9 OCTOBER

CONVOLUTION $\Longrightarrow$ LSI

$$g(x) = f(x) * h(x) = \int_{-\infty}^{+\infty} f(\alpha)\, h(x-\alpha)\, d\alpha$$

$$= h(x) * f(x)$$

$$\left(A_0 + A_1 \cos(2\pi \xi_0 x)\right) * \mathrm{RECT}\left(\frac{x}{1}\right) =$$

$$= A_0 + A_1 \underbrace{\mathrm{SINC}(\xi_0)}_{\text{MODULATION}} \cos(2\pi \xi_0 x)$$

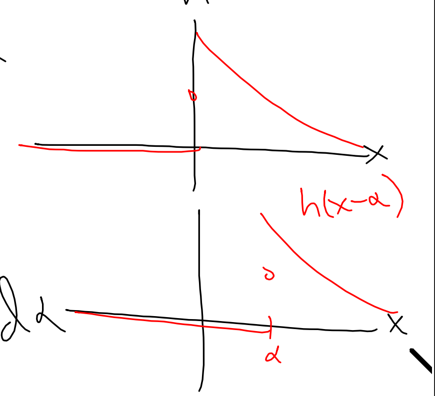
AREA OF CONVOLUTION

$$g(x) = \int_{-\infty}^{+\infty} f(\alpha)\, h(x-\alpha)\, d\alpha$$

$$\int_{-\infty}^{+\infty} g(x)\,dx = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\alpha)\, h(x-\alpha)\, d\alpha\right] dx$$

$h(x) = e^{-x}\,\text{STEP}(x)$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\alpha)\, h(x-\alpha)\, dx\, d\alpha$$



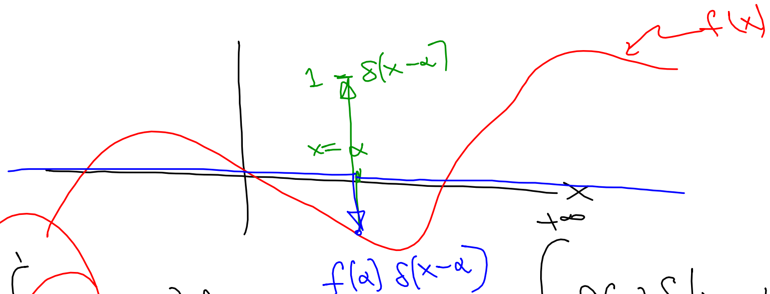
$$= \int_{-\infty}^{+\infty} f(\alpha)\underbrace{\left[\int_{-\infty}^{+\infty} h(x-\alpha)\, dx\right]} d\alpha$$

$$\int_{-\infty}^{+\infty} f(\alpha)\left[\int_{-\infty}^{+\infty} h(x)\, dx\right] d\alpha$$

AREA OF $g(x)$ = (AREA OF $h(x)$) $\cdot$ (AREA OF $f(x)$)

$A_0 + A_1\cos(2\pi\xi_1 x)$ RECT(x)

$$f(x)\,\delta(x-\alpha) = f(\alpha)\,\delta(x-\alpha)$$



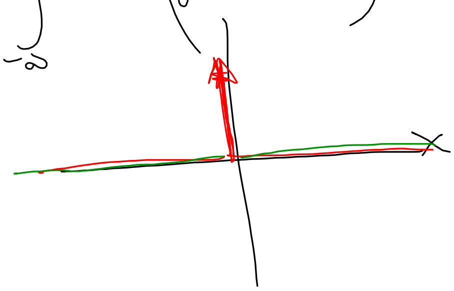
$$\int_{-\infty} f(x)\,\delta(x-\alpha)\,d\alpha = \int_{-\infty}^{+\infty} f(\alpha)\,\delta(x-\alpha)\,d\alpha = f(x) * \delta(x)$$

$$f(x)\int_{-\infty}^{+\infty}\delta(x-\alpha)\,d\alpha = f(x)\cdot 1 = \boxed{f(x) = f(x) * \delta(x)}$$

IDENTITY OPERATOR

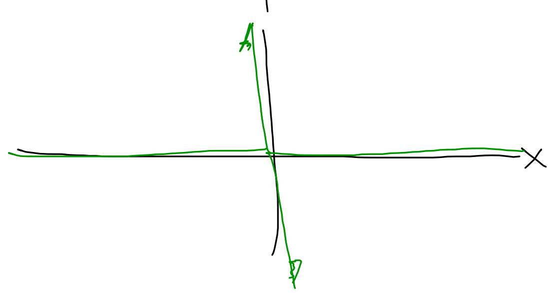
$$\frac{d}{dx}\int_{-\infty}^{+\infty} f(\alpha)\,\delta(x-\alpha)\,d\alpha = \frac{d}{dx}f(x) = f'(x)$$

$$\int_{-\infty}^{+\infty}\frac{d}{dx}\left[f(\alpha)\,\delta(x-\alpha)\right]d\alpha = \int_{-\infty}^{+\infty} f(\alpha)\left(\frac{d}{dx}\,\delta(x-\alpha)\right)d\alpha$$



⋮

$$\frac{df}{dx} = \lim_{b\to 0}\frac{f(x+b)-f(x)}{b}$$



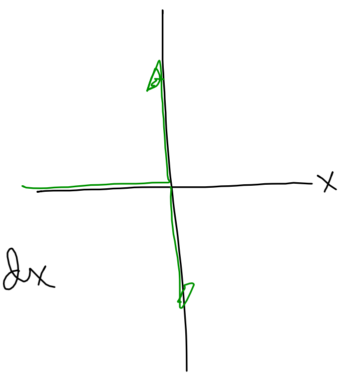
$$\frac{df}{dx} = f'(x) = \int_{-\infty}^{+\infty} f(\alpha)\ \delta'(x-\alpha)\ d\alpha = f(x) * \delta'(x)$$

$$f(x) = \int_{-\infty}^{+\infty} f(\alpha)\ \delta(x-\alpha)\ d\alpha = f(x) * \delta(x)$$

$$\frac{d^n}{dx^n} f(x) = f^{(n)}(x) = f(x) * \delta^{(n)}(x)$$

$$\frac{df}{dx} = f(x) * \delta'(x)$$

$$\int_{-\infty}^{+\infty} \frac{df}{dx}\,dx = \int_{-\infty}^{+\infty} f(x)\,dx \cdot \int_{-\infty}^{+\infty} \delta'(x)\,dx$$



$$\left(\overset{A_0 \cdot 1(x)}{A_0} + A_1 \cos[2\pi\xi_0 x]\right) \to \delta'(x)$$

$$A_0 \cdot \cancel{\frac{d}{dx} 1(x)}^{\,0} + A_1 \frac{d}{dx} \cos[2\pi\xi_0 x]$$

$$= 0 + A_1 \cdot 2\pi\xi_0 \left(-\sin[2\pi\xi_0 x]\right)$$

$$= -2\pi\xi_0 A_1 \sin(2\pi\xi_0 x)$$

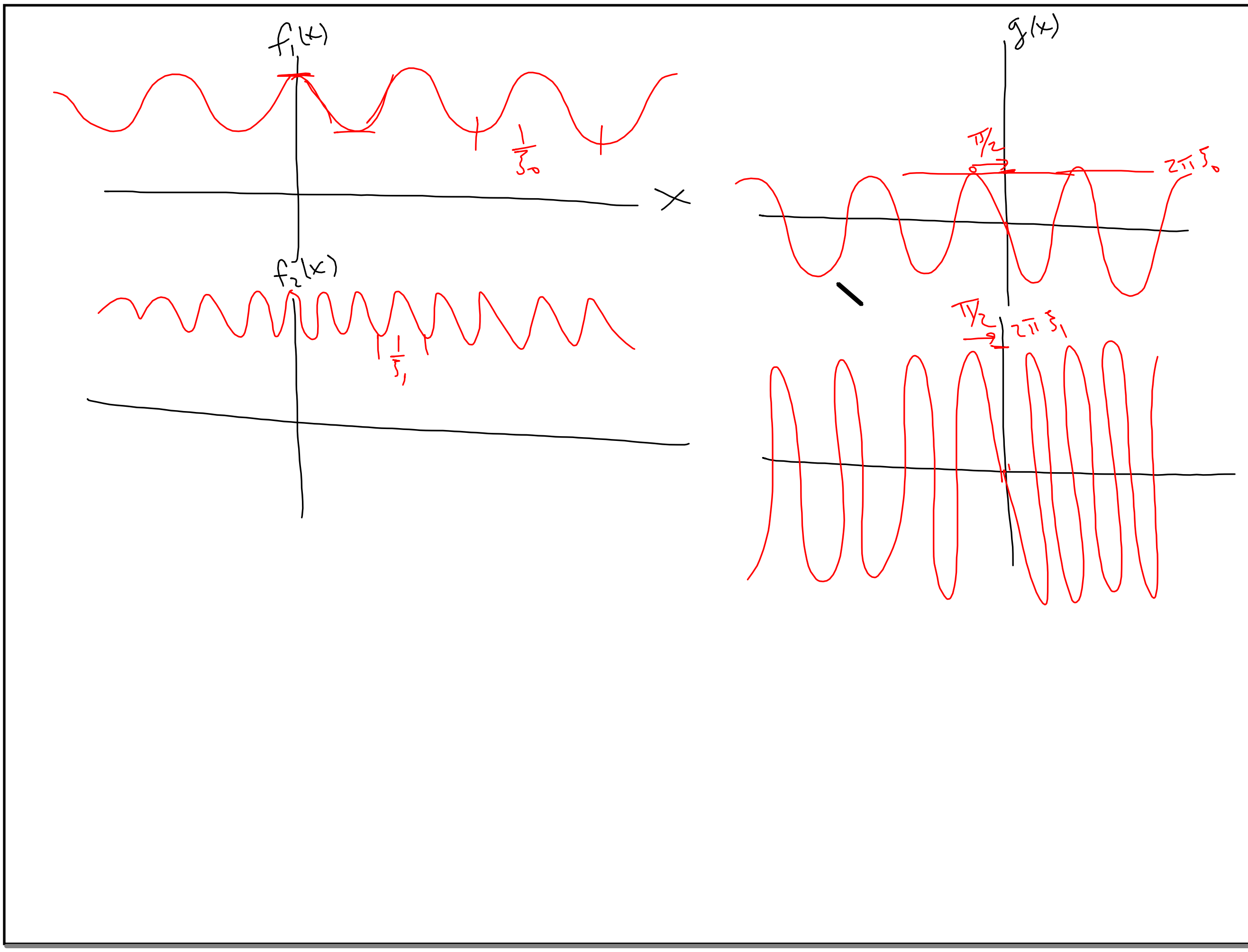
$$\cos\left(2\pi\xi_0 x + \frac{\pi}{2}\right) = \cos(2\pi\xi_0 x) \cancel{\cos\frac{\pi}{2}}^{\,0} - \sin(2\pi\xi_0 x) \cancel{\sin\frac{\pi}{2}}^{\,1}$$

$$\cos\left(2\pi\xi_0 x + \frac{\pi}{2}\right) = -\sin 2\pi\xi_0 x$$

$$A_0\, 2\pi\xi_0 \left(-\sin(2\pi\xi_0 x)\right) = A_0 \cdot 2\pi\xi_0 \cdot \cos\left[2\pi\xi_0 x + \frac{\pi}{2}\right]$$

$$f(x) = A_0 \cdot 1(x) + A_1 \cdot \cos[2\pi\xi_0 x + 0]$$

$$g(x) = A_0 \cdot 0(x) + 2\pi\xi_0 \left[A_1 \cos\left[2\pi\xi_0 x + \frac{\pi}{2}\right]\right]$$



$f_1(x)$
$\frac{1}{\xi_0}$
x
$f_2(x)$
$\frac{1}{\xi_1}$
$g(x)$
$\pi/2$
$2\pi\xi_0$
$\pi/2$
$2\pi\xi_1$

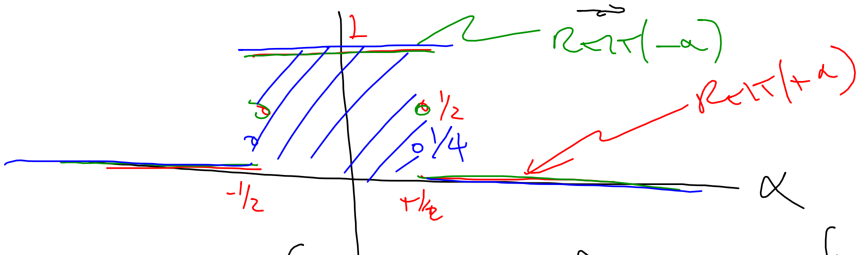
October 08, 2014

$$f(x) * h(x) = g(x)$$

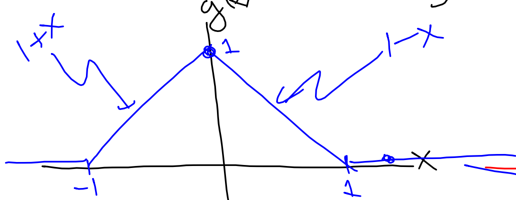
$$h(x) = \delta'(x)$$

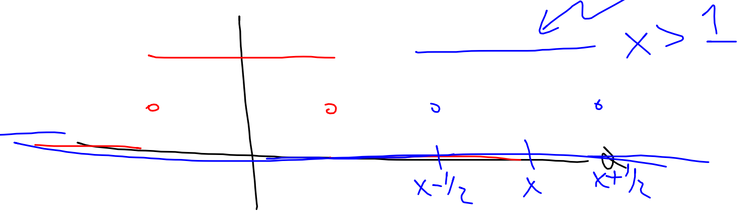
$$\underset{\substack{\uparrow \\ \text{INPUT}}}{\delta(x)} * \underset{\substack{\uparrow \\ \text{RESPONSE}}}{\delta'(x)} = \underbrace{\delta'(x)}_{\text{INPUT}} * \underbrace{\delta(x)}_{\text{SYSTEM}} = \delta'(x)$$

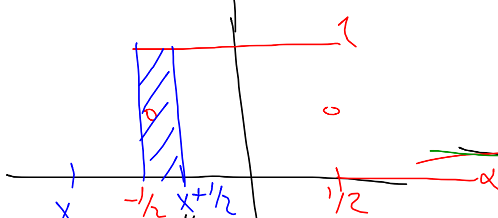
$$f(x) = \mathrm{Rect}(x) = \underset{\mathrm{Rect}(\alpha)}{h(x)} = \int_{-\infty}^{+\infty} \mathrm{Rect}(\alpha)\,\mathrm{Rect}(x-\alpha)\,d\alpha$$

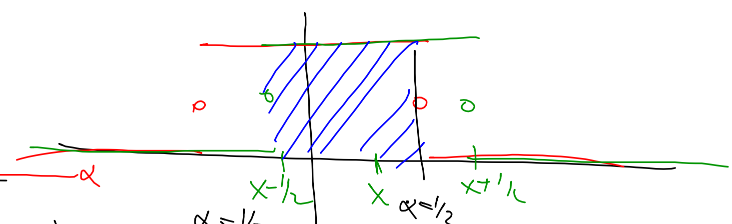


$$g(x=0) = \int \mathrm{Rect}(\alpha)\,\mathrm{Rect}(0-\alpha)\,d\alpha = \int \mathrm{Rect}(\alpha)\,\mathrm{Rect}(-\alpha)\,d\alpha$$









$$\int_{\alpha=-1/2}^{\alpha=x+1/2} 1\,d\alpha = (x+1/2)-(-1/2) = 1+x \qquad \int_{\alpha=x-1/2}^{\alpha=1/2} 1\,d\alpha = \frac{1}{2} - (x-1/2) = 1-x$$

$$\mathrm{RECT}(x) * \mathrm{RECT}(x) = \mathrm{TRI}(x)$$

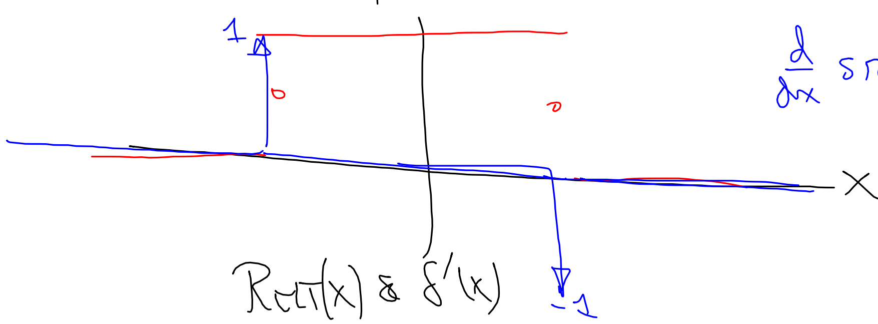
$$\text{meas} \quad \underbrace{\phantom{\mathrm{RECT}(x)}}_{1} \quad \underbrace{\phantom{\mathrm{RECT}(x)}}_{1} \quad = \quad \text{AREA} = 1$$

$$\text{SUPPORT} \quad 1 + 1 = 2$$

IF $f(x)$ & $h(x)$ HAVE COMPACT SUPPORT

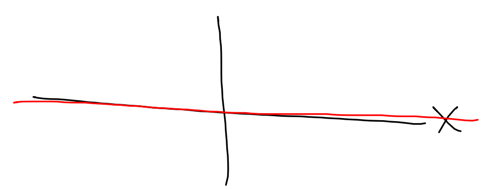
$\mathrm{RECT}(x)$

$$\frac{d}{dx}\,\mathrm{STEP}(x) = \delta(x)$$



$\mathrm{RECT}(x) * \delta'(x)$

$$1(x) * \delta'(x) = \frac{d}{dx}\,1(x) = 0(x)$$



$$f(x) = e^{+i(\pi x^2)} = \cos(\pi x^2) + i\sin(\pi x^2)$$

$$\mathbb{1}(x) \qquad \Phi\{f(x)\}$$

$$|f(x)| = \mathbb{1}(x)$$

$$\Phi\{f(x)\} = \pi x^2$$

$$h(x) = e^{-i\pi x^2} = \mathbb{1}(x)\, e^{i(-\pi x^2)}$$

$$f(x) * h(x) = g(x) = e^{+i\pi x^2} * e^{-i\pi x^2}$$

$$\int f(\alpha)\, h(x-\alpha)\, d\alpha = \int_{-\infty}^{+\infty} e^{+i\pi(\alpha^2)}\, e^{-i\pi(x-\alpha)^2}\, d\alpha$$

$$= \int_{-\infty}^{+\infty} e^{+i\pi\alpha^2}\, e^{-i\pi(x^2+\alpha^2-2x\alpha)}\, d\alpha$$

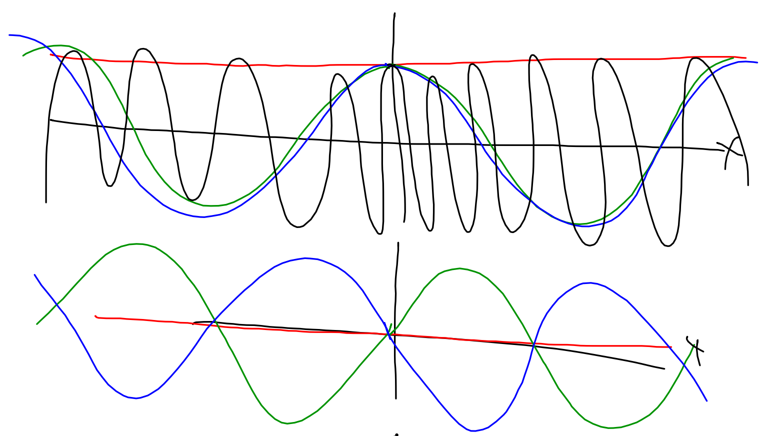
$$= \int e^{+i\pi\alpha^2}\, e^{-i\pi x^2}\, e^{-i\pi\alpha^2}\, e^{+i\pi 2x\alpha}\, d\alpha$$

$$= \int \cancel{e^{+i\pi(\alpha^2-\alpha^2)}}\, e^{-i\pi x^2}\, e^{+i\pi 2x\alpha}\, d\alpha$$

$$e^0 = 1$$

$$e^{+i\pi x^2} * e^{-i\pi\alpha^2} = e^{-i\pi x^2} \underbrace{\int_{-\infty}^{+\infty} e^{+i2\pi x\alpha}\, d\alpha}_{\delta(x)}$$

$$\int_{-\infty}^{+\infty} e^{+i2\pi x\xi}\, d\xi = \delta(x)$$



$$e^{+i\pi x^2} * e^{-i\pi x^2} = e^{-i\pi x^2} \cdot \delta(x) = g(x)$$

$$e^{-i\pi x^2}\, \delta[x-0] = \cancelto{1}{e^{-i\pi \cdot 0^2}}\, \delta(x)$$

$$f(x)\, \delta(x-x_0) = f(x_0)\, \delta(x-x_0)$$

$$\boxed{\underbrace{e^{+i\pi x^2}}_{f(x)} * \underbrace{e^{-i\pi x^2}}_{h(x)} = \delta(x)}$$

$$\left(\delta'(x) * \left(e^{+i\pi x^2}\right) * e^{-i\pi x^2}\right] = \delta'(x) * \delta(x) = \delta'(x)$$

$$f(x-x_0) \otimes h(x) = g(x-x_0)$$

$$f(x-x_0) = f(x) \otimes \delta(x-x_0)$$

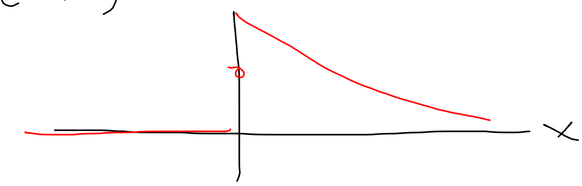
$$f(x-x_0) \otimes h(x) = f(x) \otimes \delta(x-x_0) \otimes h(x)$$

$$= \delta(x-x_0) \otimes \big(f(x) \otimes h(x) \big)$$

$$= \delta(x-x_0) \otimes g(x) = g(x-x_0)$$

$$f(x) = h(x) = e^{-x}\,\text{STEP}(x)$$

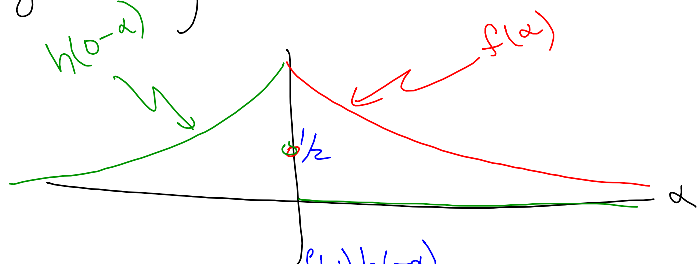
$$h(x) = f(-x)$$

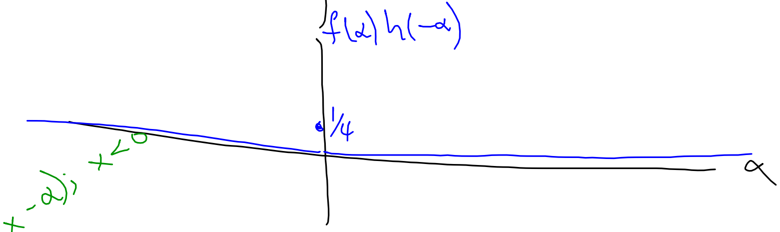


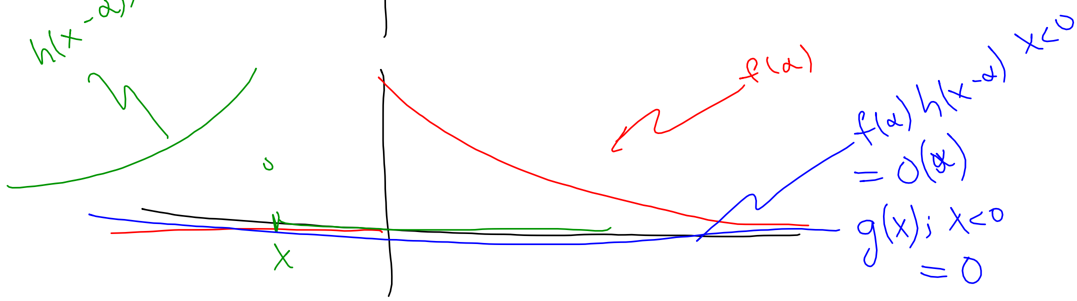
$$e^{-x}\,\text{STEP}(x) * e^{-x}\,\text{STEP}(x) =$$

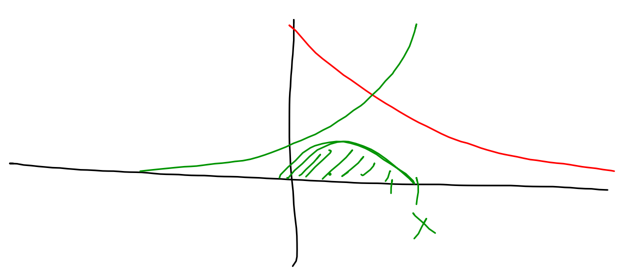
$$= \int_{\alpha=-\infty}^{\alpha=+\infty} \left[e^{-\alpha}\,\text{STEP}(\alpha)\right]\left[e^{-(x-\alpha)}\,\text{STEP}(x-\alpha)\right] d\alpha$$

$$x=0 \Rightarrow g(0) = \int e^{-\alpha}\,\text{STEP}(\alpha)\; e^{-(0-\alpha)}\,\text{STEP}(-\alpha)\, d\alpha$$









$x \le 0 \quad g(0) = 0$

$x > 0$

$$\int_{\alpha=0}^{\alpha=x} e^{-\alpha} \, e^{-(x-\alpha)} \, d\alpha$$

$$\int_{\alpha=0}^{\alpha=x} \cancel{e^{-\alpha}} \, \cancel{e^{+\alpha}} \, e^{-x} \, d\alpha$$

$$e^{-x} \int_{\alpha=0}^{x} 1 \, d\alpha = x e^{-x}$$

$$g(x) = \begin{cases} x e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases} = x e^{-x} \operatorname{STEP}(x)$$