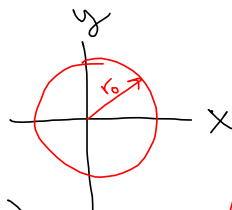
November 04, 2014

EXAM Th 11/6/2014

HANKEL TRANSFORM $f[x,y] \rightarrow f_r(r)\,1(\theta)$

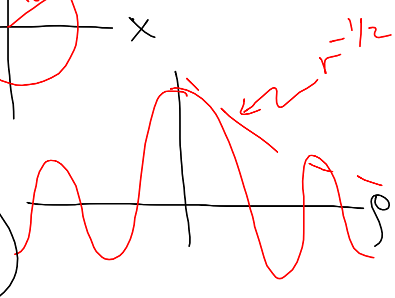
$$\mathcal{F}_2\{f_r(r)\,1(\theta)\} = \mathcal{H}_0\{f_r(r)\} = F_r(\rho)$$

$$\int_{r=0}^{\infty} f_r(r)\,2\pi r\,J_0(2\pi r\rho)\,dr$$



$$f_r(r) = \delta(r-r_0)$$

$$F_r(\rho) = 2\pi r_0\,J_0(2\pi r_0\rho)$$



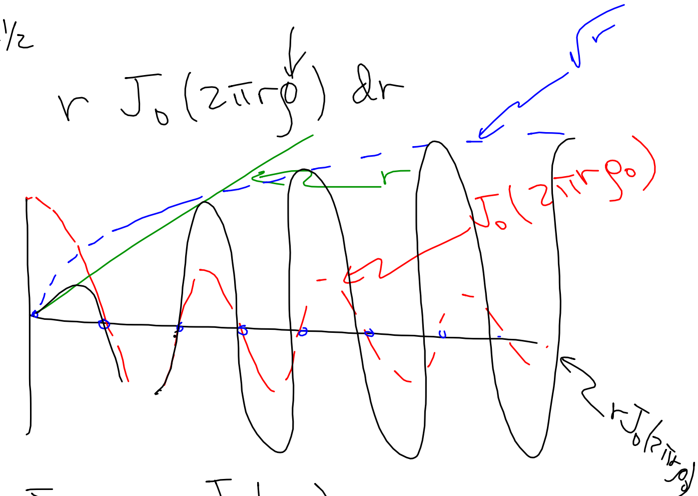
$$\mathcal{F}_2\{2\pi r\rho_0\} = \frac{1}{2\pi\rho_0}\,\delta(\rho-\rho_0)$$

$$\mathcal{H}_0^{-1}\{F_r(\rho)\} = \int_{\rho=0}^{\infty} F_r(\rho)\,2\pi\rho\,J_0(2\pi r\rho)\,d\rho$$

$$\mathcal{F}_2\left\{CYL\left(\frac{r}{d_0}\right)\right\} = \int_0^\infty CYL(r)\;2\pi r\, J_0(2\pi r\rho)\, dr$$

1

$$= 2\pi \int_0^{1/2} r\, J_0(2\pi r\rho)\, dr$$



$$\int_0^u r\, J_0(r)\, dr = u\, J_1(u)$$

$$2\pi \int_0^{1/2} r\, J_0(2\pi r\rho)\, dr$$

$$v \equiv 2\pi r\rho \Rightarrow r = \frac{v}{2\pi\rho}$$

$$dr = \frac{dv}{2\pi\rho}$$

$$r = 1/2 \Rightarrow v = 2\pi\cdot\frac{1}{2}\cdot\rho = \pi\rho$$

$$\cancel{2\pi}\int_0^{\pi\rho} \frac{v}{\cancel{2\pi}\rho}\, J_0(v)\, \frac{dv}{2\pi\rho}$$

$$\frac{1}{2\pi\rho^2}\int_0^{\pi\rho} v\, J_0(v)\, dv = \frac{1}{2\pi\rho^2}\Big[v\, J_1(v)\Big]_{v=0}^{v=\pi\rho}$$

$$= \frac{2\;1}{4}\;\frac{J_1(\pi\rho)}{\pi\rho}$$

$$\rho^{-3/2} \sim SOMB(\rho) \equiv \frac{2J_1(\pi\rho)}{\pi\rho} \sim SINC[\xi] \propto \xi^{-1}$$

$$\mathcal{F}_2\{CYL(r)\} = \frac{\pi}{4}\, SOMB(\rho)$$

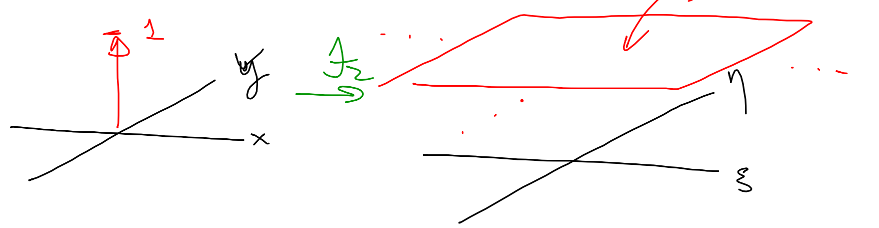
$$\mathcal{F}_2\{\mathrm{CYL}(r)\} = \frac{\pi}{4}\,\mathrm{SOMB}(\rho)$$

$$\mathcal{F}_2\{\mathrm{RECT}[x,y]\} = \mathrm{SINC}[\xi,\eta] = \mathrm{SINC}[\xi]\,\mathrm{SINC}[\eta]$$

$$\mathcal{F}\left\{\mathrm{CYL}\left(\frac{r}{d_0}\right)\right\} \longrightarrow \frac{\pi}{4}\cdot d_0^2\,\mathrm{SOMB}[d_0\rho]$$

$$\mathrm{SOMB}\left[\frac{\rho}{1/d_0}\right]$$

$$\delta(x,y) = \delta[x]\,\delta(y) = \frac{\delta(r)}{\pi r}\,1(\theta)$$



$$\int_0^\infty \frac{\delta(r)}{\cancel{\pi r}} \cdot 2\cancel{\pi r}\, J_0(2\pi r\rho)\,dr$$

$$= 2\int_0^\infty \underbrace{\delta(r)}_{\delta(r-0)} J_0(2\pi r\rho)\,dr$$

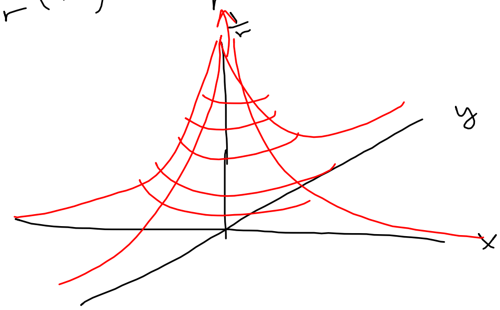
$$= 2\int_0^\infty \delta(r)\,\underbrace{J_0(2\pi\cdot 0\cdot\rho)}_{1}\,dr$$

$$= 2\int_0^\infty \delta(r)\,dr$$

$$= 1\int_{-\infty}^{+\infty} \delta(r)\,dr$$

$$= \int_{-\infty}^{+\infty} \delta[u]\,du = 1(\rho)$$

$$f_r(r) = \frac{1}{r}\,1(\theta) = \frac{1}{\sqrt{x^2+y^2}}$$



$$\int_0^\infty \cancel{\frac{1}{r}}\, 2\pi \cancel{r}\, J_0(2\pi r\rho)\, dr$$

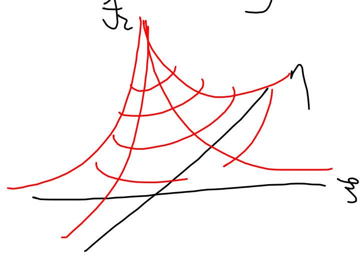
$$2\pi \int_0^\infty J_0(2\pi r\rho)\, dr$$

$$u = 2\pi r\rho \Rightarrow r = \frac{u}{2\pi\rho};\quad dr = \frac{du}{2\pi\rho}$$

$r = 0 \Rightarrow u = 0$
$r = \infty \Rightarrow u = \infty$

$$\mathcal{F}_2\left\{\frac{1}{r}\right\} = \cancel{2\pi} \cdot \int_{u=0}^{u=\infty} J_0(u) \cdot \frac{du}{\cancel{2\pi}\rho} = \frac{1}{\rho}\underbrace{\cancel{\int_0^\infty J_0(u)\, du}}_{\text{CONSTANT}} \rightarrow 1$$

$$\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}} \underset{\mathcal{F}_2^{-1}}{\overset{\mathcal{F}_2}{\rightleftarrows}} \frac{1}{\rho} = \frac{1}{\sqrt{\xi^2+\eta^2}}$$



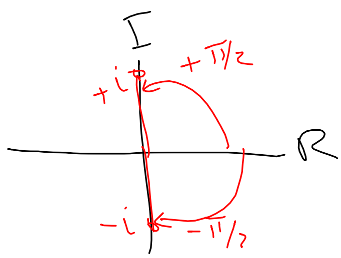
$$e^{-\pi r^2} = e^{-\pi(x^2+y^2)} = e^{-\pi x^2} e^{-\pi y^2} \longrightarrow e^{-\pi(\xi^2+\eta^2)}$$
$$= e^{-\pi\rho^2}$$

$$\int_0^\infty e^{-\pi r^2} \cdot 2\pi r\, J_0(2\pi r\rho)\, dr = e^{-\pi\rho^2}$$

$$e^{\pm i\pi(x^2+y^2)} = e^{\pm i\pi r^2}$$

$$\downarrow$$

$$e^{\pm i\pi x^2} e^{\pm i\pi y^2} \longrightarrow \left(e^{\pm i\frac{\pi}{4}} e^{\mp i\pi\xi^2}\right)\left(e^{\pm i\frac{\pi}{4}} e^{\mp i\pi\eta^2}\right)$$

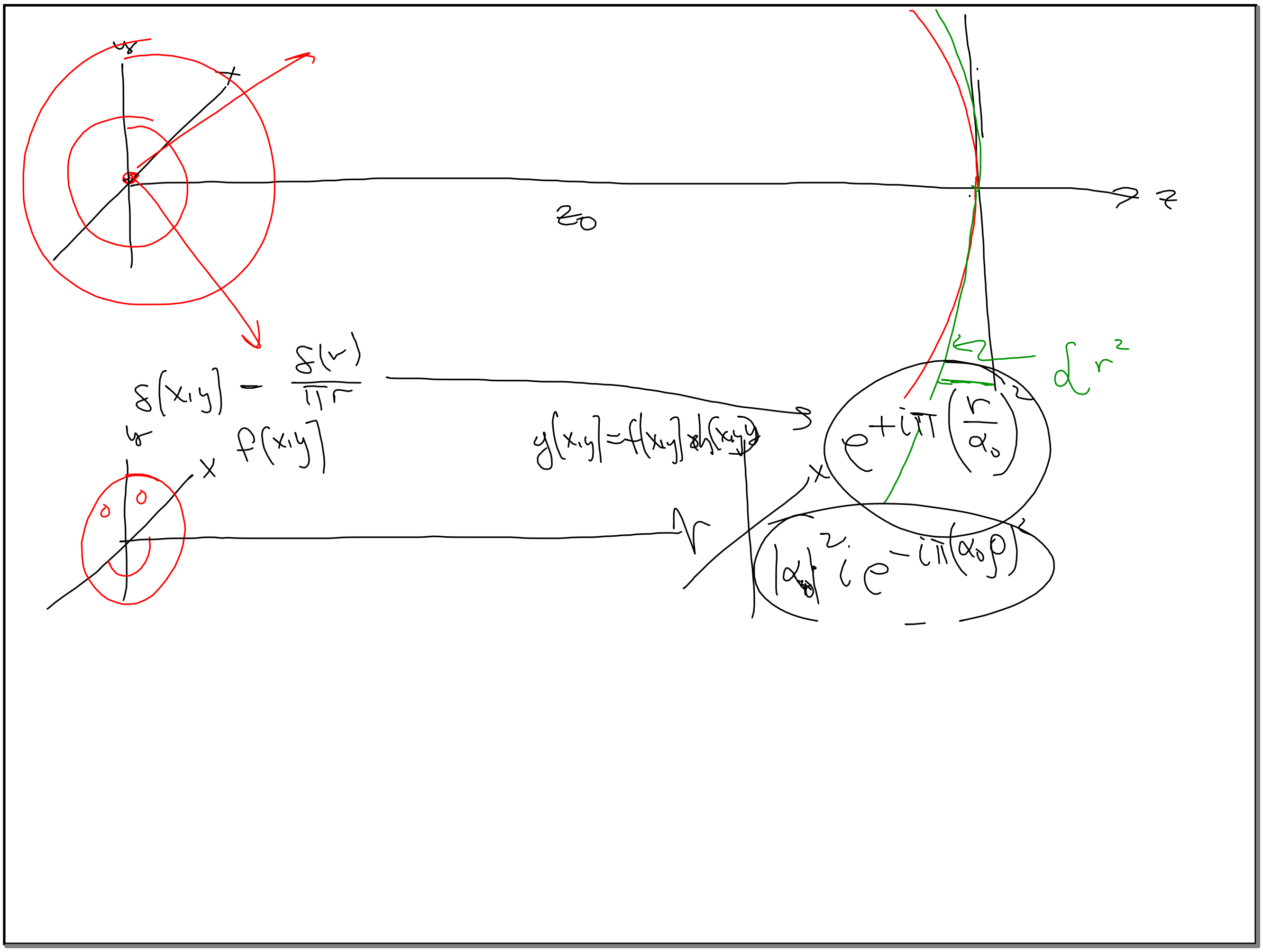


$$e^{\pm i\frac{\pi}{2}} e^{\mp i\pi(\xi^2+\eta^2)}$$

$$\pm i\, e^{\mp i\pi(\rho^2)}$$

$$\pm i\left(\cos(\pi\rho^2) \mp i\sin(\pi\rho^2)\right)$$

$$\mathcal{F}_2\left\{e^{\pm i\pi r^2}\right\} = \sin(\pi\rho^2) \pm i\cos(\pi\rho^2)$$



y
x
z
z_0
$\delta(x,y) - \frac{\delta(r)}{\pi r}$
$f(x,y)$
$g(x,y) = f(x,y) * h(x,y)$
$e^{+i\pi\left(\frac{r}{a_0}\right)^2}$
$|a_0|^2 i e^{-i\pi(a_0\rho)^2}$
$\alpha\, r^2$
r

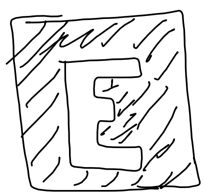
CARTESIAN SEPARABLE $\quad f(x,y) = f_1(x)\,f_2(y) \rightarrow F_1(\xi)\cdot F_2(\eta)$

$$= F(\xi,\eta)$$

POLAR SEPARABLE $\quad f(x,y) = f_r\left(\sqrt{x^2+y^2}\right)\,1(\theta)$

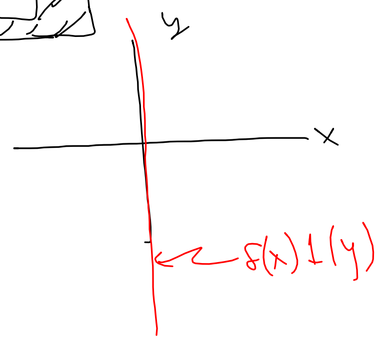
$$\longrightarrow \mathcal{H}_0\{f_r(r)\} = F_r(\rho)$$

ARBITRARY $\quad$ e.g.,



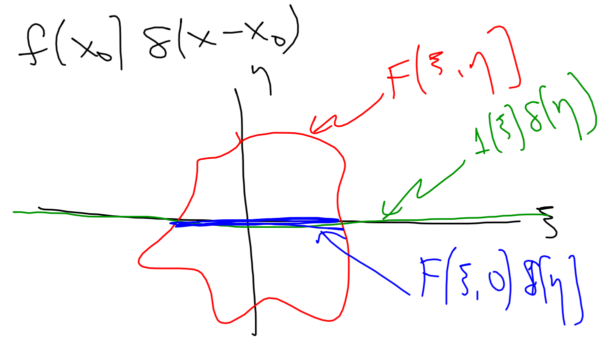
$$\mathcal{F}_2\{f(x,y) \times \delta(x)\,1(y)\}$$

$$F(\xi,\eta) \cdot 1(\xi)\,\delta(\eta)$$



$$f(x)\,\delta(x-x_0) = f(x_0)\,\delta(x-x_0)$$

$$F(\xi,0)\,1(\xi)\,\delta(\eta)$$



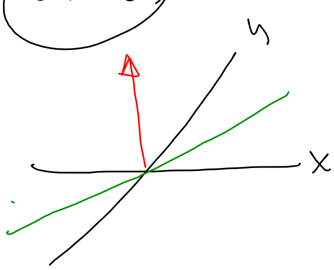
$$f(x,y) = \delta(x,y) = \delta(x)\,\delta(y)$$

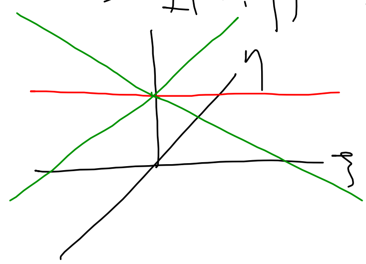
$$h(x,y) = \delta(x)\,1(y)$$

$$g(x,y) = f(x,y) * h(x,y) = \delta(x)\,1(y)$$

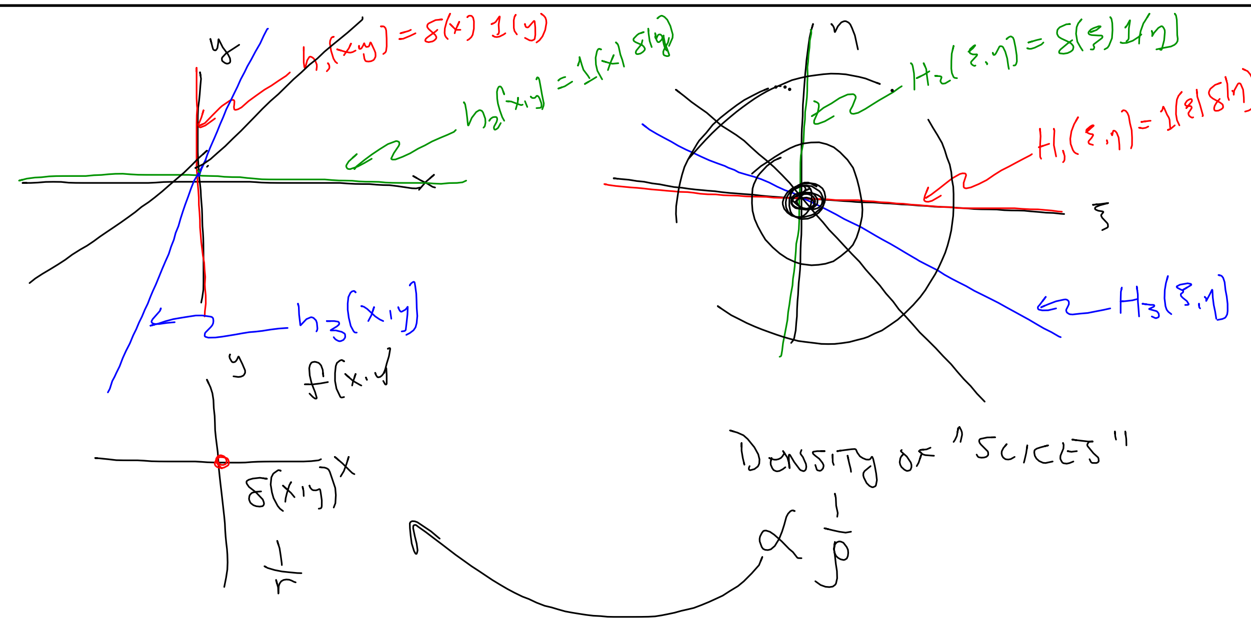
$$G(\xi,\eta) = 1(\xi)\,\delta(\eta)$$

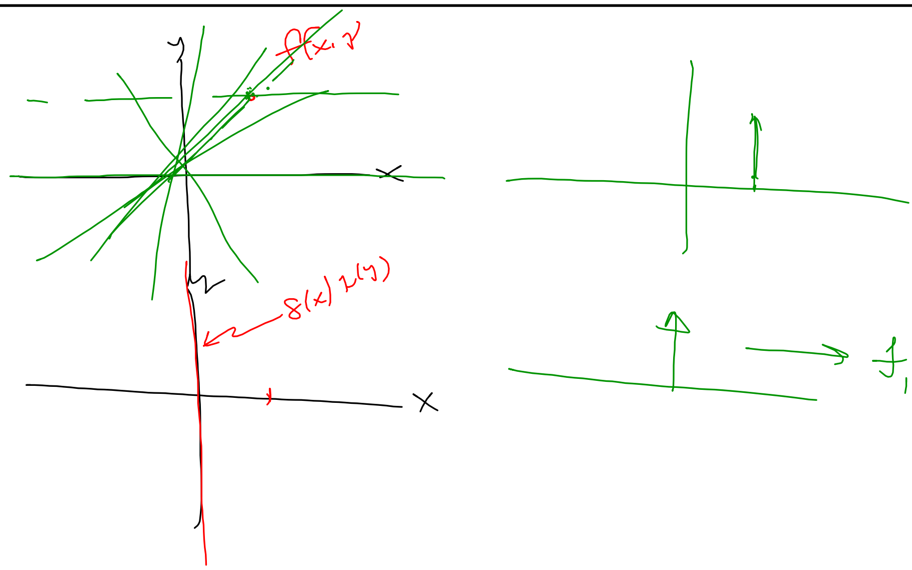
$$\delta(x,y) * 1(x)\,\delta(y) \longrightarrow 1(\xi,\eta) \cdot \delta(\xi)\,1(\eta)$$





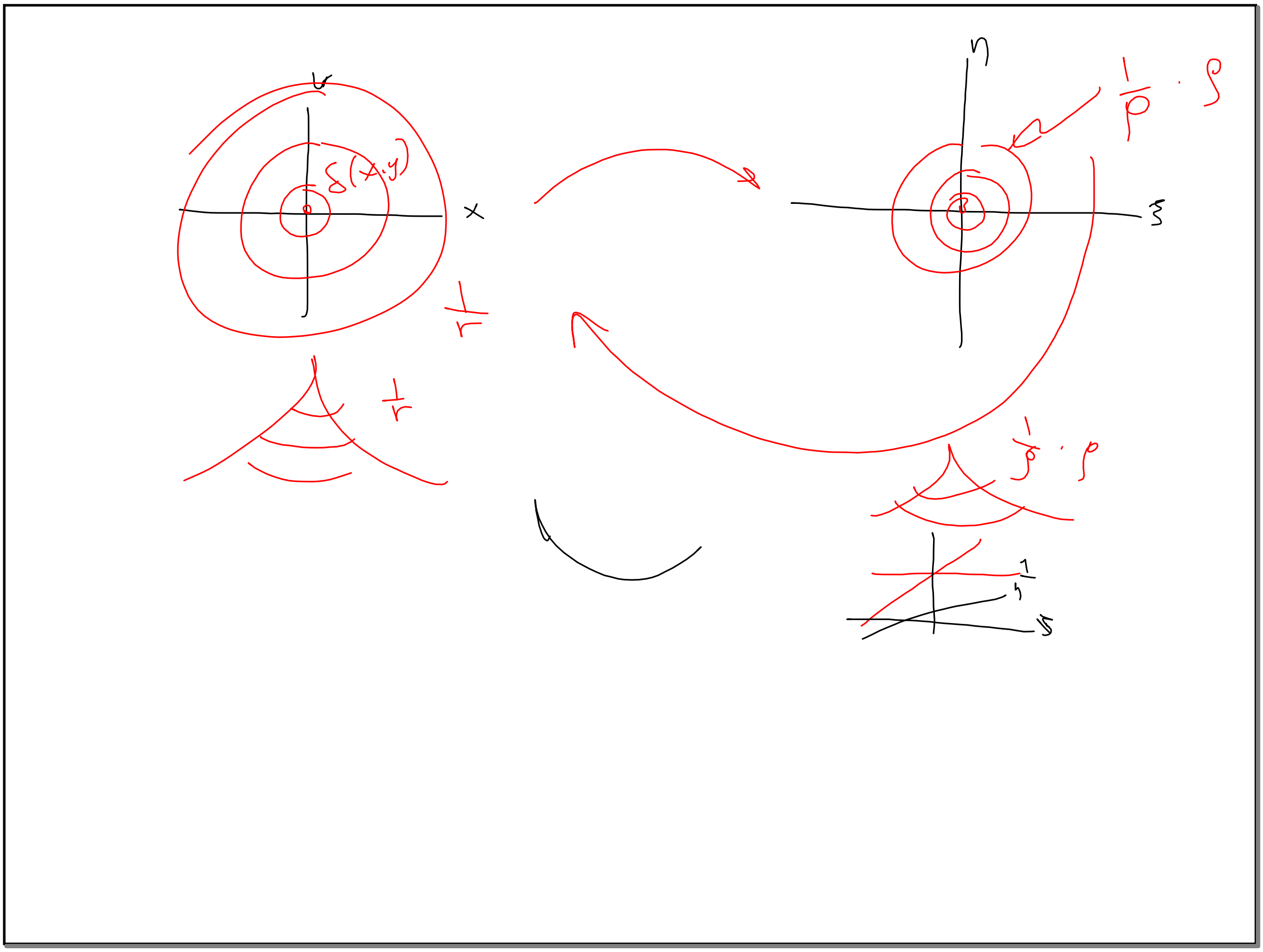
November 04, 2014



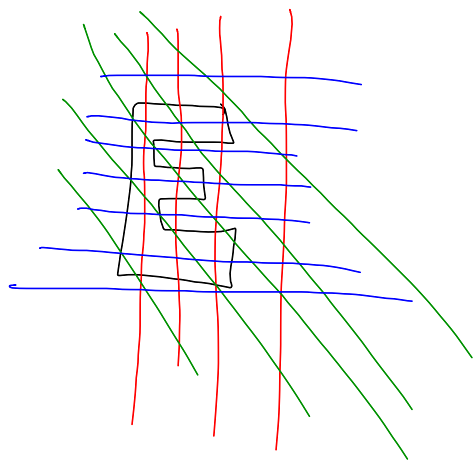


f(x, y)
δ(x) 1(y)
x
y

November 04, 2014



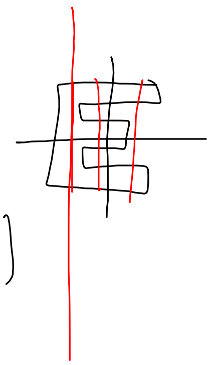
November 04, 2014

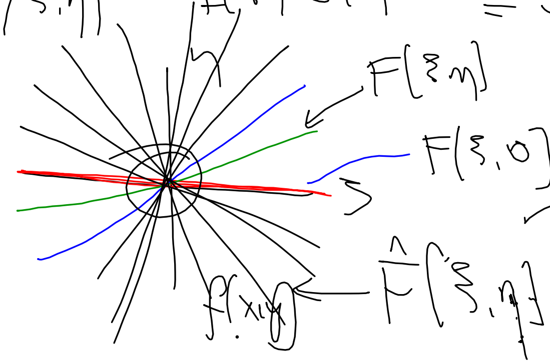


$$f(x..) * \delta(x)\, 1(y) = g(x,y)$$

$$\downarrow$$

$$F(\xi,\eta) \cdot 1(\xi)\,\delta(\eta) = G(\xi,\eta)$$





$$\hat{F}(\xi,\eta) = \left(F(\xi,\eta) \cdot \frac{1}{\rho}\right) \rho$$

$$f(x) \longrightarrow F(\xi)$$

$$f(x) * \delta'(x) = \frac{df}{dx} \longrightarrow F(\xi) \cdot i\, 2\pi \xi$$

$$\left(F(\xi, \eta) \cdot \frac{1}{\rho} \right) \cdot \rho$$