23 OCTOBER

THEOREMS

(1) LINEARITY $\mathcal{F}\left\{\sum_n \alpha_n f_n(x)\right\} = \sum_n \alpha_n F_n(\xi)$

$$\mathcal{F}\{f_n(x)\} = F_n(\xi)$$

$$\mathcal{F}\{\alpha_n f_n(x)\} = \alpha_n F_n(\xi); \quad \mathcal{F}\{-f(x)\} = -F(\xi)$$

SCALING $f(x) \longrightarrow F(\xi)$

SPECTRUM

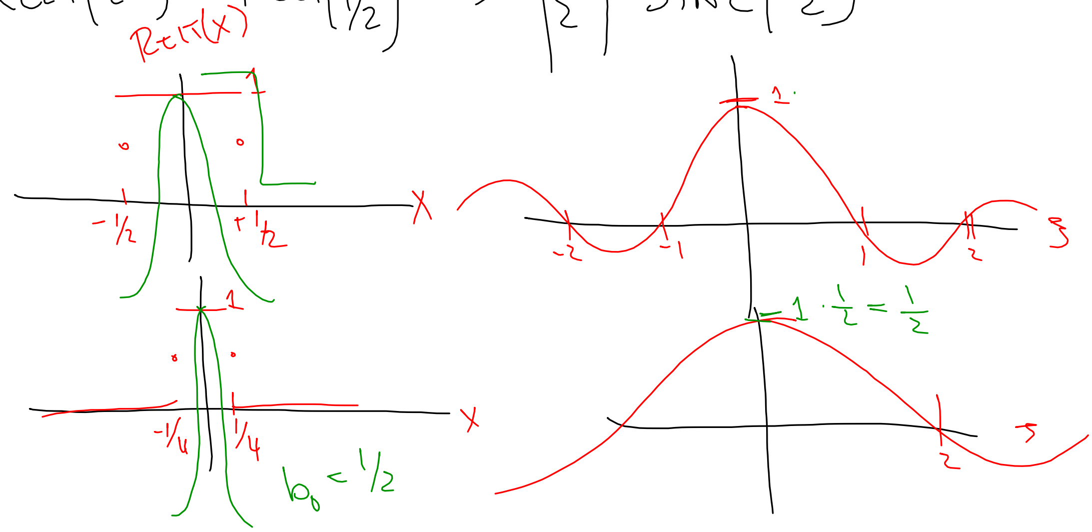
$$f\left(\frac{x}{b_0}\right) = |b_0|\, F(b_0 \xi) = |b_0|\, F\left(\frac{\xi}{1/b_0}\right)$$

$$f(-x) = |-1|\, F\left(\frac{\xi}{-1}\right) = F(-\xi)$$

n.b.

$$\mathrm{RECT}(x) \rightarrow \mathrm{SINC}(\xi)$$

$$\mathrm{RECT}(2x) = \mathrm{RECT}\left(\frac{x}{1/2}\right) \rightarrow \left|\frac{1}{2}\right| \mathrm{SINC}\left(\frac{\xi}{2}\right)$$



CENTRAL ORDINATE THEOREM

$\longleftarrow$ ABSCISSA

$\updownarrow$ ORDINATE

$$F(\xi)\Big|_{\xi=0} = F[0] = \int_{-\infty}^{+\infty} f(x)\, \cancelto{1}{e^{-i2\pi\cdot 0\cdot x}}\, dx$$

$$= \int_{-\infty}^{+\infty} f(x)\,dx = \text{AREA OF } f(x)$$

$$f(0) = \int_{-\infty}^{+\infty} F(\xi)\, \cancelto{1}{e^{+i2\pi\xi\cdot 0}}\, d\xi = \int_{-\infty}^{+\infty} F(\xi)\, d\xi$$

SHIFT THEOREM $\quad f(x) \longrightarrow F(\xi)$

$$f[x-x_0] \longrightarrow ?$$

$$\int_{-\infty}^{+\infty} f[x-x_0]\, e^{-i2\pi\xi x}\, dx \quad ; \quad u \equiv x - x_0$$

$$du = d(x-x_0) = dx - \cancel{dx_0}^{\,0}$$

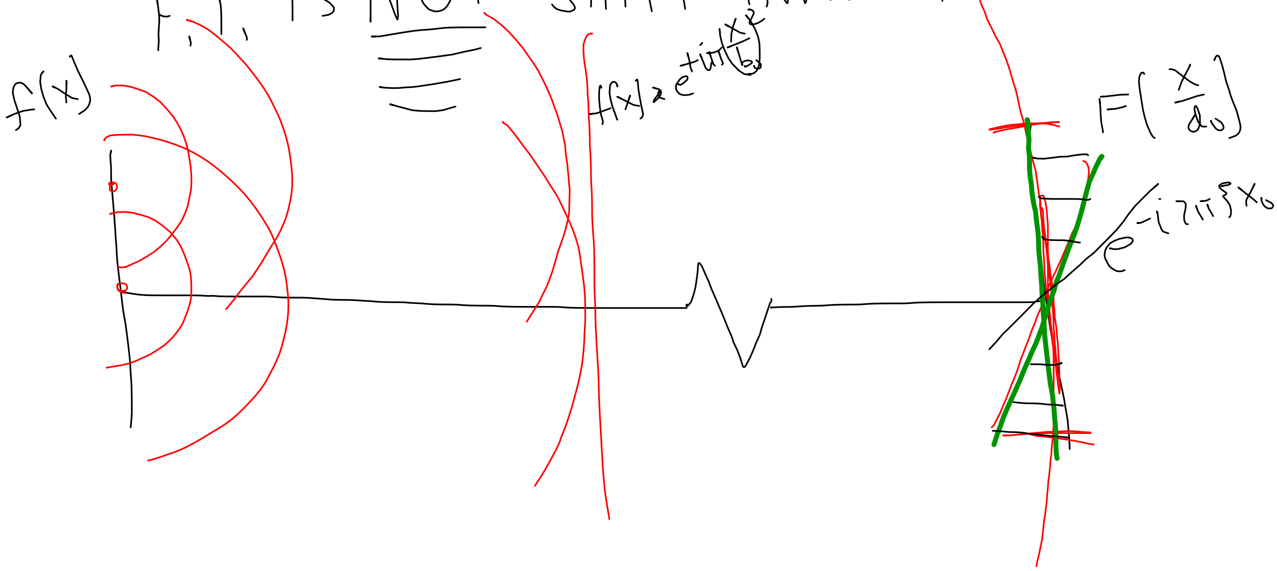
$$\int_{u=-\infty}^{u=+\infty} f(u)\, e^{-i2\pi\xi(u+x_0)}\, du \qquad x = u + x_0$$

$$x = \pm\infty \Rightarrow u = \pm\infty$$

$$\int_{-\infty}^{+\infty} f(u)\, e^{-i2\pi\xi u}\, e^{-i2\pi\xi x_0}\, du$$

$$\mathcal{F}\{f(x-x_0)\} = e^{-i2\pi\xi x_0} \underbrace{\int_{-\infty}^{\infty} f(u)\, e^{-i2\pi\xi u}\, du}_{F(\xi)}$$

F.T. IS <u>NOT</u> SHIFT INVARIANT



TRANSFORM OF TRANSFORM

RECT $\quad\quad$ SINC

$f(x) \longrightarrow F(\xi)$

SINC

$F(x) \longrightarrow ?$

$$\int F(\xi)\, e^{+i2\pi\xi x}\, d\xi$$

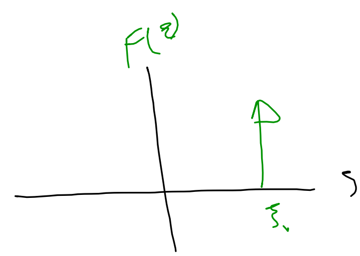
$$\int F(x)\, e^{-i2\pi\xi x}\, dx = f(-\xi)$$

$$\int F(\xi)\, e^{-i2\pi\xi(-x)}\, d\xi = f(-x)$$

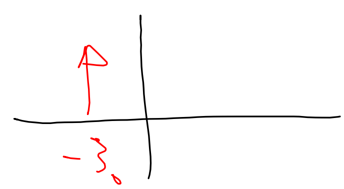
$$\mathcal{F}\{F(x)\} = f(-\xi)$$

$$\mathcal{F}\{\mathrm{SINC}(x)\} \Rightarrow \mathrm{RECT}(-\xi) = \mathrm{RECT}(+\xi)$$

$$\mathcal{F}\{e^{+i2\pi\xi_0 x}\} = \delta(\xi - \xi_0)$$



$\cancel{\delta(x-\xi_0)}$ $\quad\quad$ $\delta(\xi - (-\xi_0))$



$$\mathcal{F}\{\delta(x - x_0)\} = e^{+i2\pi(-\xi)x_0}$$

$$= e^{-i2\pi\xi x_0}$$

$$\int \delta(x-x_0)\, e^{-i2\pi\xi x}\, dx = \int \delta(x-x_0)\, e^{-i2\pi\xi x_0}\, dx$$

FILTER THEOREM

$$\mathcal{F}\{f(x) * h(x)\} = \int_{x=-\infty}^{+\infty}\left[\int_{\alpha=-\infty}^{+\infty} f(\alpha)\, h(x-\alpha)\, d\alpha\right] e^{-i2\pi\xi x}\, dx$$

$$= \int_{\alpha=-\infty}^{+\infty} f(\alpha) \underbrace{\left[\int_{x=-\infty}^{+\infty} h(\underset{x_0}{x-\alpha})\, e^{-i2\pi\xi x}\, dx\right]}_{} d\alpha$$

$$= \int_{-\infty}^{+\infty} f(\alpha) \left[H(\xi)\, e^{-i2\pi\xi\alpha}\right] d\alpha$$

$$= H(\xi) \int f(\alpha)\, e^{-i2\pi\xi\alpha}\, d\alpha \quad F(\xi)$$

$$\mathcal{F}\{f(x) * \underbrace{h(x)}_{\text{IMPULSE RESPONSE, psf}}\} = F(\xi) \cdot \underbrace{H(\xi)}_{\text{TRANSFER FUNCTION, OTF}}$$

$$\frac{|H(\xi)|}{|H(0)|} = MTF(\xi)$$

$$\underbrace{\underset{\sim}{A}}_{} \quad \underbrace{\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \delta & \alpha & \beta & \gamma \\ \gamma & \delta & \alpha & \beta \\ \beta & \gamma & \delta & \alpha \end{bmatrix}}_{h(x)} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{f(x)} = \underbrace{\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{g(x)}$$

$$\underbrace{\begin{bmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \lambda_2 & \\ 0 & & & \lambda_3 \end{bmatrix}}_{H(\xi)} \underbrace{\begin{bmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{bmatrix}}_{F(\xi)} = \underbrace{\begin{bmatrix} b_0' = \lambda_0 x_0' \\ b_1' = \lambda_0 x_1' \\ \vdots \end{bmatrix}}_{G(\xi)}$$

$$F(\xi) \cdot H(\xi) = G(\xi)$$

SPACE $\quad f(x) * h(x) = g(x)$

FREQ $\quad F(\xi) \cdot H(\xi) = G(\xi)$

$$f(n) * h(n) = g(n)$$

$$F(k) \cdot H(k) = G(k)$$

GIVEN $g(x), h(x) \Rightarrow$ FIND $f(x)$

ESTIMATE $\quad \hat{F}(\xi) = \dfrac{G(\xi)}{H(\xi)}$; IF $H(\xi_0) = 0$, $F(\xi_0) = ?$

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EIGENFUNCTIONS OF $*$

$$f(x) * h(x) = g(x)$$

$$\downarrow$$

$$f_0(x) * h(x) = g(x) = \lambda_0 \cdot f_0(x)$$

$$F_0(\xi) \cdot H(\xi) = G(\xi) = \lambda_0 \; F_0(\xi)$$

IF $\boxed{F_0(\xi) = \delta(\xi - \xi_0)}$

$$G(\xi) = \delta(\xi - \xi_0) \cdot H(\xi) = \delta(\xi_1 - \xi_0) \cdot \underbrace{H(\xi_0)}_{\lambda_0}$$

$$f_0(x) = \mathcal{F}^{-1}\{\delta(\xi - \xi_0)\} = e^{+i2\pi\xi_0 x}$$

$$e^{+i2\pi\xi x} * h(x) = e^{+i2\pi\xi_0 x} \cdot H(\xi_0) = g(x)$$

$$\delta(\xi-\xi_0) \cdot H(\xi) = \delta(\xi-\xi_0) \cdot H(\xi_0) = G(\xi)$$

$$\underset{\sim}{A}\,\underset{\sim}{x_0} = \lambda_0 \underset{\sim}{x_0}$$

$$h(x) * e^{+i2\pi\xi_0 x} \quad H(\xi_0)$$

$$\delta(x) \longrightarrow 1(\xi)$$

$$e^{+i2\pi\xi_0 x} \longrightarrow \delta(\xi-\xi_0)$$

$$e^{+i\pi\left(\frac{x}{\alpha_1}\right)^2} \longrightarrow 1(\xi)\, e^{-i\pi\alpha}$$

$$(\quad)\, e^{-i\pi\alpha_0^2\xi^2}$$

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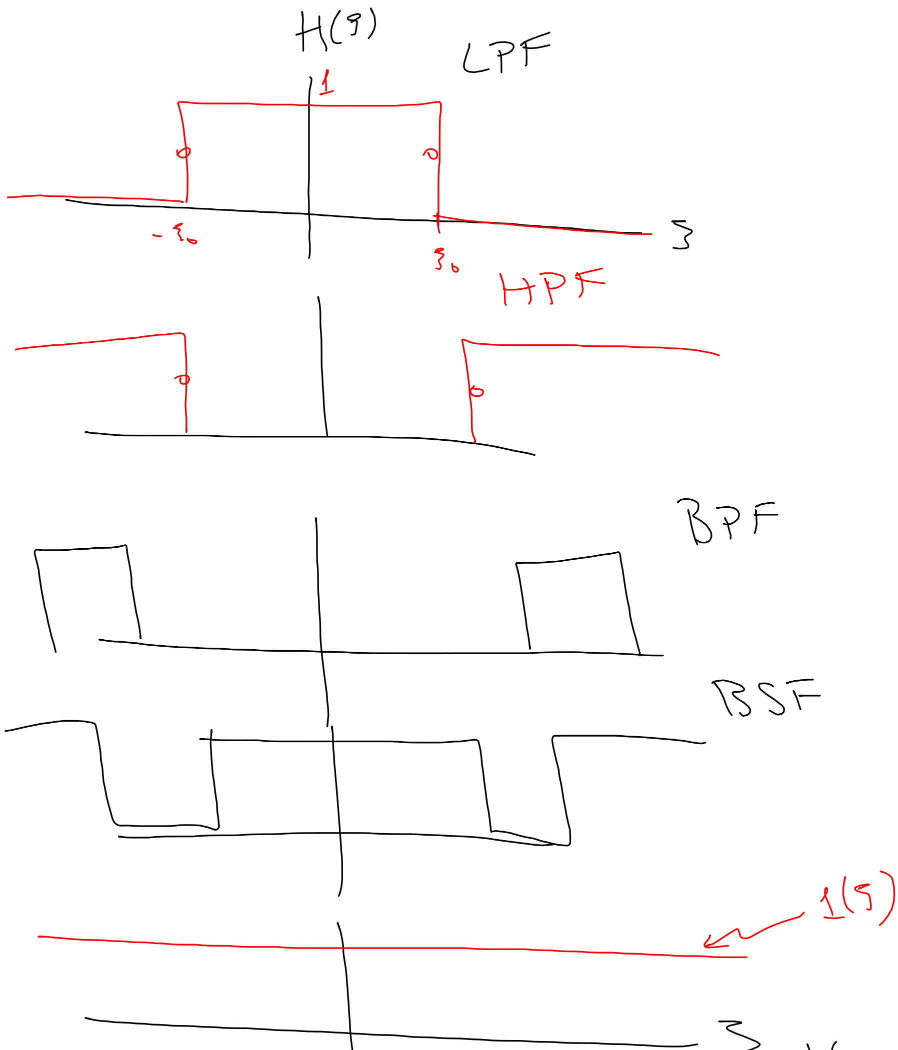


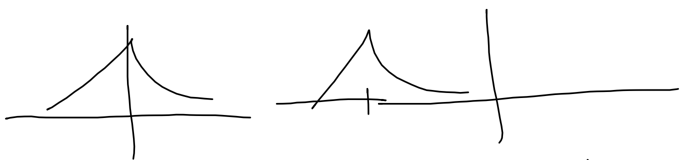
$$H(\xi) = 1(\xi) \Rightarrow h(x) = \mathcal{F}^{-1}\{1(\xi)\} = \delta(x)$$

$$H(\xi) = \cos(2\pi\xi x_0) + i\sin(2\pi\xi x_0) = 1(\xi)\, e^{+i2\pi\xi x_0}$$

$$\Phi[H(\xi)] = +2\pi\xi x_0$$

LINEAR PHASE

$$h(x) = \delta(x + x_0)$$



$$H(\xi) = 1(\xi)\, e^{+i\pi(\alpha_0 \xi)^2}$$

MODULATION THEOREM

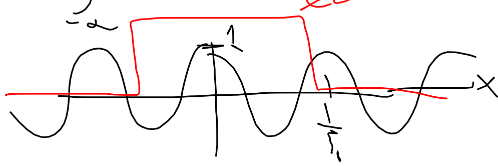
$$\mathcal{F}\{f(x)\cdot m(x)\} = F(\xi) * M(\xi)$$

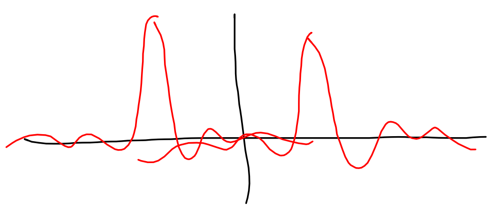
$$\int_{-\infty}^{+\infty} F(u)\, e^{+i2\pi u x}\, du \qquad \int M(v)\, e^{+i2\pi v x}\, dv$$

$\mathrm{Rect}\left(\frac{x}{b_0}\right)$

$$\cos(2\pi\xi_0 x) \longrightarrow \frac{1}{2}\delta(\xi+\xi_0) + \frac{1}{2}\delta(\xi-\xi_0)$$

$$\mathrm{Rect}\left(\frac{x}{b_0}\right) \longrightarrow |b_0|\,\mathrm{SINC}\left(\frac{\xi}{1/b_0}\right)$$





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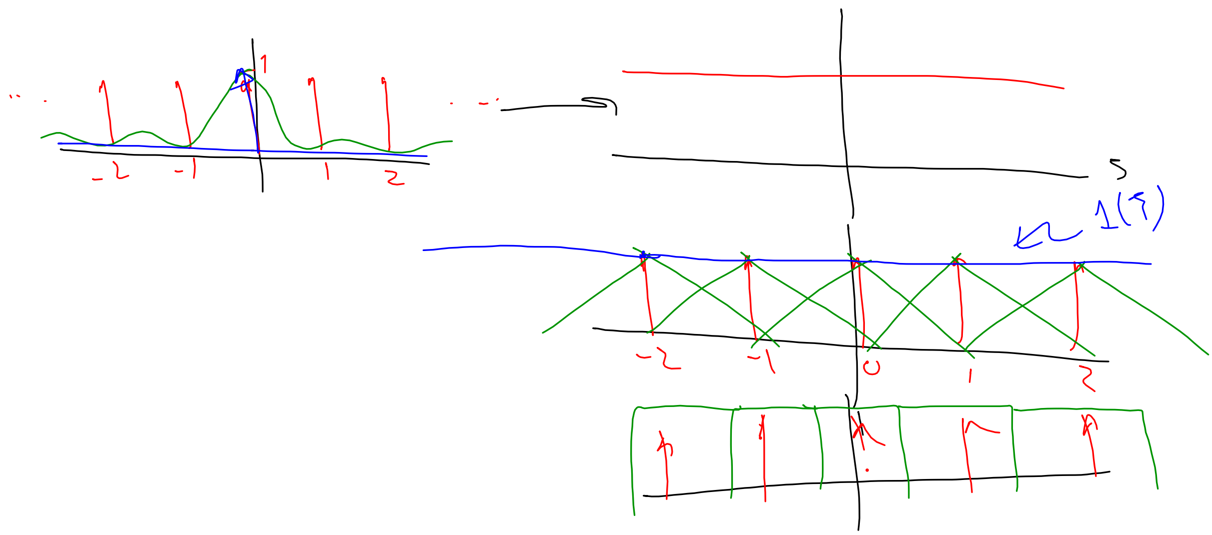
$\mathrm{comb}(x) \cdot \mathrm{sinc}(x) \longrightarrow \delta(x) \rightarrow \mathrm{comb}(\xi) * \mathrm{rect}(\xi)$

$\mathrm{comb}(x) \cdot \mathrm{sinc}^2(x) \longrightarrow \mathrm{comb}(\xi) * \mathrm{tri}(\xi)$



$$\mathcal{F}\left\{\frac{df}{dx}\right\} = ?$$

$$\delta'(x) * f(x) \longrightarrow F(\xi)\cdot \mathcal{F}\{\delta'(x)\}$$

$$\frac{df}{dx} = \frac{d}{dx}\left\{\mathcal{F}^{-1}\{F(\xi)\}\right\}$$

$$= \frac{d}{dx}\int_{-\infty}^{+\infty} F(\xi)\, e^{+i2\pi\xi x}\, d\xi$$

$$= \int \frac{d}{dx}\left\{F(\xi)\, e^{+i2\pi\xi x}\right\} d\xi$$

$$= \int F(\xi)\left(\frac{d}{dx} e^{+i2\pi\xi x}\right) d\xi$$

$$= \int \left[F(\xi)\cdot(+i2\pi\xi)\right] e^{+i2\pi\xi x}\, d\xi$$

$$\mathcal{F}^{-1}\{\underbrace{+i2\pi\xi}\cdot F(\xi)\} = \frac{df}{dx}$$

$$+i2\pi\xi = H(\xi) = \mathcal{F}\{\delta'(x)\}$$

$$H(\xi) = +i2\pi\xi = 0(\xi) + i\left[2\pi\xi\right]$$