9/4

CIRCULANT

$$\underset{\sim}{A} = \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \delta & \alpha & \beta & \gamma \\ \gamma & \delta & \alpha & \beta \\ \beta & \gamma & \delta & \alpha \end{bmatrix}$$

e.g.,

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \\ 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

?

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 3 \\ 5 \\ -5 \\ 3 \end{pmatrix}}_{\underset{\sim}{x}} = \begin{pmatrix} +2 \\ -10 \\ +8 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ +1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} +2 \\ -10 \\ +8 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \underset{\sim}{0}$$

NULL VECTOR

"ZERO VECTOR"

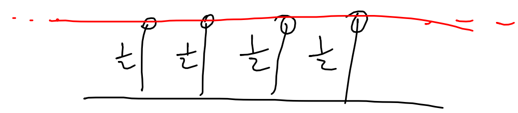
$$\begin{pmatrix} & \underset{\sim}{A} & \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} & \underset{\sim}{A} & \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = 0 \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

EIGENVECTOR

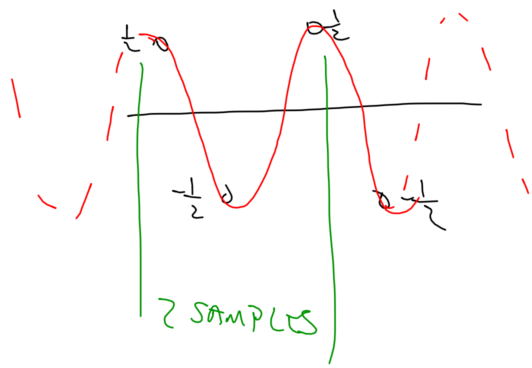
EIGENVALUE

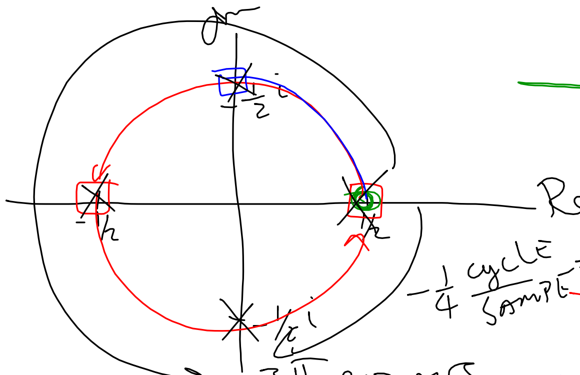


$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = (-2) \begin{pmatrix} +1/2 \\ -1/2 \\ +1/2 \\ -1/2 \end{pmatrix}$$

EIGENVECTOR

EIGENVALUE





$-\frac{1}{4}\frac{\text{cycle}}{\text{SAMPLE}} = -\frac{\pi}{2}\frac{\text{RAD}}{\text{SAM}}$

$\frac{3}{4}$ cycle $= \frac{3\pi}{2}$ RADIANS

$\xi = 0 \ \frac{\text{cycles}}{\text{SAMPLE}};\ 1 \ \frac{\text{cycle}}{\text{SAMPLE}}$

"XI"

$\xi = \frac{1}{2} \ \frac{\text{cycle}}{\text{SAMPLE}}$

$\xi = \frac{1}{4} \ \frac{\text{cycle}}{\text{SAMPLE}}$

$$\hat{\underset{\sim}{x}}_{\frac{1}{4}} = \begin{bmatrix} 1/2 \\ +i/2 \\ -1/2 \\ -i/2 \end{bmatrix}$$

$$\sqrt{\begin{pmatrix} \frac{1}{2} \\ +i/2 \\ -1/2 \\ -i/2 \end{pmatrix} \circ \begin{pmatrix} \frac{1}{2} \\ +i/2 \\ -1/2 \\ -i/2 \end{pmatrix}} = \underset{\sim}{x}_{\frac{1}{4}} = \sqrt{\left(\frac{1}{2}\right)^{*}\left(\frac{1}{2}\right) + \left(+\frac{i}{2}\right)^{*}\left(\frac{i}{2}\right) + \left(-\frac{1}{2}\right)^{*}\left(-\frac{1}{2}\right) + \left(-\frac{i}{2}\right)^{*}\left(-\frac{i}{2}\right)}$$

$$= \sqrt{\frac{1}{4} + \frac{(-i)(+i)}{4} + \frac{(-1)(-1)}{4} + \frac{(+i)(-i)}{4}}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = 1$$

$$\hat{\underset{\sim}{x}}_{\frac{3}{4}} = \frac{1}{2}\begin{pmatrix} +1 \\ -i \\ -1 \\ +i \end{pmatrix}$$

$\frac{3}{4}$ cycles/SAMPLE

September 04, 2014

$$\underset{\sim}{A}\ \hat{\underset{\sim}{x}}_{\frac{1}{2}} = \lambda_{\frac{1}{2}}\ \hat{\underset{\sim}{x}}_{\frac{1}{2}}$$

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \delta & \alpha & \beta & \gamma \\ \gamma & \delta & \alpha & \beta \\ \beta & \gamma & \delta & \alpha \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 \\ +i \\ -1 \\ -i \end{pmatrix} = \lambda_{\frac{1}{2}} \cdot \frac{1}{2} \begin{pmatrix} 1 \\ +i \\ -1 \\ -i \end{pmatrix}$$

$$\cancel{\tfrac{1}{2}}\left(\alpha + i\beta - \gamma - i\delta\right) = \lambda_{1/2} \cdot \cancel{\tfrac{1}{2}} \cdot 1$$

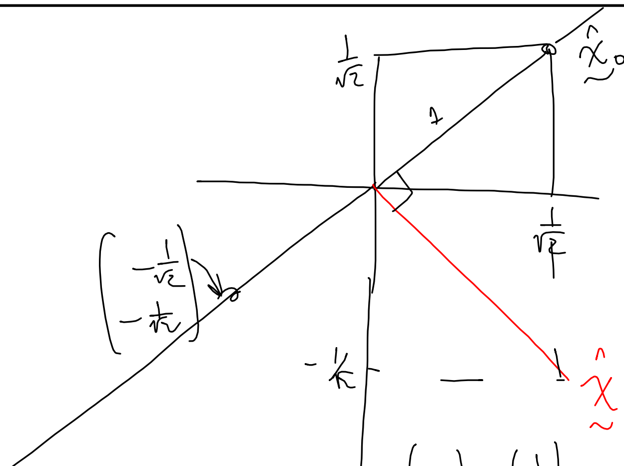
EIGENVALUE OF $\underset{\sim}{A}$ ASSOCIATED WITH $\hat{\underset{\sim}{X}}_1 \rightarrow \frac{1}{4}$ CYCLE PER SAMPLE

$$\lambda_1 = \alpha + i\beta - \gamma - i\delta$$

$$(-i)\ \cancel{\tfrac{1}{2}}\left(\delta + i\alpha - \beta - i\gamma\right) = \lambda_1 \cdot \cancel{\tfrac{1}{2}} \cdot i\ (-i)$$

$$-i\delta + \alpha + i\beta - \gamma = \lambda_1 \cdot 1$$

$$\underset{\sim}{\hat{x}}_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$



$$\underset{\sim}{\hat{x}}_0 = \begin{pmatrix} \frac{1}{\sqrt{N}} \\ \frac{1}{\sqrt{N}} \\ \vdots \\ \frac{1}{\sqrt{N}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\underset{\sim}{\hat{x}_0} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \circ \underset{\sim}{\hat{x}_1} = \begin{pmatrix} \frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{pmatrix} = \left(\frac{1}{2}\right)^* \cdot \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^* \left(+\frac{i}{2}\right) + \left(\frac{1}{2}\right)^* \left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)^* \left(-\frac{i}{2}\right)$$

$$= \frac{1}{4} + \frac{i}{4} - \frac{1}{4} - \frac{i}{4} = 0$$

$\underset{\sim}{\hat{x}_1} \circ \underset{\sim}{\hat{x}_3}$

$$\frac{1}{2}\begin{bmatrix} 1 \\ +i \\ -1 \\ -i \end{bmatrix} \circ \frac{1}{2}\begin{bmatrix} 1 \\ -i \\ -1 \\ +i \end{bmatrix} = \frac{1}{4}\left[1 \cdot 1 + (+i)^*(-i) + (-1)^*(-1) + (-i)^*(+i)\right]$$

$$= \frac{1}{4}\left[(1)^2 + (-i)^2 + (-1)^2 + (+i)^2\right]$$

$$= \frac{1}{4}\left(1 - 1 + 1 - 1\right) = 0$$

$$\underset{\sim}{A}\begin{bmatrix} \hat{\underset{\sim}{x}}_0 & & & \end{bmatrix} = \begin{bmatrix} \lambda_0 \hat{\underset{\sim}{x}}_0 & & & \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \delta & \alpha & \beta & \gamma \\ \gamma & \delta & \alpha & \beta \\ \beta & \gamma & \delta & \alpha \end{bmatrix}}_{\underset{\sim}{A}} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & +\frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & +\frac{i}{2} \end{bmatrix}}_{\underset{\sim}{D}} = \begin{bmatrix} \lambda_0 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} & \lambda_1 \begin{pmatrix} \frac{1}{2} \\ +\frac{i}{2} \\ -\frac{1}{2} \\ -\frac{i}{2} \end{pmatrix} & \lambda_2 \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} & \lambda_3 \begin{pmatrix} \frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ +\frac{i}{2} \end{pmatrix} \end{bmatrix}$$

$$\underset{\sim}{D} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & +\frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & +\frac{i}{2} \end{bmatrix} \underset{\underset{\sim}{\Lambda}}{\begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}}$$

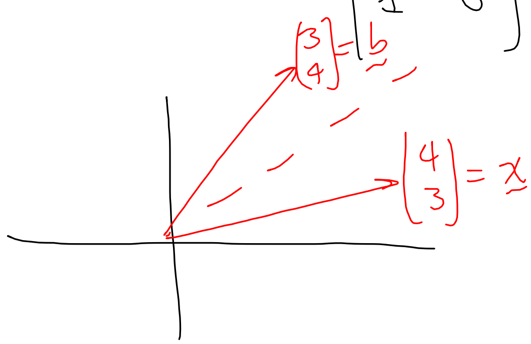
$$\underset{\sim}{A}\,\underset{\sim}{D} = \underset{\sim}{D}\,\underset{\sim}{\Lambda}$$

NORMALIZED
MATRIX OF EIGENVECTORS

$$\underset{\sim}{D} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & +i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & +i \end{pmatrix}$$

$$\underset{\sim}{D}\,\underset{\sim}{D}^{-1} = \underset{\sim}{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{A}^{-1} = \underset{\sim}{I} \qquad \underset{\sim}{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \underset{\sim}{A}\,\underset{\sim}{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$



$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

$$\underbrace{\underset{\sim}{A}^{-1}\underset{\sim}{A}}_{\underset{\sim}{I}}\,\underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{b}$$

$$\underset{\sim}{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \underset{\sim}{A}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \overset{\underset{\sim}{I}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

$$\underset{\sim}{A} = \overset{\underset{\sim}{A}}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}} \overset{\underset{\sim}{x}}{\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}} = \overset{\underset{\sim}{b}}{\begin{bmatrix} y \\ z \\ t \\ x \end{bmatrix}}$$

$$\overset{\underset{\sim}{A}^{-1}}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} y \\ z \\ t \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$$\underset{\sim}{A}^{-1} = \underset{\sim}{A}^{T}$$

$$A D = D \Lambda$$

$$\Lambda = \begin{bmatrix} \lambda_0 & & & & 0 \\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \lambda_3 & \\ 0 & & & & \ddots \end{bmatrix}$$

$$D^{-1} A D = \underbrace{\left(D^{-1} D \right)}_{I} \Lambda$$

WHAT IS D^{-1} ?

$$D = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & +i \end{bmatrix}$$

$$D^{-1} D = I$$

$$\underbrace{\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{bmatrix}}_{D^{-1}} \underbrace{\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & +i \end{bmatrix}}_{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D^{-1} = D^{\dagger}$$

$$\underset{\sim}{A}\,\underset{\sim}{D} = \underset{\sim}{D}\,\underset{\sim}{\Lambda}$$

$$\underset{\sim}{D}^{-1}\underset{\sim}{A}\,\underset{\sim}{D} = \underset{\sim}{\Lambda} = \begin{bmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_{N-1} \end{bmatrix}$$

KNOWN BECAUSE $\underset{\sim}{A}$ IS CIRCULANT

CIRCULANT

$$\underset{\sim}{D}^{-1} = \underset{\sim}{D}^{*}$$

$$\underset{\sim}{A} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \frac{x+y+z+t}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{\Lambda} = \begin{pmatrix} 1 & & \text{O} & \\ & 0 & & \\ & & 0 & \\ \text{O} & & & 0 \end{pmatrix}$$