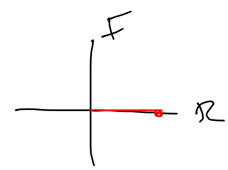
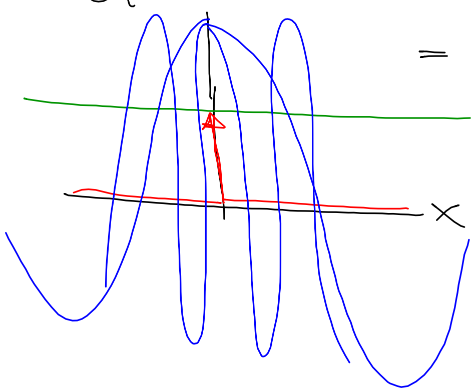
October 21, 2014

21 OCTOBER

$$\delta[x] + i \cdot 0[x] = 1[\xi] + i \cdot 0[\xi]$$



$$= 1[\xi]\, e^{i \cdot 0[\xi]}$$



$$\delta[x - x_0] + i\, 0[x] \longrightarrow e^{-i 2\pi \xi x_0} = \cos[2\pi\xi x_0] - i \sin[2\pi \xi x_0]$$

$$1[\xi]\, e^{+i\underbrace{(-2\pi\xi x_0)}_{\Phi\{F[\xi]\}}}$$

$F(\xi) \rightarrow$ "SPECTRUM"

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$\delta(x-x_0) \longrightarrow e^{-i2\pi\xi x_0}$

$\longleftarrow \quad \delta(\xi-\xi_0)$

$$e^{+i2\pi\xi_0 x} = \int_{-\infty}^{+\infty} \delta(\xi-\xi_0)\, e^{+i2\pi\xi x}\, d\xi$$

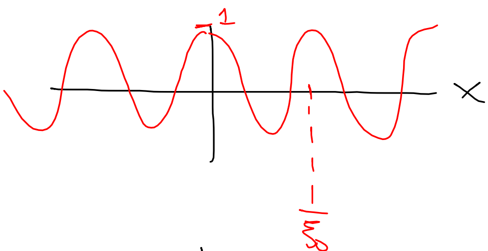
$$= \int \delta(\xi-\xi_0)\, e^{+i2\pi(\xi_0)x}\, d\xi$$

$$= e^{+i2\pi\xi_0 x} \underbrace{\int_{-\infty}^{+\infty} \delta(\xi-\xi_0)\, d\xi}_{1}$$

$$\delta(x-x_0) \rightleftharpoons e^{-i2\pi\xi x_0}$$

$$e^{+i2\pi\xi_0 x} \rightleftharpoons \delta(\xi-\xi_0)$$

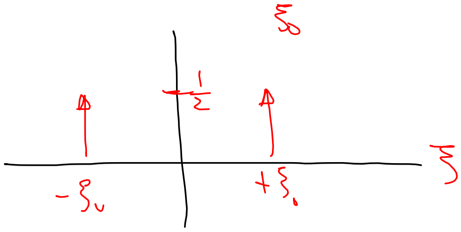
$$\cos(2\pi \xi_0 x) = \frac{1}{2}\left[e^{+i2\pi\xi_0 x} + e^{-i2\pi\xi_0 x}\right]$$



$$= \frac{1}{2}\left[e^{+i2\pi(+\xi_0)x} + e^{+i2\pi(-\xi_0)x}\right]$$

$$= \frac{1}{2}\left[\delta(\xi-\xi_0) + \delta(\xi-(-\xi_0))\right]$$

$$= \frac{1}{2}\left[\delta(\xi-\xi_0) + \delta(\xi+\xi_0)\right]$$



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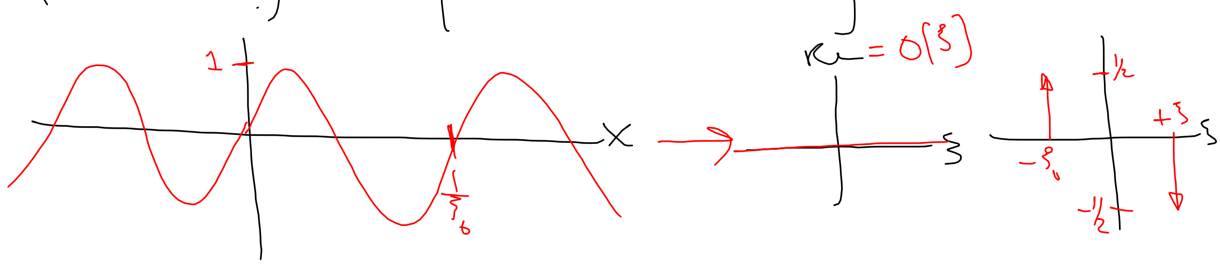
$$\sin[2\pi\xi_0 x] = \frac{1}{2i}\left[e^{+i2\pi(+\xi_0)x} - e^{+i2\pi(-\xi_0)x}\right]$$

$$\frac{1}{i} = \frac{-i}{-i}\cdot\frac{1}{i} = \frac{-i}{-(i^2)} = \frac{-i}{-(-1)} = -i$$

$$= i \cdot \frac{1}{2}\left[-e^{+i2\pi(+\xi_0)x} + e^{+i2\pi(-\xi_0)x}\right]$$

$$\mathcal{F}\{\sin(2\pi\xi_0 x)\} = i\cdot\frac{1}{2}\left[\delta(\xi+\xi_0) - \delta(\xi-\xi_0)\right]$$

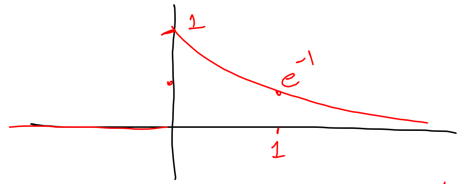


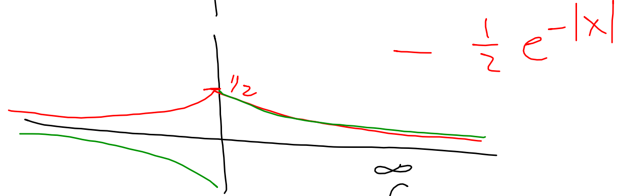
$$\mathcal{F}\left\{ i\cos(2\pi\xi_0 x) \right\} = i\,\mathcal{F}\left\{ \cos(2\pi\xi_0 x) \right\}$$

$$= i\cdot\left(\frac{1}{2}\delta(\xi+\xi_0) + \frac{1}{2}\delta(\xi-\xi_0) \right)$$

$$\mathcal{F}\left\{ i\sin(2\pi\xi_0 x) \right\} = \frac{i^2}{2}\left[\delta(\xi+\xi_0) - \delta(\xi-\xi_0) \right]$$

$e^{-x}\,\mathrm{STEP}(x)$





$$\int_{-\infty}^{+\infty} e^{-x}\,\mathrm{STEP}(x)\, e^{-i2\pi\xi x}\,dx = \int_{x=0}^{\infty} e^{-x(1+i2\pi\xi)}\,dx$$

$$= \left.\frac{e^{-x(1+i2\pi\xi)}}{-(1+i2\pi\xi)}\right|_{x=0}^{x=\infty} = \frac{0-1}{-(1+i2\pi\xi)} = \frac{1}{1+i2\pi\xi}$$

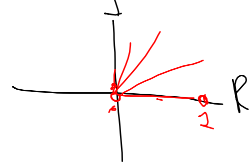
$$= \frac{1}{1+i2\pi\xi}\cdot\frac{1-i2\pi\xi}{1-i2\pi\xi} = \frac{1-i2\pi\xi}{1+(2\pi\xi)^2}$$

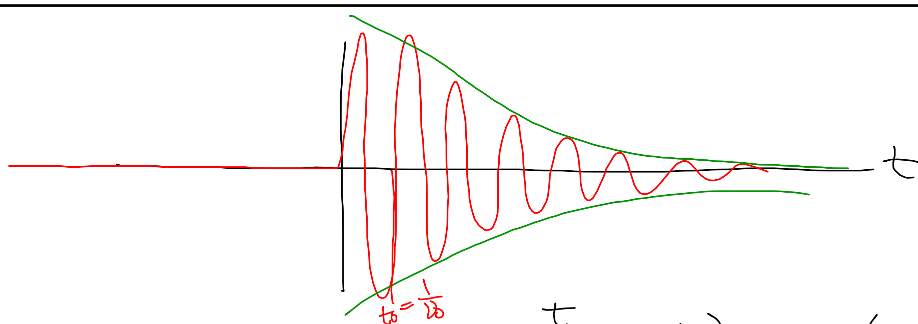
$$= \underbrace{\frac{1}{1+(2\pi\xi)^2}}_{\text{even}} + i\underbrace{\left(\frac{-2\pi\xi}{1+(2\pi\xi)^2}\right)}_{\text{ODD}} = \sqrt{\left(\frac{1}{1+(2\pi\xi)^2}\right)^2 + \frac{+4\pi^2\xi^2}{(1+(2\pi\xi)^2)^2}}$$

$$|F(\xi)|$$

$$\sqrt{\frac{\cancel{1+(2\pi\xi)^2}}{(1+(2\pi\xi)^2)^{\cancel{2}}}} = \sqrt{\frac{1}{1+(2\pi\xi)^2}}$$

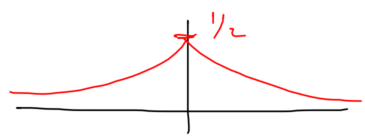
$$\phi\{F(\xi)\} = \mathrm{TAN}^{-1}\left[\frac{\left(\frac{-2\pi\xi}{\cancel{1+(2\pi\xi)^2}}\right)}{\frac{1}{\cancel{1+(2\pi\xi)^2}}}\right] = \mathrm{TAN}^{-1}\left[-2\pi\xi\right]$$

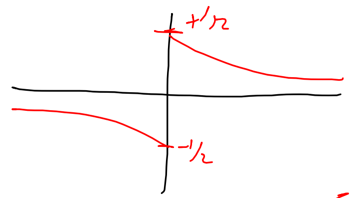




$$f(t) = e^{-\frac{t}{\alpha}} \operatorname{STEP}(t) \cdot \sin(2\pi\nu_0 t)$$

$$e^{-x}\,\mathrm{step}(x) = \frac{1}{2}e^{-|x|} + \frac{1}{2}e^{-|x|}\,\mathrm{sgn}(x)$$





$$\frac{1}{2}e^{-|x|} \xrightarrow{\mathcal{F}} \frac{1}{1+(2\pi\xi)^2} \qquad \frac{1}{2}e^{-|x|}\,\mathrm{sgn}(x) \longrightarrow \frac{-2\pi\xi}{1+(2\pi\xi)^2}$$

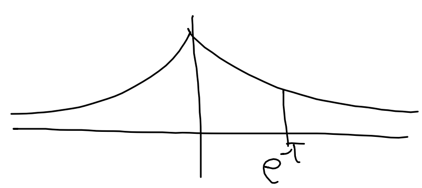
$$\underset{\text{ODD}}{\mathrm{SGN}(x)} \equiv \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$

$$\mathrm{STEP}(x) = \underbrace{\frac{1}{2}\,1(x)}_{\text{even}} + \underbrace{\frac{1}{2}\,\mathrm{SGN}(x)}_{\text{ODD}}$$

?

$$\mathcal{F}\{\mathrm{SGN}(x)\} = \int_{-\infty}^{+\infty} \mathrm{SGN}(x)\, e^{i2\pi\xi x}\, dx = \int_{-\infty}^{0} -e^{-i2\pi\xi x}\, dx + \int_{0}^{\infty} +e^{-i2\pi\xi x}\, dx$$

$$\mathrm{SGN}(x) = \lim_{\tau\to 0} \mathrm{SGN}(x)\, e^{-\tau|x|} \qquad \tau > 0$$



$$\mathcal{F}\left\{\lim_{\tau\to 0} e^{-\tau|x|}\,\mathrm{sgn}(x)\right\} = \lim_{\tau\to 0}\left\{\int_{-\infty}^{+\infty} e^{-\tau|x|}\,\mathrm{sgn}(x)\, e^{-i2\pi\xi x}\,dx\right\}$$

$x<0 \Rightarrow |x| = -x$

$$\lim_{\tau\to 0}\left\{-\int_{-\infty}^{0} e^{-\tau(-x)}\, e^{-i2\pi\xi x}\,dx + \int_{0}^{\infty} +1\, e^{-\tau(+x)}\, e^{-i2\pi\xi x}\,dx\right\}$$

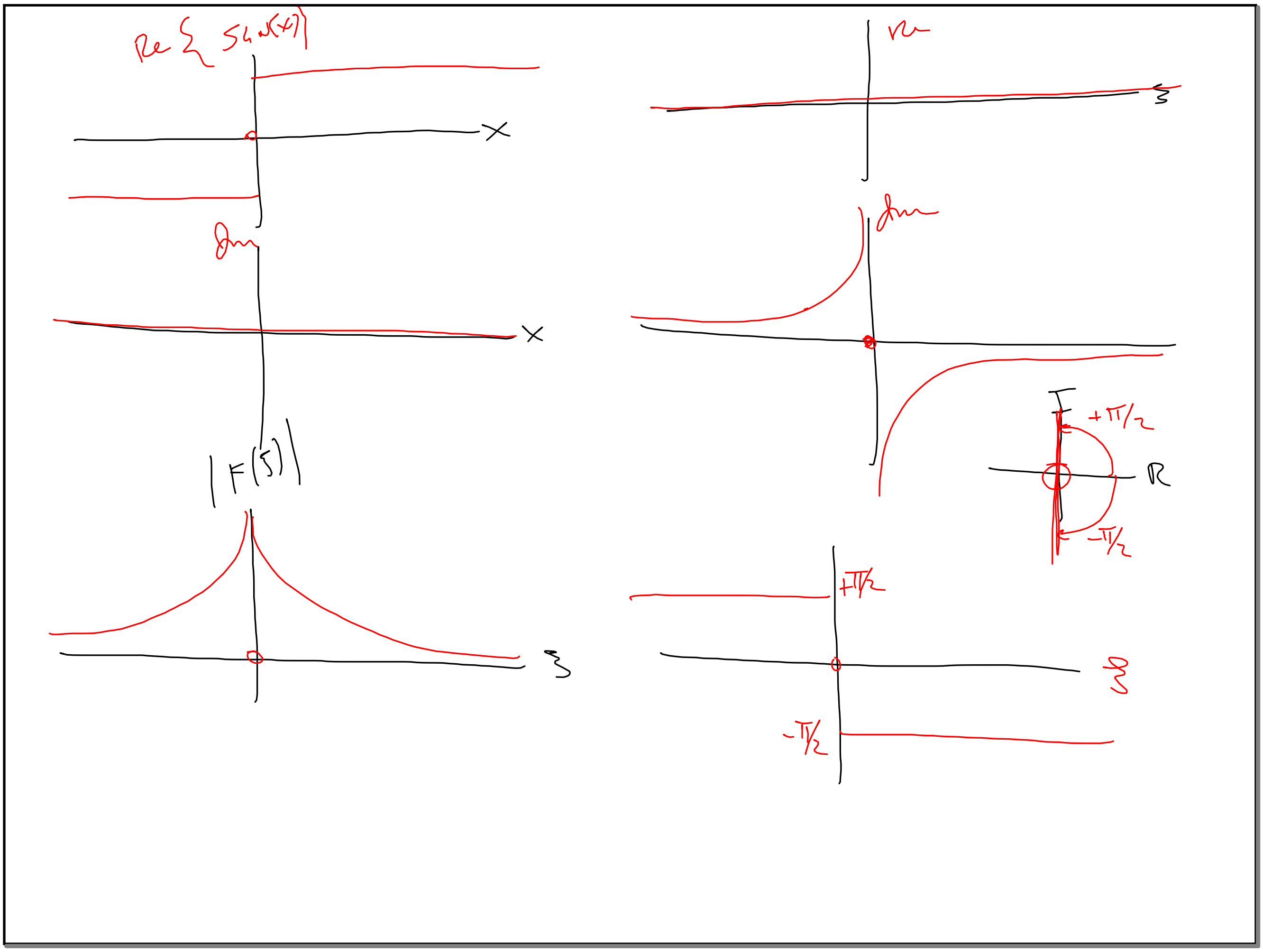
$$\lim_{\tau\to 0}\left\{-\int_{-\infty}^{0} e^{x(\tau-i2\pi\xi)}\,dx + \int_{0}^{\infty} e^{-x(\tau+i2\pi\xi)}\,dx\right\}$$

$$\lim_{\tau\to 0}\left\{-\left[\frac{e^{(\tau-i2\pi\xi)x}}{\tau-i2\pi\xi}\Bigg|_{x=-\infty}^{x=0}\right] + \left[\frac{e^{-x(\tau+i2\pi\xi)}}{-(\tau+i2\pi\xi)}\Bigg|_{x=0}^{x=\infty}\right]\right\}$$

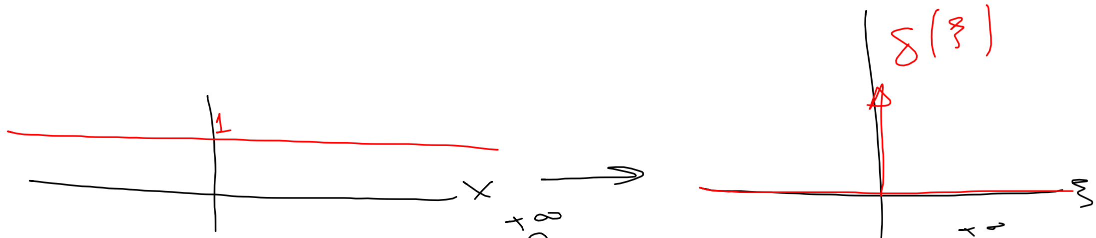
$$\lim_{\tau\to 0}\left(-\frac{1}{\tau-i2\pi\xi} + \left[\frac{-1}{-(\tau+i2\pi\xi)}\right]\right) = \frac{1}{+i(2\pi\xi)} + \frac{1}{i(2\pi\xi)}$$

$$= \frac{\cancel{2}}{i\,\cancel{2}\pi\xi} = \frac{1}{i\pi\xi}$$

$$= 0[\xi] + i\left[-\frac{1}{\pi\xi}\right]$$


Re { sign(x) }
x
Im
x
|F(ξ)|
ξ
Re
ξ
Im
I
+π/2
R
-π/2
+π/2
-π/2
ξ

$$\mathrm{step}(x) = \frac{1}{2}\,\mathbb{1}(x) + \frac{1}{2}\cdot \mathrm{sgn}(x)$$

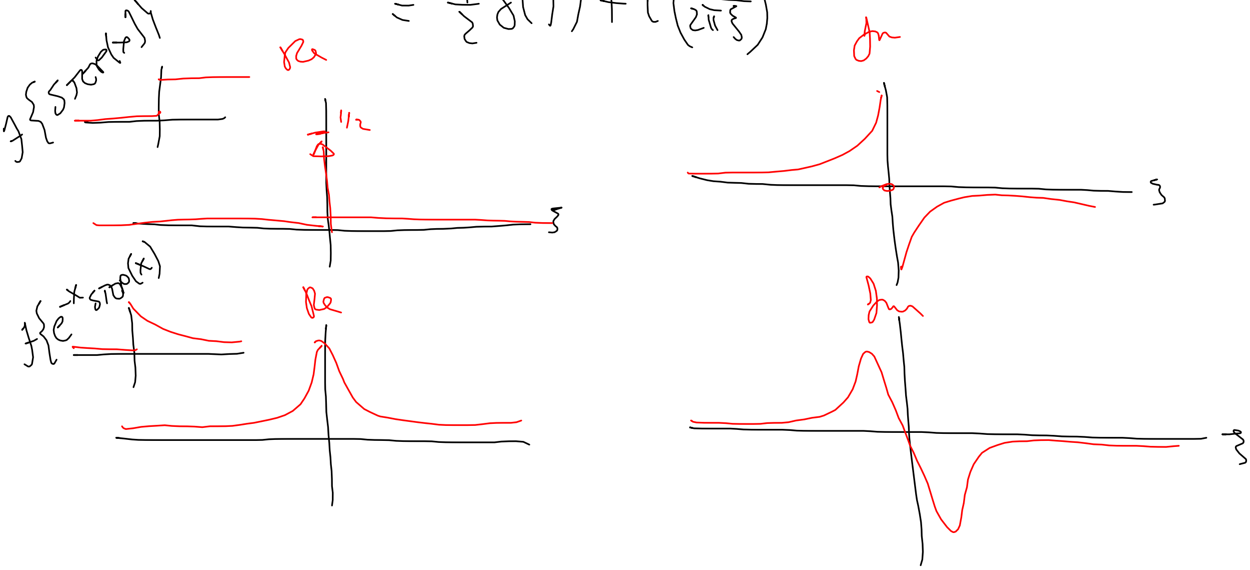


$$\mathcal{F}^{-1}\{\delta(\xi)\} = \int_{-\infty}^{+\infty} \delta(\xi)\, e^{+i2\pi\xi x}\, d\xi = \int_{-\infty}^{+\infty} \delta(\xi)\, e^{+i2\pi\cdot 0 x}\, d\xi = 1 \int \delta(\xi)\, d\xi = \mathbb{1}(x)$$

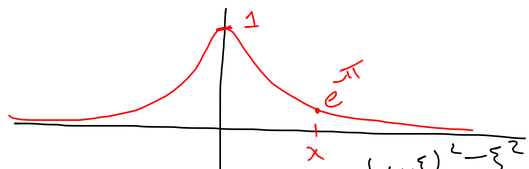
$$\mathcal{F}\{\mathrm{step}(x)\} = \mathcal{F}\left\{\frac{1}{2}\cdot \mathbb{1}(x)\right\} + \mathcal{F}\left\{\frac{1}{2}\cdot \mathrm{sgn}(x)\right\}$$

$$= \frac{1}{2}\cdot\delta(\xi) + \frac{1}{2}\left(\frac{1}{i\pi\xi}\right)$$

$$= \frac{1}{2}\delta(\xi) + i\left(\frac{-1}{2\pi\xi}\right)$$



$$\mathrm{GAUS}(x) = e^{-\pi x^2}$$



$$e^{\pm i\pi x^2}$$

$(x+\xi)^2 - \xi^2$

$x^2 + 2x\xi + \xi^2 - \xi^2$

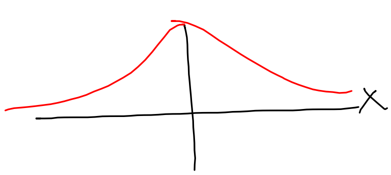
$$\int_{-\infty}^{+\infty} e^{-\pi x^2}\, e^{-i2\pi\xi x}\, dx = \int e^{-\pi\left(\left[x^2 + i2\xi x + (i\xi)^2\right] - (i\xi)^2\right)} dx$$

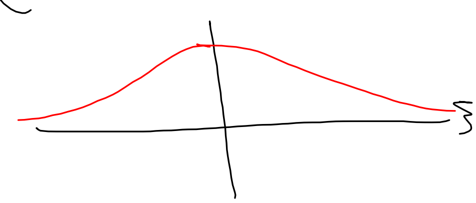
$$= \int e^{-\pi(x+i\xi)^2}\, e^{+\pi(+i\xi)^2}\, dx$$

$$e^{-\pi\xi^2}$$

$$= e^{-\pi\xi^2}\left[\int_{-\infty}^{+\infty} e^{-\pi(x+i\xi)^2}\, dx\right] \rightarrow \text{CONSTANT}$$

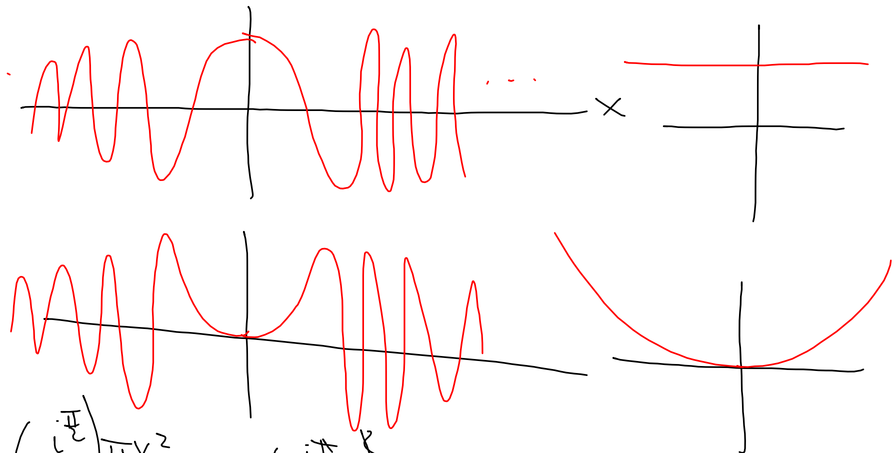
$$e^{-\pi x^2} \longrightarrow e^{-\pi\xi^2} + i\cdot 0(\xi)$$





$e^{+i\pi x^2}$

$= \cos(\pi x^2) + i\sin(\pi x^2)$



$i = e^{+i\pi/2}$

$e^{\left(e^{i\frac{\pi}{2}}\right)\pi x^2}$

$e^{\pi\left(e^{i\frac{\pi}{4}}x\right)^2}$

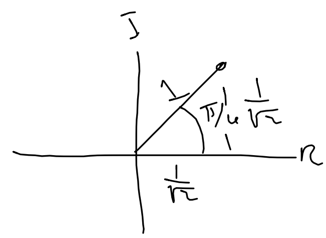
$$\int_{-\infty}^{+\infty} e^{+i\pi x^2} e^{-i2\pi\xi x}\,dx$$

$$\int_{-\infty}^{+\infty} e^{+i\pi\left(x^2-2\xi x+\xi^2-\xi^2\right)}\,dx$$

$$\int_{-\infty}^{+\infty} e^{+i\pi\left(\underline{x^2-2\xi x+\xi^2}\right)} e^{-i\pi\xi^2}\,dx$$

$$\mathcal{F}\left\{e^{+i\pi x^2}\right\} = e^{-i\pi\xi^2}\int_{-\infty}^{+\infty} e^{+i\pi(x-\xi)^2}\,dx \longrightarrow e^{+i\frac{\pi}{4}}$$

$$\mathcal{F}\left\{e^{+i\pi x^2}\right\} = e^{+i\frac{\pi}{4}}\, e^{-i\pi\xi^2}$$



$$= \left(\frac{1+i}{\sqrt{2}}\right) e^{-i\pi\xi^2}$$

$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\left(\pi\xi^2\right) - i\sin\left(\pi\xi^2\right)\right)$$

$$e^{+i\frac{\pi}{4}} e^{-i\pi x^2} \longrightarrow \cancel{e^{-i\frac{\pi}{4}}} e^{+i\pi\xi^2} \cancel{e^{+i\frac{\pi}{4}}}$$

$$e^{+i\frac{\pi}{4}} e^{-i\pi x^2} \longrightarrow e^{+i\pi\xi^2}$$

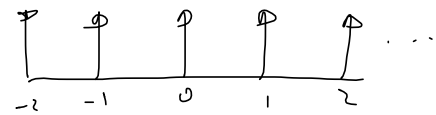
$$e^{+i\pi\left(\frac{x}{\alpha_0}\right)^2} \quad e^{-i2\pi\xi x} \quad (\alpha_0\xi)\left(\frac{x}{\alpha}\right) \longrightarrow e^{-i\pi(\alpha_0\xi)^2}$$

$$\left(\frac{\xi}{1/\alpha_0}\right)^2$$

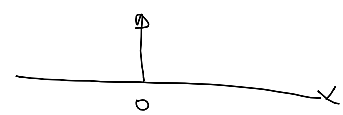
α_0 IS A LENGTH ("CHIRP RATE")

PERIODIC, DISCRETE

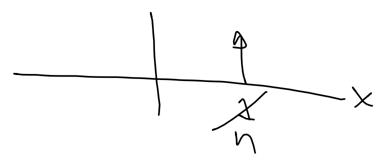
$$\mathrm{COMB}[x] \equiv \sum_{n=-\infty}^{+\infty} \delta(x-n) \ldots$$



$$\mathcal{F}\{\mathrm{comb}(x)\} = \int_{-\infty}^{+\infty} \sum_n \delta(x-n)\, e^{-i2\pi\xi x}\, dx$$



$$= \sum_n \int \delta(x-n)\, e^{-i2\pi\xi x}\, dx$$

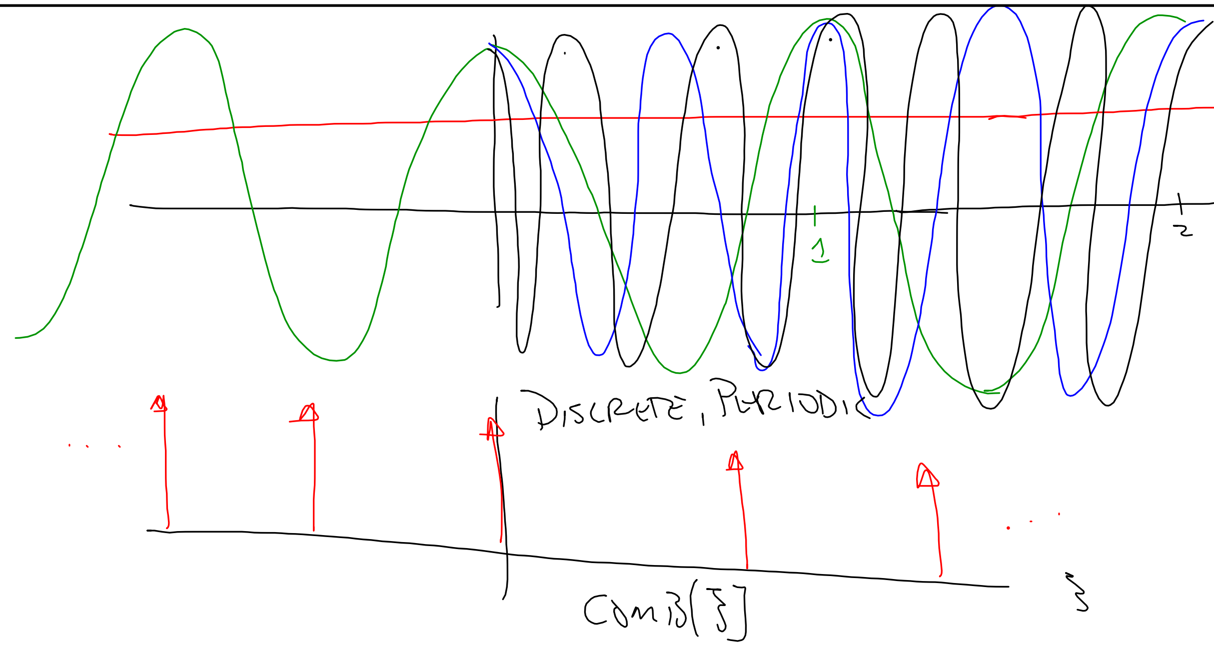


$$= \sum_n \int_{-\infty}^{+\infty} \delta(x-n)\, e^{-i2\pi\xi \cdot n}\, dx$$

$$= \sum_n e^{-i2\pi\xi n} \underbrace{\cancel{\int_{-\infty}^{+\infty} \delta(x-n)\, dx}}_{1}$$

$$= \sum_n e^{-i2\pi\xi n} = 1 + \left[e^{-i2\pi\xi \cdot 1} + e^{-i2\pi\xi(-1)}\right] + \cdots$$

$$= 1 + 2\cos\left[2\pi\xi\right] + 2\cos\left(2\pi \frac{\xi}{1/2}\right) + 2\cos\left(2\pi \frac{\xi}{1/3}\right) + \cdots$$



$$\mathrm{COMB}(x) \longrightarrow \mathrm{COMB}[\xi]$$

THEOREMS

$$f(x) \longrightarrow F(\xi)$$

$$f\left(\frac{x-x_0}{b_0}\right)$$

$$F(x) \longrightarrow ?$$

$$\overset{f}{\mathrm{RECT}(x)} \longrightarrow \overset{F}{\mathrm{SINC}(\xi)}$$

$$\overset{F}{\mathrm{SINC}(x)} \longrightarrow ?$$

1. CHANGE OF VARIABLE
2. SUBSTITUTE $\mathcal{F}^{-1}\{F(\xi)\}$ FOR $f(x)$

$$f(x) \longrightarrow F(\xi)$$

$$m(x) \longrightarrow M(\xi)$$

$$f(x)\cdot m(x) \longrightarrow ?$$

SCALING THEOREM

$$\mathcal{F}\left\{f\left(\frac{x}{b_0}\right)\right\} = ?$$

$b_0 > 0$ $\quad$ $0 < b_0 < 1$ MINIFYING

$b_0 < 0$ $\quad$ $1 < b_0 < \infty$ MAGNIFYING

$b_0 \neq 0$ $\quad$ $-1 < b_0 < 0$

$$\int f\left(\frac{x}{b_0}\right) e^{-i2\pi\xi x}\, dx$$

$b_0 > 0$

$u \equiv \dfrac{x}{b_0}$

$b_0 < 0$