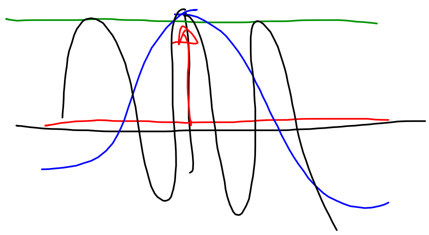
SCALING

SHIFTING

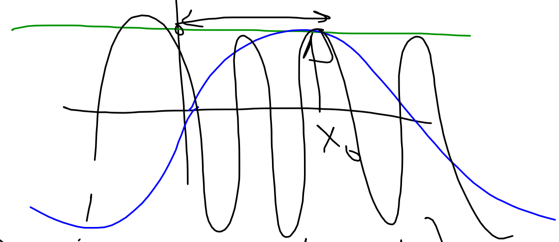
$$f(x) \rightarrow F(\xi)$$

$$f(x-x_0) \rightarrow \cancel{F(\xi-x_0)} \quad F(\xi)\, e^{-i2\pi\xi x_0}$$

$$\Phi = -2\pi\xi x_0$$

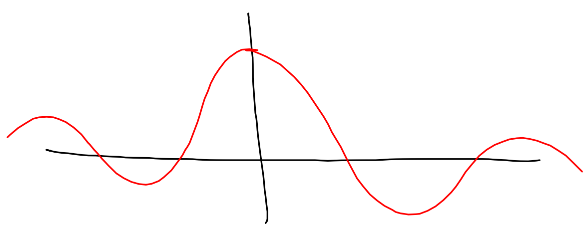


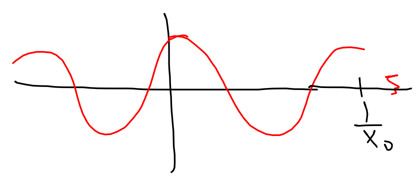
$x_0 \propto \xi x_0$



$$f(x) = \mathrm{RECT}(x) \qquad g(x) = \mathrm{RECT}(x-1)$$

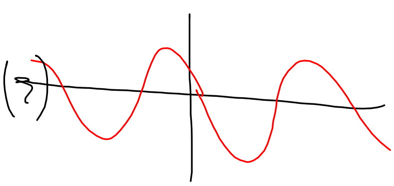
$$F(\xi) = \mathrm{SINC}(\xi) \qquad G(\xi) = \mathrm{SINC}(\xi)\, e^{-i2\pi\xi\cdot 1}$$





$$|G(\xi)| = |\mathrm{SINC}(\xi)|\, 1(\xi)$$

$$\Phi(\xi) = (-2\pi\xi x_0)$$



COMPLEX CONJUGATE

$$f(x) \to F(\xi)$$

$$f^*(x) = \left(f(x)\right)^* \to ?$$

$$\mathcal{F}\{f^*(x)\} = \int f^*(x)\, e^{-i2\pi\xi x}\, dx$$

$$= \int f^*(x)\left(e^{+i2\pi\xi x}\right)^* dx$$

$$= \int \left(f(x)\, e^{+i2\pi\xi x}\right)^* dx$$

$$= \left(\int f(x)\, e^{+i2\pi\xi x}\, dx\right)^*$$

$$\left(\underbrace{\int f(x)\, e^{-i2\pi(-\xi)x}\, dx}_{F(-\xi)}\right)^*$$

$$\mathcal{F}\{f^*(x)\} = F^*(-\xi)$$

$$\mathcal{F}\{f^*(-x)\} = F^*(\xi)$$

F.T. OF $f(x) \star m(x) = f(x) * m^*(-x)$

$$\mathcal{F}\{f(x) \star m(x)\} = F(\xi) \cdot M^*(\xi)$$

AUTOCORRELATION

$$\mathcal{F}\{f(x) \star f(x)\} = \mathcal{F}\{f(x) * f^*(-x)\}$$

$$= F(\xi) \cdot F^*(\xi)$$

$$= |F(\xi)|^2$$

WIENER-KHINTCHIN THEROM

$$f(x) \star f(x) = f(x) * f^*(-x)$$

$$= \int_{-\infty}^{+\infty} f(\alpha)\, f^*(\alpha - x)\, d\alpha$$

$$\mathcal{F}\{f(x) \star f(x)\} = |F(\xi)|^2$$

$$\mathcal{F}\{f(x) \star m(x)\} = F(\xi)\, M^*(\xi)$$

$$\int_{-\infty}^{+\infty} f(x)\, r^*(x)\, dx = \int_{-\infty}^{+\infty} F(\xi)\, R^*(\xi)\, d\xi$$ RAYLEIGH'S THEOREM

$$r(x) \longrightarrow f(x) \quad \boxed{\int_{-\infty}^{+\infty} |f(x)|^2\, dx = \int_{-\infty}^{+\infty} |F(\xi)|^2\, d\xi}$$ PARSEVAL'S THEOREM

INTEGRATED POWER INTEGRATED POWER

$|f(x)|^2 \Rightarrow$ "POWER" OF $f(x)$

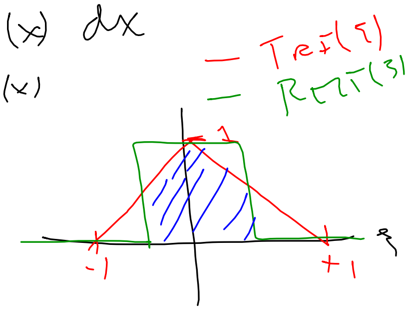
$$\int_{-\infty}^{+\infty} \mathrm{RECT}(x)\, dx = 1 \qquad \int_{-\infty}^{+\infty} |\mathrm{SINC}(\xi)|^2\, d\xi$$

$$\int |\mathrm{RECT}(x)|^2\, dx = 1 \qquad = \int \mathrm{SINC}^2(\xi)\, d\xi$$

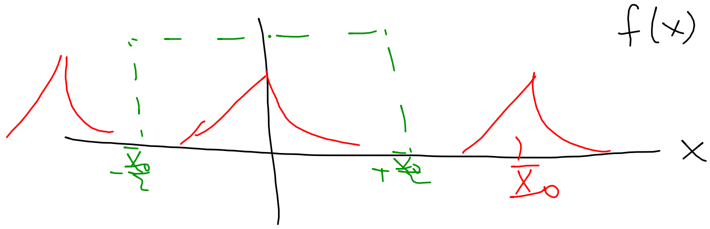
$$= 1$$

$$\int \mathrm{SINC}^3(x)\, dx = \int \underbrace{\mathrm{SINC}^2(x)}_{f(x)} \cdot \underbrace{\mathrm{SINC}(x)}_{r^*(x)}\, dx$$

$$\longrightarrow \quad F(\xi) = \mathrm{TRI}(\xi)$$
$$R(\xi) = \mathrm{RECT}(\xi)$$



F.T. OF PERIODIC FUNCTION
DISCRETE ("SAMPLED") FUNCTION



$f(x)$ PERIODIC OVER X_0

$$w(x) \equiv f(x) \cdot \mathrm{RECT}\left(\frac{x}{X_0}\right)$$

$$f(x) = w(x) * \frac{1}{|X_0|}\mathrm{COMB}\left(\frac{x}{X_0}\right)$$

$$\mathrm{COMB}\left(\frac{x}{X_0}\right) = \sum_{n=-\infty}^{+\infty} \delta\left(\frac{x}{X_0} - n\right) = \sum_n \delta\left(\frac{x - nX_0}{X_0}\right)$$

$$= \sum_n |X_0|\, \delta(x - nX_0)$$

$$f(x) = w(x) * \frac{1}{|X_0|}\mathrm{COMB}\left(\frac{x}{X_0}\right)$$

$$= W(\xi) \cdot \frac{\cancel{|X_0|}}{\cancel{|X_0|}}\mathrm{COMB}[X_0\xi]$$

$$= W(\xi) \cdot \sum_k \delta(X_0\xi - k)$$

$$= W(\xi) \cdot \sum_k \delta\left(X_0\left(\xi - \frac{k}{X_0}\right)\right)$$

$$= \sum_k W(\xi)\, \frac{1}{|X_0|}\, \delta\left(\xi - \frac{k}{X_0}\right)$$

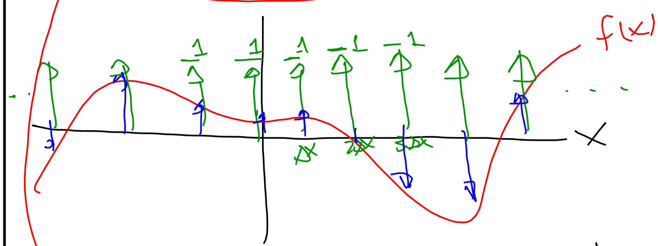
$$= \frac{1}{X_0}\sum_{k=-\infty}^{+\infty} \underline{W(\xi)\, \delta\left(\xi - \frac{k}{X_0}\right)}$$

PERIODIC $f(x) \longrightarrow$ DISCRETE $F(\xi)$

$$\cos(2\pi\xi_0 x) \longrightarrow \frac{1}{2}\delta(\xi + \xi_0) + \frac{1}{2}\delta(\xi - \xi_0)$$

$$f(x)\cdot \frac{1}{\Delta x}\,\mathrm{comb}\left(\frac{x}{\Delta x}\right) = f(x)\cdot\sum_n \delta(x-n\cdot\Delta x)$$

$$= \sum_n f(n\cdot\Delta x)\,\delta(x-n\cdot\Delta x)$$



— $f(x)\cdot\frac{1}{\Delta x}\,\mathrm{comb}\left(\frac{x}{\Delta x}\right)$

$$\longrightarrow F(\xi) * \cancel{\frac{1}{\Delta x}\cdot\Delta x}\,\mathrm{comb}(\Delta x\,\xi)$$

$$= F(\xi) * \sum_k \delta(\Delta x\,\xi - k)$$

$$= F(\xi) * \sum_k \delta\left(\Delta x\left(\xi - \frac{k}{\Delta x}\right)\right)$$

$$= \frac{1}{\Delta x}\sum_k F(\xi) * \delta\left(\xi - \frac{k}{\Delta x}\right)$$

$f(x) * \delta(x-x_0) = f(x-x_0)$

$$= \frac{1}{\Delta x}\sum_k F\left(\xi - \frac{k}{\Delta x}\right)$$

PERIODIC

PERIOD $= \Xi_0 = \frac{1}{\Delta x}$

Ξ_0

→ "SOURCE" OF ALIASING

$f(x) \rightarrow F(\xi)$

LSI $f(x) * h(x) \Rightarrow F(\xi) \cdot H(\xi) = G(\xi)$

$F(\xi_0) = 0 \Rightarrow G(\xi_0) = 0$

$H(\xi_0) = 0 \Rightarrow G(\xi_0) = 0$ EVEN IF $F(\xi_0) \neq 0$

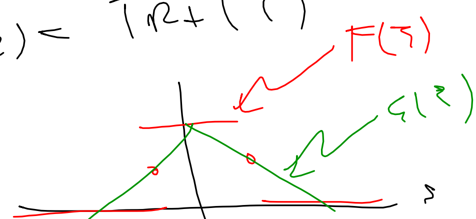
$\mathcal{O}\{f(x)\} = g(x)$

$g(x) = (f(x))^2$

$G(\xi) = \mathcal{F}\{(f(x))^2\} = \mathcal{F}\{f(x) \cdot f(x)\} = F(\xi) * F(\xi)$

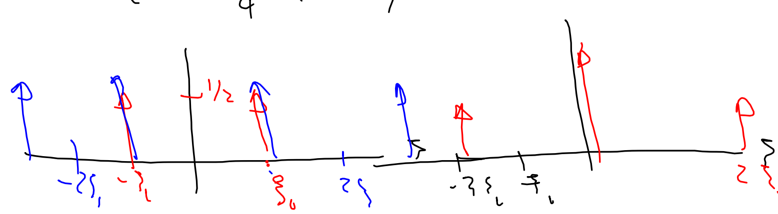
$f(x) = \mathrm{SINC}(x) \longrightarrow F(\xi) = \mathrm{RECT}(\xi)$

$g(x) = \mathrm{SINC}^2(x) \longrightarrow G(\xi) = \mathrm{TRI}(\xi)$



$f(x) = \cos(2\pi\xi_0 x) \longrightarrow F(\xi) = \frac{1}{2}\delta(\xi+\xi_0) + \frac{1}{2}\delta(\xi-\xi_0)$

$g(x) = \cos^2(2\pi\xi_0 x) \longrightarrow G(\xi) = \frac{1}{4}\delta(\xi+2\xi_0) + \frac{1}{2}\delta(\xi) + \frac{1}{4}\delta(\xi-2\xi_0)$

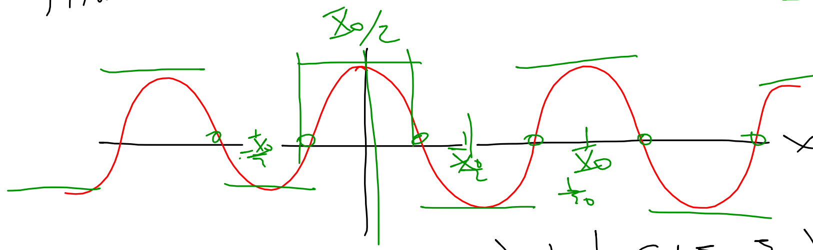


THRESHOLDING OPERATOR

$f(x)$

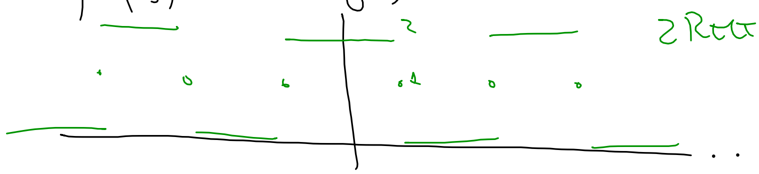
$g(x) = \mathrm{SGN}(f(x))$

$\cancel{G(\xi) = \frac{1}{\pi \xi} * F(\xi)}$



$$F(\xi) = \frac{1}{2}\delta(\xi+\xi_0) + \frac{1}{2}\delta(\xi-\xi_0)$$

$g(x)$ + 1



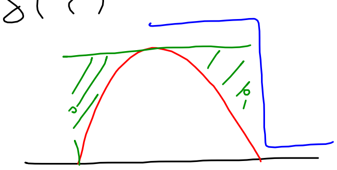
$$2\,\mathrm{RECT}\left(\frac{x}{X_0/2}\right) * \frac{1}{X_0}\mathrm{COMB}\left(\frac{x}{X_0}\right) - 1(x)$$

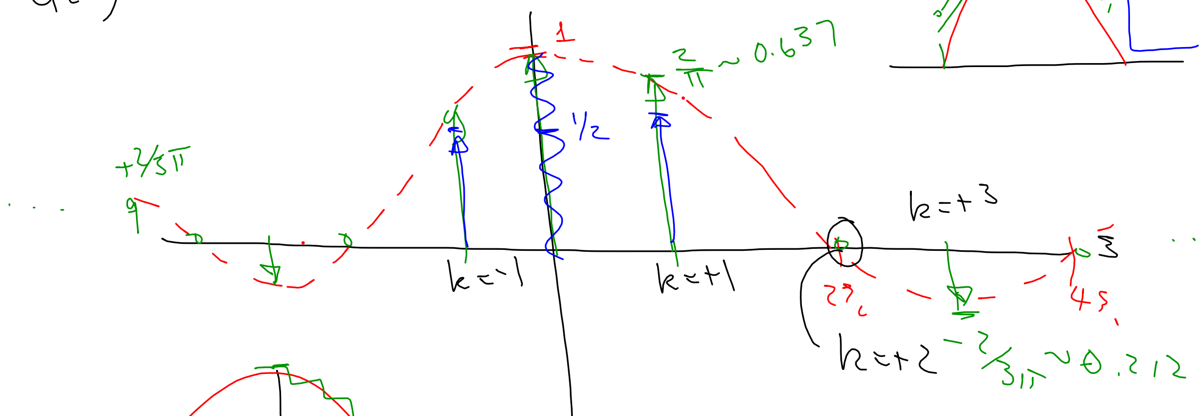
$$g(x) = 2\,\mathrm{RECT}(2\xi_0 x) * \xi_0\,\mathrm{COMB}[\xi_0 x] - 1[x]$$

$$= \mathrm{SGN}[\cos(2\pi\xi_0 x)]$$

$$G(\xi) = \cancel{2}\,\frac{1}{\cancel{2}\xi_0}\,\mathrm{SINC}\left[\frac{\xi}{2\xi_0}\right]\cdot \cancel{\xi_0}\cdot\frac{1}{\cancel{\xi_0}}\mathrm{COMB}\left(\frac{\xi}{\xi_0}\right) - \delta(\xi)$$

$$G(\xi) = \mathrm{SINC}\left(\frac{\xi}{2\xi_0}\right)\cdot\frac{1}{\xi_0}\mathrm{COMB}\left(\frac{\xi}{\xi_0}\right) - \delta(\xi)$$





Power Law $g(x) = \left(f(x)\right)^{\gamma}$

$$f(x) = e^{+i2\pi\xi_0 x} \Rightarrow F(\xi) = \delta(\xi - \xi_0)$$

$$g(x) = \left(e^{+i2\pi\xi_0 x}\right)^{\gamma} = e^{+i2\pi(\gamma\xi_1)x} \rightarrow G(\xi) = \delta(\xi - \gamma\xi_1)$$

$$\left(f(x)\right)^{1/2} \rightarrow G(\xi) = \delta\left(\xi - \frac{\xi_0}{2}\right)$$

$$g(x) = \left(f(x)\right)^{\gamma}$$

$$f(x) = \alpha_1 e^{+i2\pi\xi_1 x} + \alpha_2 e^{+i2\pi\xi_2 x}$$

$$F(\xi) = \alpha_1\,\delta(\xi - \xi_1) + \alpha_2\,\delta(\xi - \xi_2)$$

$$\alpha_1 > \alpha_2 \Rightarrow \alpha_1 e^{+i2\pi\xi_1 x}\left[1 + \frac{\alpha_2}{\alpha_1} e^{+i2\pi(\xi_2 - \xi_1)x}\right] = f(x)$$

$$g(x) = \underbrace{\alpha_1^{\gamma} e^{+i2\pi(\gamma\xi_1)x}}_{\delta(\xi - \gamma\xi_1)} \cdot \underbrace{\left[1 + \frac{\alpha_2}{\alpha_1} e^{+i2\pi(\xi_2 - \xi_1)x}\right]^{\gamma}}$$

$$(1+u)^{\gamma} = 1 + \frac{\gamma}{1!}u^1 + \frac{\gamma(\gamma-1)}{2!}u^2 + \frac{\gamma(\gamma-1)(\gamma-2)}{3!}u^3 + \cdots$$

$$|u| < 1$$