace{r\left(\frac{x-x_0}{b_0}\right) e^{+i\pi\left(\frac{x}{\alpha_0}\right)^2}}_{f(x)}\right) e^{-i\,2\pi\xi x}\, dx = F(\xi)$$

$$e^{i\xi\mu(x)} = e^{i\xi\left(\underbrace{\pi\frac{x^2}{\alpha_0^2\xi} - 2\pi x}_{\mu(x)}\right)}$$

$$\mu'(x) = 2\pi\frac{x}{\alpha_0^2\xi} - 2\pi$$

$$\mu'(x_0) = 0\ ;\ \mu'(x_0) = 2\pi\left(\frac{x_0}{\alpha_0^2\xi} - 1\right) = 0 \Rightarrow \frac{x_0}{\alpha_0^2\xi} = 1 \qquad \boxed{x_0 = \alpha_0^2\xi}$$

$$\mu(x_0) = \pi\frac{x_0^2}{\alpha_0^2\xi} - 2\pi x_0 = \pi\frac{\alpha_0^{4}\xi^{2}}{\cancel{\alpha_0^2\xi}} - 2\pi\alpha_0^2\xi = \pi\alpha_0^2\xi - 2\pi\alpha_0^2\xi$$

$$\boxed{\mu(x_0) = -\pi\alpha_0^2\xi}$$

$$\mu''(x) = \frac{2\pi}{\alpha_0^2\xi} = \mu''(x_0)$$

$$\hat{F}(\xi) = e^{i\frac{\pi}{4}} \sqrt{\frac{2\pi}{\mu''(x_0)\cdot\xi}} \; r\left(\frac{x_0 - x_1}{b_0}\right) e^{i\xi\mu(x_0)}$$

$$\mu(x_0) = -\pi\alpha_0^2\xi \; ; \quad e^{i\xi\mu(x_0)} = e^{i\xi(-\pi\alpha_0^2\xi)} = e^{-i\pi(\alpha_0\xi)^2}$$

$$r\left(\frac{x_0 - x_1}{b_0}\right) = r\left(\frac{\alpha_0^2\xi - x_1}{b_0}\right) = r\left(\frac{\left(\frac{\alpha_0^2\xi - x_1}{\alpha_0^2}\right)}{\left(\frac{b_0}{\alpha_0^2}\right)}\right) = r\left[\frac{\xi - \frac{x_1}{\alpha_0^2}}{\left(\frac{b_0}{\alpha_0^2}\right)}\right]$$

$$\mu''(x_0) = \frac{2\pi}{\alpha_0^2\xi}$$

$$\sqrt{\frac{2\pi}{\mu''(x_0)\xi}} = \sqrt{\frac{\cancel{2\pi}}{\frac{\cancel{2\pi}}{\alpha_0^2\cancel{\xi}}\cdot\cancel{\xi}}} = \sqrt{\alpha_0^2} = |\alpha_0|$$

$$\mathcal{F}\left\{ r\left(\frac{x - x_1}{b_0}\right) e^{i\pi\left(\frac{x}{\alpha_0}\right)^2} \right\} \cong |\alpha_0| e^{i\frac{\pi}{4}} \; r\left[\frac{\xi - \frac{x_1}{\alpha_0^2}}{\left(\frac{b_0}{\alpha_0^2}\right)}\right] e^{-i\pi(\alpha_0\xi)^2}$$

(1) $r\left(\frac{x-x_1}{b_0}\right) = 1(x)$

$$f(x) = e^{i\pi\left(\frac{x}{\alpha_0}\right)^2}$$

$$F(\xi) = |\alpha_0|\, e^{i\frac{\pi}{4}}\, e^{-i\pi(\alpha_0\xi)^2}$$

$$\hat{F}(\xi) = |\alpha_0|\, e^{i\frac{\pi}{4}}\, 1(\xi)\, e^{-i\pi(\alpha_0\xi)^2}$$