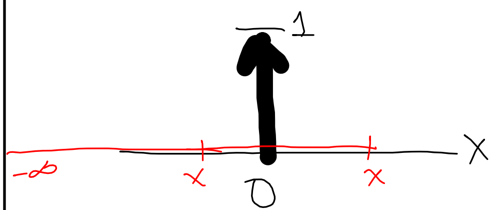
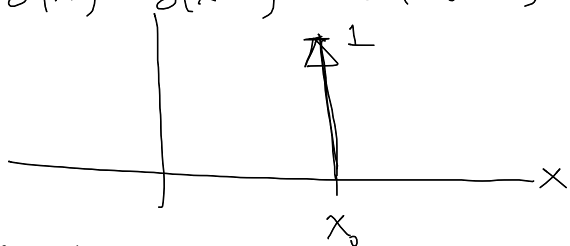
9/25

EXAM NEXT THURSDAY, 9:30 – 10:45

(NO HOMEWORK UNTIL AFTER EXAM)

DIRAC DELTA FUNCTION $\delta(x)$ $\delta(x-x_0) = \delta(x_0-x)$





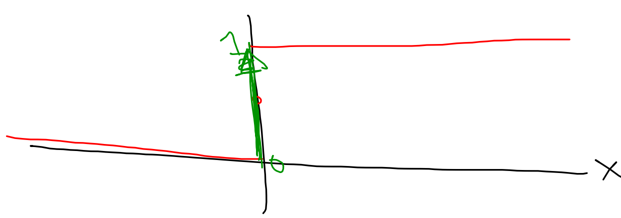
$\delta(-x) = \delta(+x) \Rightarrow$ even

$$\int_{-\infty}^{x} \delta(\alpha)\, d\alpha = \begin{cases} 1 & x>0 \\ \frac{1}{2} & x=0 \\ 0 & x<0 \end{cases} \quad STEP(x)$$

$$\frac{d}{dx}\int_{-\infty}^{x} \delta(\alpha)\, d\alpha = \frac{d}{dx} STEP(x)$$

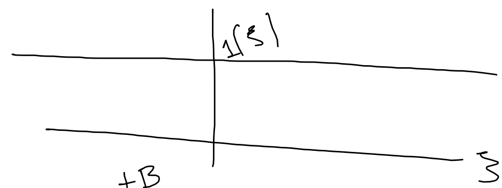
FUNDAMENTAL THM. OF CALCULUS

$$\delta(x) - \delta(-\infty) = \delta(x) - 0 = \boxed{\delta(x) = \frac{d}{dx} STEP(x)}$$



$$\int_{-\infty}^{+\infty} e^{+i2\pi\xi x}\, d\xi = \int_{-\infty}^{+\infty} \left(1[\xi]\right) e^{+i2\pi\xi x}\, d\xi$$

n.b.



$$\int_{-\infty}^{+\infty} e^{+i2\pi\xi x}\, d\xi = \lim_{B\to\infty} \int_{-B}^{+B} e^{+i2\pi\xi x}\, d\xi$$

$$\int_a^b e^{Ax}\, dx = \frac{1}{A} e^{Ax}\Big|_a^b \quad ; \quad A = +i2\pi x$$

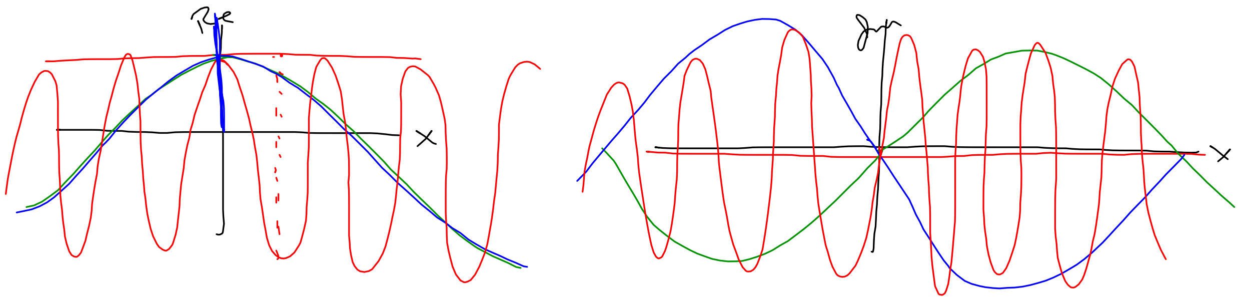
$$\lim_{B\to\infty} \frac{e^{+i2\pi\xi x}}{+i2\pi x}\Bigg|_{\xi=-B}^{+B} = \lim_{B\to\infty} \frac{e^{+i2\pi x\cdot B} - e^{+i2\pi x(-B)}}{(+2i)(\pi x)}$$

$$= \lim_{B\to\infty} \left(\frac{e^{+i2\pi xB} - e^{-i2\pi xB}}{2i}\right)\cdot\frac{1}{\pi x}$$

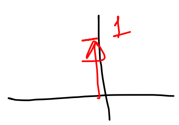
$$= \lim_{B\to\infty} 2B\, \frac{\sin(2\pi xB)}{2B\pi x} = \lim_{B\to\infty} 2B\, \mathrm{SINC}[2xB]$$

$$\mathrm{SINC}\left[\frac{x}{1/2B}\right]$$

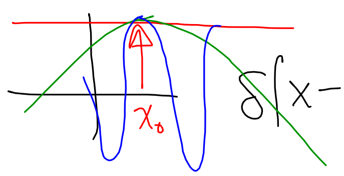
$$\int_{-\infty}^{+\infty} e^{+i(2\pi\xi x + 0)}\, d\xi = \delta[x]$$



$$e^{+i2\pi x\cdot\xi} = \cos(2\pi x\xi) + i\sin(2\pi x\xi)$$



$\delta(x) \longleftarrow \underbrace{1(\xi)}$ FOURIER SYNTHESIS OF $\delta(x)$



$\delta(x-x_0)$

EQUAL AMPLITUDES AT ALL ξ

$$\delta(x) = \int_{-\infty}^{+\infty} e^{+i2\pi\xi x}\, dx$$

$$\int_{-\infty}^{+\infty} e^{+i2\pi\xi(-x)}\, d\xi = \delta(-x) = \delta(x)$$

$$\int_{-\infty}^{+\infty} e^{+i2\pi\frac{x}{b_0}\cdot\xi}\, d\xi = \delta\left(\frac{x}{b_0}\right)$$

$$\int_{\xi=-\infty}^{+\infty} e^{+i2\pi x\cdot\left(\frac{\xi}{b_0}\right)}\, d\xi\, ; \quad u \equiv \frac{\xi}{b_0} \Rightarrow \xi = u\cdot b_0$$

$$d\xi = d(u\cdot b_0) = |b_0|\, du$$

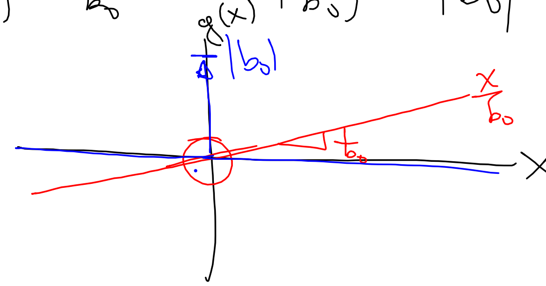
$$\xi = \pm\infty \Rightarrow u = \pm\infty$$

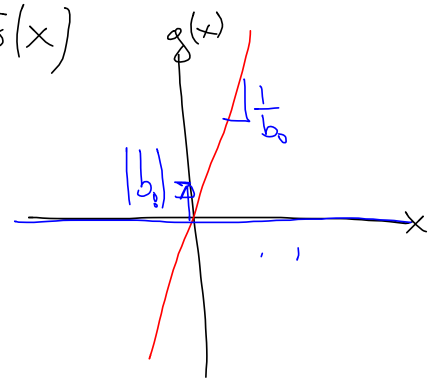
$$\int_{u=-\infty}^{+\infty} e^{+i2\pi x\cdot u}\, |b_0|\, du = |b_0| \int_{u=-\infty}^{u=+\infty} e^{+i2\pi x u}\, du$$

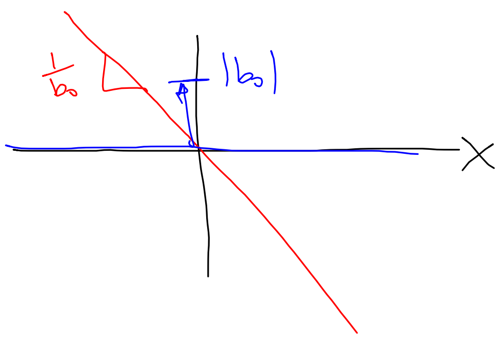
$$\delta\left(\frac{x}{b_0}\right) = |b_0|\, \delta(x)$$

$\delta[g(x)]$

$$g(x) = \frac{x}{b_0} \Rightarrow \delta\left(\frac{x}{b_0}\right) = |b_0|\,\delta(x)$$







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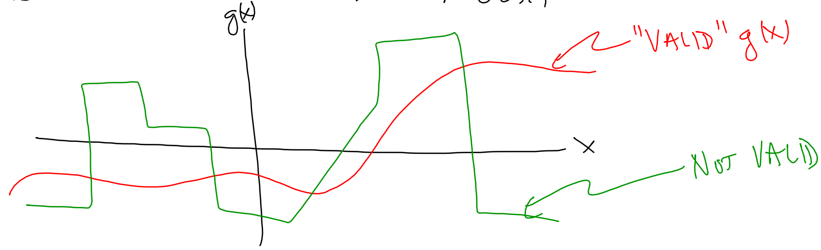
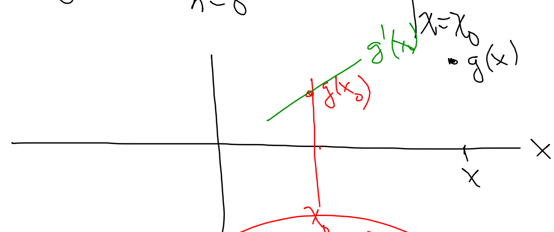
$\delta[g(x)]$

$g(x)$ IS "WELL BEHAVED" $\Longrightarrow$ "SMOOTH"

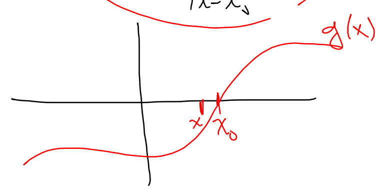


$$g(x) = \sum_{n=0}^{\infty} a_n x^n \longrightarrow \sum a_n (x-x_0)^n$$

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n g}{dx^n}\right|_{x=x_0} (x-x_0)^n$$



$$g(x) = \cancel{\frac{1}{0!} g(x_0)}^{\,0} + \frac{1}{1!} \left.\frac{dg}{dx}\right|_{x=x_0} (x-x_0)^1 + \cancel{\frac{1}{2!} \left.\frac{d^2 g}{dx^2}\right|_{x_0} (x-x_0)^2 + \cdots}$$



$g(x_0) = 0$

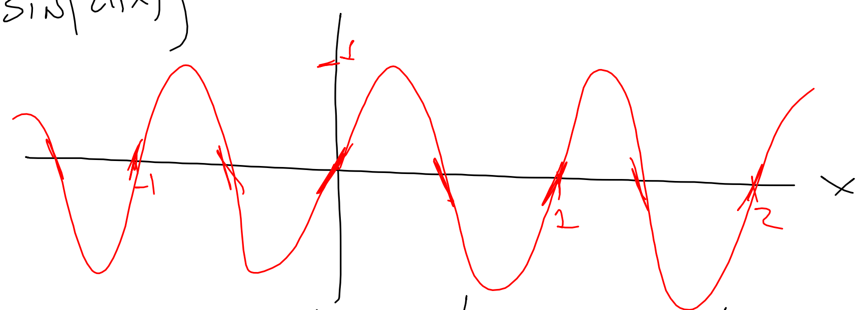
x IS CLOSE TO x_0 $\quad (x-x_0) > (x-x_0)^2$
$> (x-x_0)^3$

$$\delta[g(x)] \simeq \delta[g'(x_0)(x-x_0) + \cdots]$$

$$\delta\left(\frac{x}{b_0}\right) = |b_0|\,\delta(x) \Rightarrow \boxed{\delta[g(x)] = \frac{1}{|g'(x_0)|}\,\delta(x-x_0)}$$

WHERE $g(x_0) = 0$

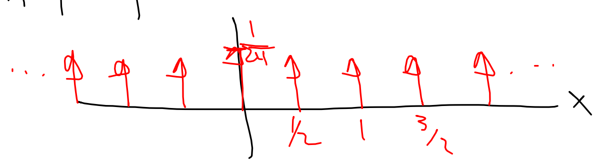
$\delta[\sin(2\pi x)]$



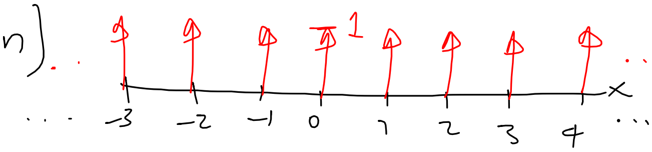
$$g(x) = \sin(2\pi x) \Rightarrow g'(x) = 2\pi \cos(2\pi x)$$

$$g'(0) = 2\pi, \quad g'(\tfrac{1}{2}) = -2\pi$$

$$\delta[\sin(2\pi x)] = \sum_n \frac{1}{|\pm 2\pi|} \delta(x - n \cdot \tfrac{1}{2})$$



$$\text{COMB}(x) \equiv \sum_{n=-\infty}^{+\infty} \delta(x-n)$$

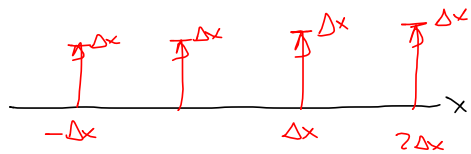


$$\text{COMB}\left(\frac{x}{\Delta x}\right) = \sum_{n=-\infty}^{+\infty} \delta\left(\frac{x}{\Delta x}-n\right) = \sum_{n} \delta\left(\frac{x-n\cdot\Delta x}{\Delta x}\right)$$

SPACING

$$\delta\left(\frac{x}{b_0}\right) = |b_0|\,\delta(x)$$

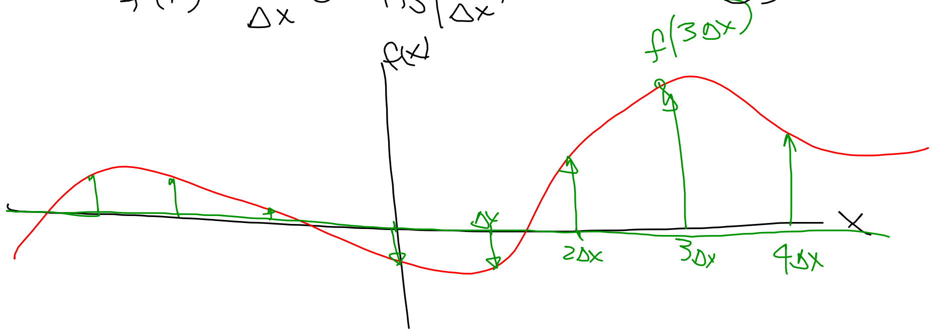
$$\text{COMB}\left(\frac{x}{\Delta x}\right) = |\Delta x| \sum_{n} \delta(x-n\cdot\Delta x)$$



$$\frac{1}{\Delta x}\,\text{COMB}\left(\frac{x}{\Delta x}\right) = \sum_{n} \delta(x-n\cdot\Delta x)$$

SPACING

$$f(x)\cdot\frac{1}{\Delta x}\,\text{COMB}\left(\frac{x}{\Delta x}\right) = f_s(x;\Delta x)$$



$$e^{+i(2\pi \xi_0 x)} = \cos[2\pi \xi_0 x] + i \sin(2\pi \xi_0 x)$$

$$\Phi[x] = 2\pi \xi_0 x \quad [\text{RADIANS}]$$

$$\frac{d\Phi}{dx} = 2\pi \xi_0 \quad \left(\frac{\text{RADIANS}}{\text{LENGTH}}\right)$$

$$\frac{1}{2\pi}\frac{d\Phi}{dx} = \xi_0 \quad \left(\frac{\text{CYCLES}}{\text{LENGTH}}\right)$$

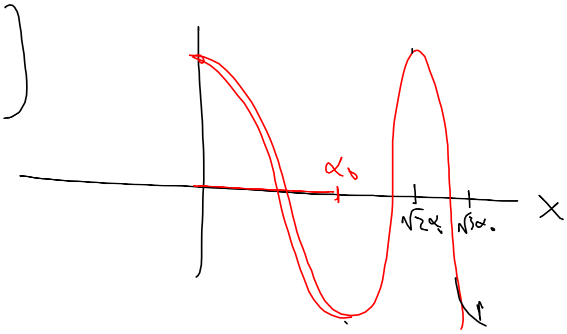
$$e^{+i\pi\left(\frac{x}{\alpha_0}\right)^2} = \cos\left[\pi \frac{x^2}{\alpha_0^2}\right] + i \sin\left[\pi \frac{x^2}{\alpha_0^2}\right]$$

$$\Phi = \pi\left(\frac{x}{\alpha_0}\right)^2 = \pi \frac{x^2}{\alpha_0^2}$$

$$\frac{d\Phi}{dx} = \pi \cdot \frac{2x}{\alpha_0^2} = 2\pi \frac{x}{\alpha_0^2} \quad \left(\frac{\text{RADIANS}}{\text{LENGTH}}\right)$$

$$\xi = \frac{1}{2\pi}\frac{d\Phi}{dx} = \frac{x}{\alpha_0^2} \quad \left(\frac{\text{cycles}}{\text{LENGTH}}\right)$$

α_0 = "CHIRP LENGTH"



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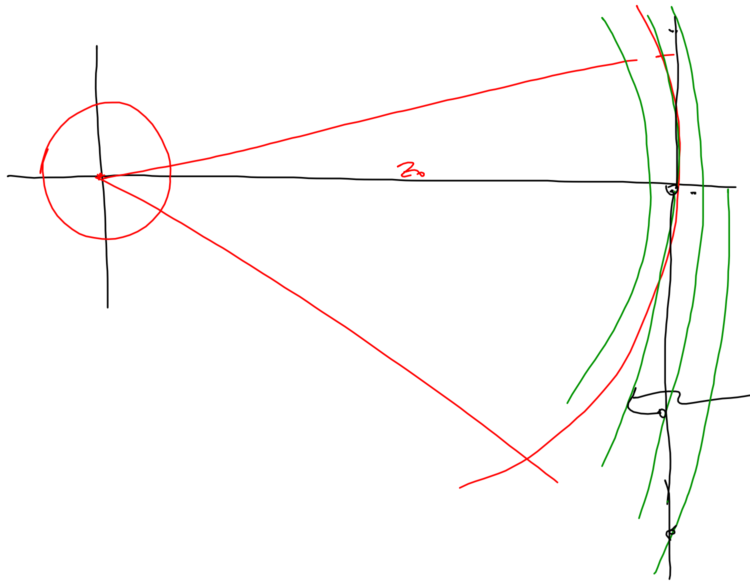


$$e^{+i\pi \frac{x^2}{\lambda_0 z_0}} \qquad e^{+i\pi \left(\frac{x}{\sqrt{\lambda_0 z_0}}\right)^2}$$

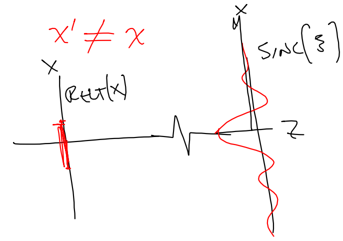
α_0

§ 8 OPERATIONS $\mathcal{O}\{f(x)\} = g(x)$

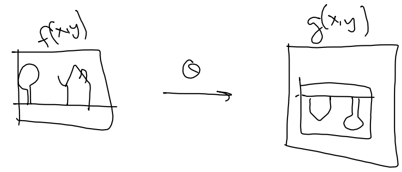
LENGTH "POSITION" $x' \neq x$

$\mathcal{F}\{f(x)\} = F(\xi)$

LENGTH (mm) FREQ $\left(\frac{\text{cycles}}{\text{mm}}\right)$



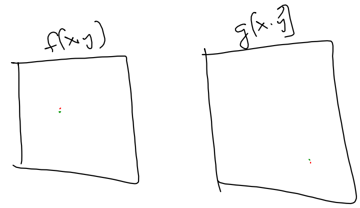
$\mathcal{O}\{f(x)\} = g(x)$



$\mathcal{O} \rightarrow$ ARBITRARY

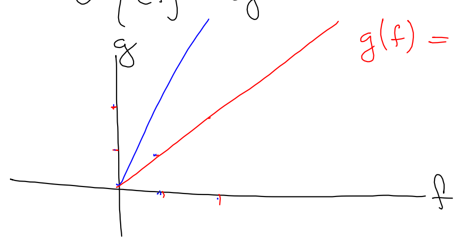
$\mathcal{O} \rightarrow [x]$
$\rightarrow f$

CONSTRAINED



(1) $\mathcal{O}$ ON AMPLITUDE f

LINEARITY $\mathcal{O}\{f\} = g$
$\mathcal{O}\{2f\} = 2g$



$g(f)$ = LOOKUP TABLE