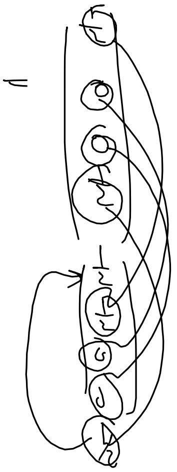
$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

$\underset{\sim}{A}$ IS CIRCULANT $\Longrightarrow$ "SHIFT-INVARIANT"

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$



CONVOLUTION

$$\begin{bmatrix} \alpha & \beta & \gamma & \cdots & & \omega \\ \omega & \alpha & \beta & \gamma & \cdots & \xi \\ \\ \beta & \gamma & \cdots & & \omega & \alpha \end{bmatrix}$$

LINEAR, SHIFT-INVARIANT

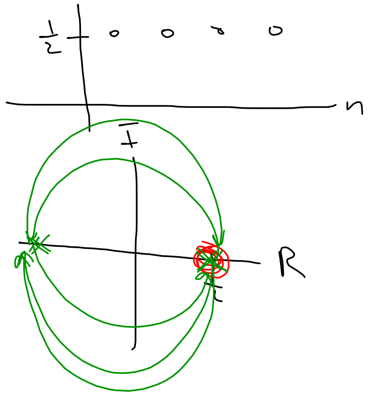
(CIRCULANT)

(DISCRETE CONVOLUTION)

EIGENVECTORS FOR CIRCULANT MATRICES

4 × 4

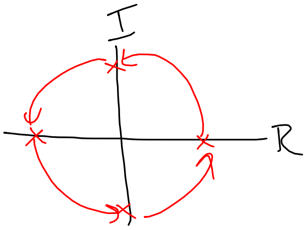
$$\hat{\underset{\sim}{x}}_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$



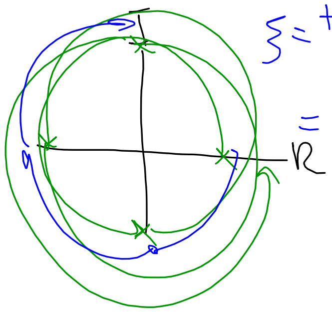
$$\hat{\underset{\sim}{x}}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$\xi = \pm\frac{1}{2}\ \frac{\text{cycle}}{\text{pixel}}$$

$$\hat{\underset{\sim}{x}}_1 = \begin{pmatrix} \frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{pmatrix}$$



$$\hat{\underset{\sim}{x}}_3 = \begin{pmatrix} 1/2 \\ -i/2 \\ -1/2 \\ +i/2 \end{pmatrix}$$



$\xi = +\frac{3}{4}$

$= -\frac{1}{4}$

$$\underset{\sim}{\overline{X}} = \begin{bmatrix} \left(\underset{\sim}{\hat{x}_0}\right) & \left(\underset{\sim}{\hat{x}_1}\right) & \cdots & \left(\underset{\sim}{\hat{x}_{N-1}}\right) \end{bmatrix} \equiv \underset{\sim}{D}$$

MATRIX OF NORMALIZED e'VECTORS

$$\underset{\sim}{A}\,\underset{\uparrow}{\underset{\sim}{\overline{X}}}$$

$$N=4 \qquad \frac{1}{2}\begin{array}{c} \begin{array}{cccc} \xi=0 & 1/4 & 1/2 & 3/4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & +i \end{bmatrix} \end{array}$$

$$N=2 \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\boxed{\underset{\sim}{A}\,\underset{\sim}{D} = \underset{\sim}{D}\,\underset{\sim}{\Lambda}}$$

$$\underset{\sim}{D} \longrightarrow \underset{\sim}{D}^{-1} \Rightarrow \underset{\sim}{D}\,\underset{\sim}{D}^{-1} = \underset{\sim}{I} = \underset{\sim}{D}^{-1}\underset{\sim}{D}$$

$$\underset{\sim}{D}^{-1} = \left(\underset{\sim}{D}^{T}\right)^{*}$$ BECAUSE $\underset{\sim}{D}$ IS ORTHONORMAL

$$\underset{\sim}{D} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & +i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & +i \end{pmatrix}$$

$$\underset{\sim}{D}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{pmatrix}$$

$$\underset{\sim}{D}\,\underset{\sim}{D}^{-1} = \begin{pmatrix} \boxed{1} & \boxed{0} & 0 & 0 \\ \boxed{0} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{D}^{-1}\underset{\sim}{D} = \underset{\sim}{I}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \underset{\sim}{D} = \underset{\sim}{D}^{-1} = \underset{\sim}{D}^{\dagger} = \underset{\sim}{D}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

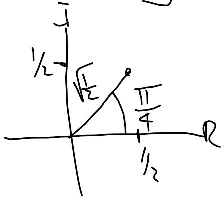
$$\underset{\sim}{A}\,\underset{\sim}{D} = \underset{\sim}{D}\,\underset{\sim}{\Lambda}$$

$$\underset{\sim}{\Lambda} = \begin{bmatrix} \lambda_0 & & & & \\ & \lambda_1 & & O & \\ & & \lambda_2 & & \\ & O & & \ddots & \\ & & & & \lambda_{N-1} \end{bmatrix}$$

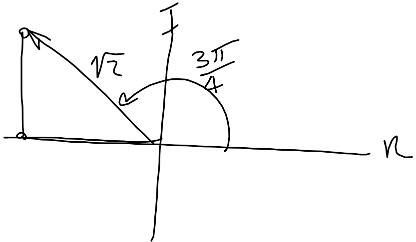
$$\underset{\sim}{D}^{-1}\underset{\sim}{A}\,\underset{\sim}{D} = \overset{\underset{\sim}{I}}{\overbrace{\underset{\sim}{D}^{-1}\underset{\sim}{D}}}\,\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}$$

$$\boxed{\underset{\sim}{D}^{-1}\underset{\sim}{A}\,\underset{\sim}{D} = \underset{\sim}{\Lambda}}$$

$$\lambda_1 = \frac{1+i}{2}$$



$\lambda_1 = -1 + i$



$$(\alpha)\begin{pmatrix} 1 \\ +i \\ -1 \\ -i \end{pmatrix}$$

$$\underset{\sim}{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} ; \quad \underset{\sim}{b} = \begin{pmatrix} -2 \\ +2 \\ -2 \\ +2 \end{pmatrix}$$

$\lambda_2 = -2 = 2e^{\pm i\pi}$

"CANONICAL"

"PIXEL"

$$\underset{\sim}{A}\ \underset{\sim}{x} = \underset{\sim}{b}$$

$$\underset{\sim}{D}\underset{\sim}{\Lambda}\underset{\sim}{D}^{-1} \uparrow \quad \downarrow \underset{\sim}{D}^{-1}\underset{\sim}{A}\underset{\sim}{D} \quad - \quad - \quad -$$

"FREQUENCY REPRESENTATION"

$$\underset{\sim}{\Lambda}$$

$$\underset{\sim}{D}^{-1}\underset{\sim}{A}\underset{\sim}{D}$$

$$\underset{\sim}{D}^{-1}\underset{\sim}{A}\underset{\sim}{D} = \underset{\sim}{\Lambda}$$

$$\cancel{\underset{\sim}{D}\underset{\sim}{D}^{-1}}\underset{\sim}{A}\underset{\sim}{D} = \underset{\sim}{D}\underset{\sim}{\Lambda}$$

$$\underset{\sim}{A}\underset{\sim}{D}\underset{\sim}{D}^{-1} = \underset{\sim}{D}\underset{\sim}{\Lambda}\underset{\sim}{D}^{-1}$$

$$\boxed{\underset{\sim}{A} = \underset{\sim}{D}\underset{\sim}{\Lambda}\underset{\sim}{D}^{-1}}$$

$$\overset{\text{SYSTEM}}{\underset{\sim}{A}}\ \overset{\text{INPUT}}{\underset{\sim}{x}} = \overset{\text{OUTPUT}}{\underset{\sim}{b}}$$

$$\underset{\sim}{A}\,\underset{\sim}{I}\,\underset{\sim}{x}$$

$$\underset{\sim}{D}^{-1}\underset{\sim}{A}\left(\underset{\sim}{D}\,\underset{\sim}{D}^{-1}\right)\underset{\sim}{x} = \underset{\sim}{D}^{-1}\underset{\sim}{b}$$

$$\left(\underbrace{\underset{\sim}{D}^{-1}\underset{\sim}{A}\,\underset{\sim}{D}}\right)\left(\underset{\sim}{D}^{-1}\underset{\sim}{x}\right) = \left(\underset{\sim}{D}^{-1}\underset{\sim}{b}\right)$$

$$\underset{\sim}{\Lambda}\,\underset{\sim}{x}' = \underset{\sim}{b}'$$

$$\begin{tikzcd}
\underset{\sim}{A} \arrow[d, "\underset{\sim}{D}^{-1}\underset{\sim}{A}\underset{\sim}{D}"'] & \underset{\sim}{x} \arrow[d, "\underset{\sim}{D}^{-1}\underset{\sim}{x}"] & = & \underset{\sim}{b} \arrow[d, "\underset{\sim}{D}^{-1}\underset{\sim}{b}"] & \\
\boxed{\underset{\sim}{\Lambda}} & \underset{\sim}{x}' \arrow[d, bend left] & = & \underset{\sim}{b}' \arrow[u, bend right] & \underset{\sim}{b}' \arrow[r] & \underset{\sim}{b}
\end{tikzcd}$$

$$\begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix}$$

$$\cancel{\underset{\sim}{D}\,\underset{\sim}{D}^{-1}}\underset{\sim}{b} = \underset{\sim}{D}\,\underset{\sim}{b}'$$

$$\underset{\sim}{D}\,\underset{\sim}{b}' = \underset{\sim}{b}$$

$$\underset{\sim}{x}' = \underset{\sim}{D}^{-1}\underset{\sim}{x} \longrightarrow \text{FROM "PIXEL" TO "FREQUENCY"}$$

$$\underset{\sim}{x} = \underset{\sim}{D}\,\underset{\sim}{x}' \longrightarrow \text{FROM FREQ. TO PIXEL}$$

$$\underset{\sim}{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad \underset{\sim}{\Lambda} = \begin{pmatrix} 2 & & & \bigcirc \\ & 2 & & \\ & & 2 & \\ \bigcirc & & & 2 \end{pmatrix}$$

$$\underset{\sim}{x} \longrightarrow \underset{\sim}{b} = 2\underset{\sim}{x}$$

$$\underset{\sim}{A} = \underset{\sim}{D}\, \underset{\underset{\underset{\sim}{I}}{\uparrow}}{\underset{\sim}{\Lambda}}\, \underset{\sim}{D}^{-1}$$

$$(\underset{\sim}{D}\,\underset{\sim}{I})\,\underset{\sim}{D}^{-1} = \underset{\sim}{D}\,\underset{\sim}{D}^{-1} = \underset{\sim}{I}$$

$$\underset{\sim}{\Lambda} = \begin{pmatrix} 2 & & & 0 \\ & 2 & & \\ & & 2 & \\ 0 & & & 2 \end{pmatrix} ; \quad \underset{\sim}{\Lambda}^{-1} = \begin{pmatrix} \frac{1}{2} & & & 0 \\ & \frac{1}{2} & & \\ & & \frac{1}{2} & \\ 0 & & & \frac{1}{2} \end{pmatrix}$$

$$\underset{\sim}{\Lambda} = \begin{pmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \lambda_2 & \\ 0 & & & \lambda_3 \end{pmatrix} ; \quad \underset{\sim}{\Lambda}^{-1} = \begin{pmatrix} \frac{1}{\lambda_0} & & & 0 \\ & \frac{1}{\lambda_1} & & \\ & & \frac{1}{\lambda_2} & \\ 0 & & & \frac{1}{\lambda_3} \end{pmatrix}$$

$$\begin{tikzcd}
\underset{\sim}{A} \arrow[d, "\underset{\sim}{D}^{-1}\underset{\sim}{A}\underset{\sim}{D}"] & \underset{\sim}{A}^{-1} \\
\underset{\sim}{\Lambda} \arrow[r] & \underset{\sim}{\Lambda}^{-1} \arrow[u, "\underset{\sim}{D}\underset{\sim}{\Lambda}^{-1}\underset{\sim}{D}^{-1}"']
\end{tikzcd}$$

$$\underset{\sim}{A} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

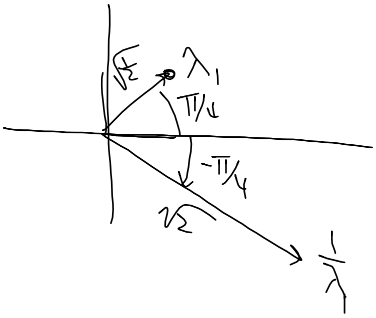
$$\underset{\sim}{\underline{\Lambda}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

PSEUDOINVERTIBLE

$\underset{\sim}{\underline{\Lambda}}^{-1}$ DOES NOT EXIST

$\underset{\sim}{\underline{\Lambda}}^{\dagger}$ PSEUDOINVERSE: $\frac{1}{\lambda_n}$ IF $\lambda_n \neq 0$

$$\underset{\sim}{A}^{\dagger} = \underset{\sim}{D}\,\underset{\sim}{\underline{\Lambda}}^{\dagger}\,\underset{\sim}{D}^{-1}$$



$$\underset{\sim}{x} \longrightarrow \underset{\sim}{x}' = \underset{\sim}{D}^{-1}\underset{\sim}{x}$$

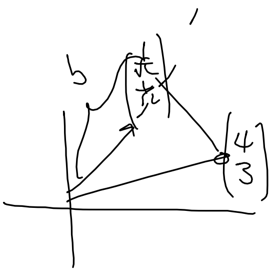
$$\underset{\sim}{A} \longrightarrow \underset{\sim}{\Lambda} = \underset{\sim}{D}^{-1}\underset{\sim}{A}\,\underset{\sim}{D}$$

$$\boxed{\underset{\sim}{D}^{-1} = \underset{\sim}{D}^{*}}$$

$$\overset{8}{\downarrow}$$

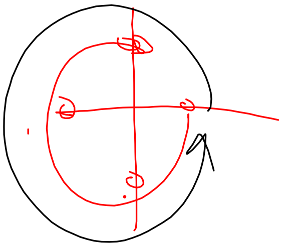
$$\hat{\underset{\sim}{a}} \circ \underset{\sim}{x} = b \leftarrow \text{PROJECTION OF } \underset{\sim}{x} \text{ ONTO } \hat{\underset{\sim}{a}}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & +\frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & +\frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} \begin{array}{l} \text{CONSTAN} \\ \xi = +\frac{1}{4} \\ \xi = +\frac{1}{2} \\ \xi = -\frac{1}{4} \\ \quad +\frac{3}{4} \end{array}$$



$$x_0' = \frac{x_0 + x_1 + x_2 + x_3}{2}$$

$$\frac{4}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$



$$x_1' = \frac{1}{2}\left(x_0 - ix_1 - x_2 + ix_3\right)$$

$$= \text{HOW MUCH OF } \underset{\sim}{x} \text{ POINTS IN DIRECTION OF } \frac{1}{2}\begin{pmatrix} 1 \\ \textcircled{+} i \\ -1 \\ \textcircled{-} i \end{pmatrix}$$

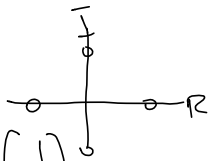
$$\left|\underset{\sim}{x}\right| = \sqrt{\underset{\sim}{x} \circ \underset{\sim}{x}} = \sum_{n=0}^{N-1} \left(\underset{\sim}{x}^{*}\right)_n \left(\underset{\sim}{x}\right)_n$$

$$\underset{\sim}{a} \circ \underset{\sim}{x} = \sum_{n=0}^{N} \left(\underset{\sim}{a}^{*}\right)_n \left(\underset{\sim}{x}\right)_n = b$$

$N=2$ $\quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$N=4$

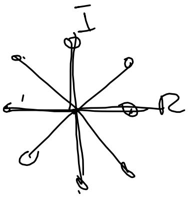
$$\frac{1}{\sqrt{4}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad \frac{1}{\sqrt{4}}\begin{pmatrix}1\\+i\\-1\\-i\end{pmatrix} \quad \frac{1}{\sqrt{4}}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \quad \frac{1}{\sqrt{4}}\begin{pmatrix}1\\-i\\-1\\+i\end{pmatrix}$$



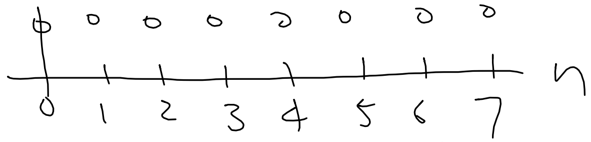
$N=8$

$$\hat{\underset{\sim}{x}}_0 = \frac{1}{\sqrt{8}}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}; \quad \hat{\underset{\sim}{x}}_1 = \frac{1}{\sqrt{8}}\begin{pmatrix}1\\e^{i\frac{\pi}{4}}\\e^{i\frac{\pi}{2}}\\e^{i\frac{3\pi}{4}}\\e^{i\pi}=-1\\e^{i\frac{5\pi}{4}}\\e^{i\frac{3\pi}{2}}\\e^{i\frac{7\pi}{4}}\end{pmatrix}; \quad \hat{\underset{\sim}{x}}_2 = \frac{1}{\sqrt{8}}\begin{pmatrix}1\\i\\-1\\-i\\1\\i\\-1\\-i\end{pmatrix}; \quad \underset{\sim}{x}_3 = \frac{1}{\sqrt{8}}\begin{pmatrix}1\\e^{i\frac{3\pi}{4}}\\-i\\e^{i\frac{\pi}{4}}\\-1\\e^{-i\frac{\pi}{4}}\\+i\\e^{-i\frac{3\pi}{4}}\end{pmatrix}$$

$\xi = 0 \frac{\text{cycles}}{\text{pixel}}$ $\quad \xi = \frac{1}{8}\frac{\text{cycle}}{\text{pixel}}$ $\quad \xi = \frac{1}{4}\frac{\text{cycle}}{\text{pixel}}.$



$$\underset{\sim}{D} = \begin{array}{c} \xi,\ k \\ \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & \\ 1 & e^{i\frac{\pi}{4}} & \\ 1 & e^{i\frac{\pi}{2}} & \\ 1 & e^{i\frac{3\pi}{4}} & \frac{1}{\sqrt{8}} e^{+i\cdot 2\pi n \frac{k}{8}} \\ \vdots & \vdots & \\ 1 & 1 & \end{array}\right] \end{array} \begin{array}{l} n=0 \\ n=1 \\ n=2 \\ \vdots \\ n=7 \end{array}$$



$$\underset{\sim}{D} = \begin{array}{c} k=0 \ \cdots \ 7 \\ \left[\frac{1}{\sqrt{8}} e^{+i\cdot 2\pi \cdot \boxed{\frac{n\cdot k}{8}}} \right] \end{array} \begin{array}{l} n=0 \\ \vdots \\ 7 \end{array}$$

$$\underset{\sim}{D}^{-1} = \left[\frac{1}{\sqrt{8}} \left(e^{+i 2\pi \frac{nk}{8}} \right)^{\cancel{8}} \ \ddots \right]$$

N-D VECTOR

$$\underset{\sim}{D} = \left[\frac{1}{\sqrt{N}} e^{+i 2\pi \frac{nk}{N}} \right]$$

$$\underset{\sim}{x} \longrightarrow \underset{\sim}{x}' = \underset{\sim}{D}^{-1} \underset{\sim}{x}$$

DISCRETE FOURIER TRANSFORM

(DFT)

$$\underset{\sim}{D}^{-1} = \frac{1}{\sqrt{N}} e^{-i 2\pi \frac{nk}{N}}$$

$$\underset{\sim}{D}^{-1} \underset{\sim}{x} = \sum_{n=0}^{N-1}$$

$$\underset{\sim}{D}^{-1} = \frac{1}{\sqrt{N}} \begin{array}{cccc|c} n=0 & n=1 & \cdots & n=N-1 & \\ & & & & k=0 \\ & e^{-i2\pi\frac{nk}{N}} & & & k=1 \\ & & & & \vdots \\ & & & & k=N-1 \end{array} \quad \begin{array}{c|c} x_0 & n=0 \\ x_1 & n=1 \\ \vdots & \\ x_{N-1} & n=N-1 \end{array}$$

$$\boxed{\sum_{n=0}^{N-1} \underset{f(n)}{x_n} \left(\underbrace{\frac{1}{\sqrt{N}} e^{-i2\pi\frac{nk}{N}}}\right) = \underset{\underset{F(k)}{\uparrow}}{x'_k}} = \begin{pmatrix} x'_0 \\ x'_1 \\ \vdots \\ \vdots \\ x'_{N-1} \end{pmatrix} = \underset{\sim}{x}'$$

$\underset{\sim}{x}$ = PIXEL BASIS; REPRESENTATION

$\underset{\sim}{x}'$ = FREQUENCY REPRESENTATION

ANALYSIS

$$f(n) \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}} F(k)$$

SYNTHESIS