9/2

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

ENSEMBLE OF REFERENCE VECTORS

$$\begin{bmatrix} \hat{\underset{\sim}{a}}_0^T \\ \hat{\underset{\sim}{a}}_1^T \\ \vdots \\ \hat{\underset{\sim}{a}}_{N-1}^T \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{N-1} \end{bmatrix}$$

4x4 CASE

$$\underset{\sim}{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\left\| \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \right\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 0^2 + 0^2} = \sqrt{\frac{1}{2}}$$

$$\underset{\sim}{A} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$\lambda = 1$

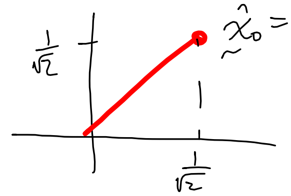
NULL VECTOR

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

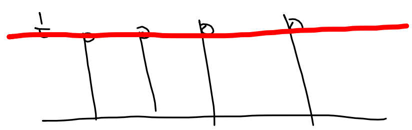
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = 0 \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

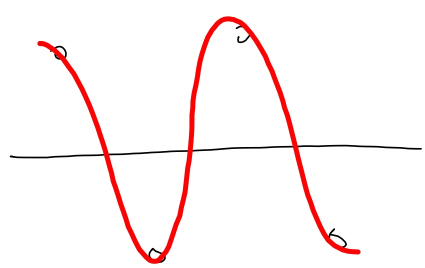
$$\hat{\underset{\sim}{x}}_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$



$$\hat{\underset{\sim}{x}}_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



$$\hat{\underset{\sim}{x}}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$



$$\underset{\sim}{x}_0 \circ \underset{\sim}{x}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \circ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = 0$$

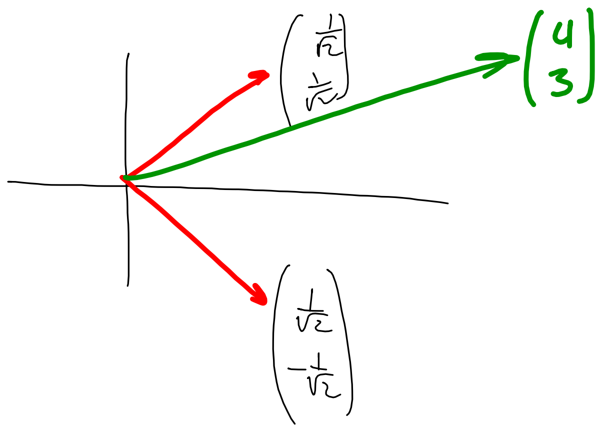
CONSTANT ξ ("XI") $= 0$

$\xi = \frac{1}{2} \frac{\text{CYCLE}}{\text{SAMPLE}}$

$$\hat{\underset{\sim}{x}}_0 \circ \hat{\underset{\sim}{x}}_0 = 1$$

$$\hat{\underset{\sim}{x}}_2 \circ \hat{\underset{\sim}{x}}_2 = 1$$

$$\hat{\underset{\sim}{x}}_0 \circ \hat{\underset{\sim}{x}}_2 = 0$$



$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
$\begin{pmatrix} 4 \\ 3 \end{pmatrix}$
$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

COMPLEX NUMBERS

$$i \equiv \sqrt{-1}$$

$$z = x + iy$$

x = REAL PART = $\mathrm{Re}\{z\}$

y = IMAGINARY PART = $\mathrm{Im}\{z\}$

$$z^* \equiv x - iy$$

$$z + z^* = (x + x) + i(y - y) = 2x = 2\,\mathrm{Re}\{z\}$$

$$\mathrm{Re}\{z\} = \frac{z + z^*}{2}$$

$$z - z^* = 2\,\mathrm{Im}\{z\} \Rightarrow \mathrm{Im}\{z\} = \frac{z - z^*}{2}$$

$$\left.\begin{aligned} z_1 &= x_1 + i y_1 \\ z_2 &= x_2 + i y_2 \end{aligned}\right\} \quad \begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2) \\ z_1 - z_2 &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

$$\begin{aligned} z_1 \circ z_2 &= (x_1 + i y_1)(x_2 + i y_2) \\ &= x_1 x_2 + i y_1 \cdot i y_2 + x_1 \cdot i y_2 + x_2 \cdot i y_1 \\ &= \underbrace{(x_1 x_2 - y_1 y_2)}_{\text{Re}} + i \underbrace{(x_1 y_2 + x_2 y_1)}_{\text{Im}} \end{aligned}$$

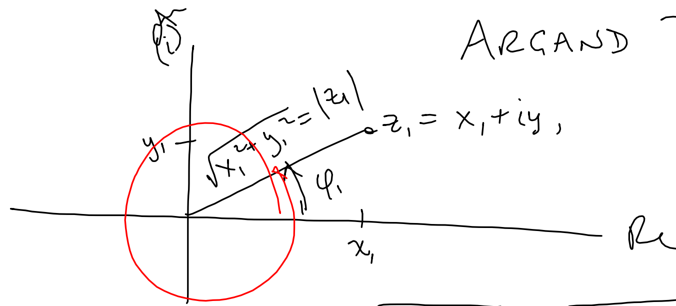
$$y_2 \Rightarrow \frac{1}{y_2} = \frac{1}{x_2 + i y_2} \qquad \frac{1}{z_2} = \frac{1}{z_2} \cdot 1 = \frac{1}{z_2} \cdot \frac{z_2^*}{z_2^*}$$

$$(x_2 + i y_2)(x_2 - i y_2) = x_2^2 + y_2^2 \geq 0 \qquad = \frac{z_2^*}{z_2 \cdot z_2^*} = \frac{z_2^*}{|z_2|^2}$$

$$\frac{1}{z_2} = \frac{1}{x_2 + i y_2} = \frac{x_2 - i y_2}{x_2^2 + y_2^2} = \underbrace{\left(\frac{x_2}{x_2^2 + y_2^2}\right)}_{\text{Re}\left\{\frac{1}{z_2}\right\}} + i \underbrace{\left(\frac{-y_2}{x_2^2 + y_2^2}\right)}_{\text{Im}\left\{\frac{1}{z_2}\right\}}$$

$$\frac{z_1}{z_2} = \frac{z_1 \cdot z_2^*}{z_2 \cdot z_2^*} = \frac{(x_1 + i y_1)(x_2 - i y_2)}{x_2^2 + y_2^2} = \underbrace{\frac{(x_1 x_2 + y_1 y_2)}{x_2^2 + y_2^2}}_{\text{Re}\left\{\frac{z_1}{z_2}\right\}} + i \underbrace{\left(\frac{(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}\right)}_{\text{Im}\left\{\frac{z_1}{z_2}\right\}}$$

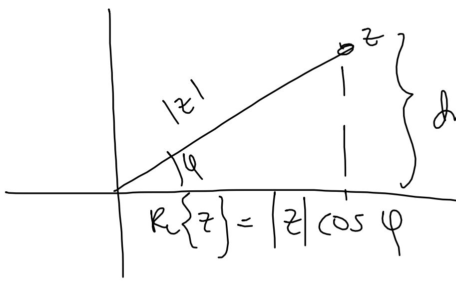
ARGAND DIAGRAM



$$|z_1| = \sqrt{z_1 z_1^*} = \sqrt{z_1^* z_1}$$

$$\varphi_1 = \tan^{-1}\left[\frac{\mathrm{Im}\{z_1\}}{\mathrm{Re}\{z_1\}}\right] + 2\pi l$$

$$(l = 0, \pm 1, \pm 2, \dots)$$



$$\mathrm{Im}\{z\} = |z| \sin\varphi$$

$$\mathrm{Re}\{z\} = |z| \cos\varphi$$

$$z = \mathrm{Re}\{z\} + i\,\mathrm{Im}\{z\}$$

$$= |z|\cos\varphi + i|z|\sin\varphi$$

$$z = |z| \underbrace{(\cos\varphi + i\sin\varphi)}$$

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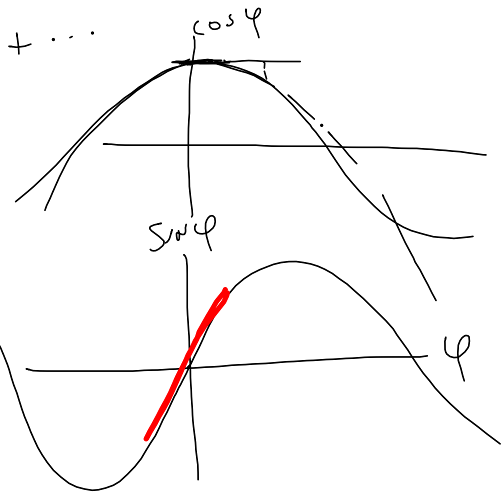
$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \cdots$$

$$= \frac{\varphi^0}{0!} - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots$$

$$\sin\varphi = \frac{\varphi^1}{1!} - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots$$

$$-1 = i^2 = i^6$$



$$\cos\varphi = \frac{(i\varphi)^0}{0!} + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^4}{4!} + \cdots$$

$$+ \quad i\sin\varphi = \frac{(i\varphi)^1}{1!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^5}{5!} + \cdots$$

$$\cos\varphi + i\sin\varphi = \frac{(i\varphi)^0}{0!} + \frac{(i\varphi)^1}{1!} + \cdots = \sum_{n=0}^{\infty} \frac{(i\varphi)^n}{n!} = e^{i\varphi}$$

$$\boxed{e^{i\phi} = \cos\varphi + i\sin\varphi}$$

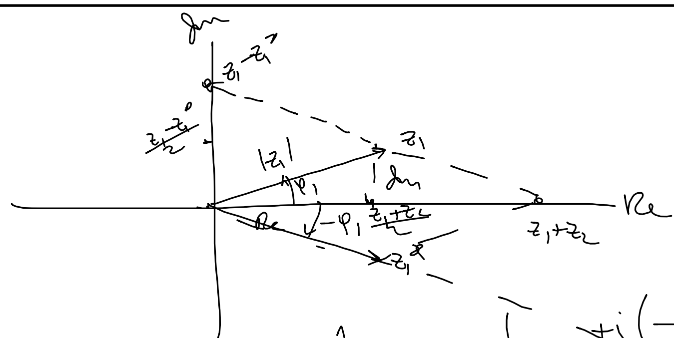
ϕ, φ

EULER RELATION "OY LER"

$$z = x + iy = |z| \cos\varphi + i\,|z| \sin\varphi = |z|\, e^{i\varphi}$$

$$\begin{aligned} z_1 \cdot z_2 &= |z_1| e^{i\varphi_1} \cdot |z_2| e^{i\varphi_2} \\ &= \underbrace{\left(|z_1||z_2|\right)}_{\text{MULTIPLY}} e^{i(\underbrace{\varphi_1+\varphi_2}_{\text{ADD AZIMUTH ANGLES}})} \end{aligned}$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{i(\varphi_1-\varphi_2)}$$



$$\frac{1}{z_1} = \frac{1}{|z_1| e^{i\varphi_1}} = \frac{1}{|z_1|} e^{+i(-\varphi_2)}$$

September 02, 2014

COMPLEX FUNCTIONS

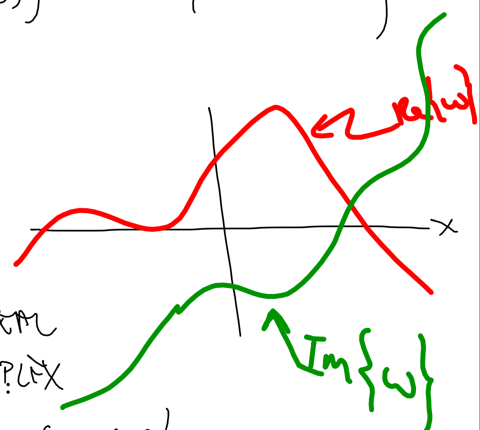
$$w[z] = \mathrm{Re}\{w[x+iy]\} + i\,\mathrm{Im}\{w[x+iy]\}$$

COMPLEX
VALUED

$$w[x+i\cdot 0] = w[x]$$

REAL

COMPLEX



$$f[x] = \mathrm{Re}\{f[x]\} + i\,\mathrm{Im}\{f[x]\}$$

REAL

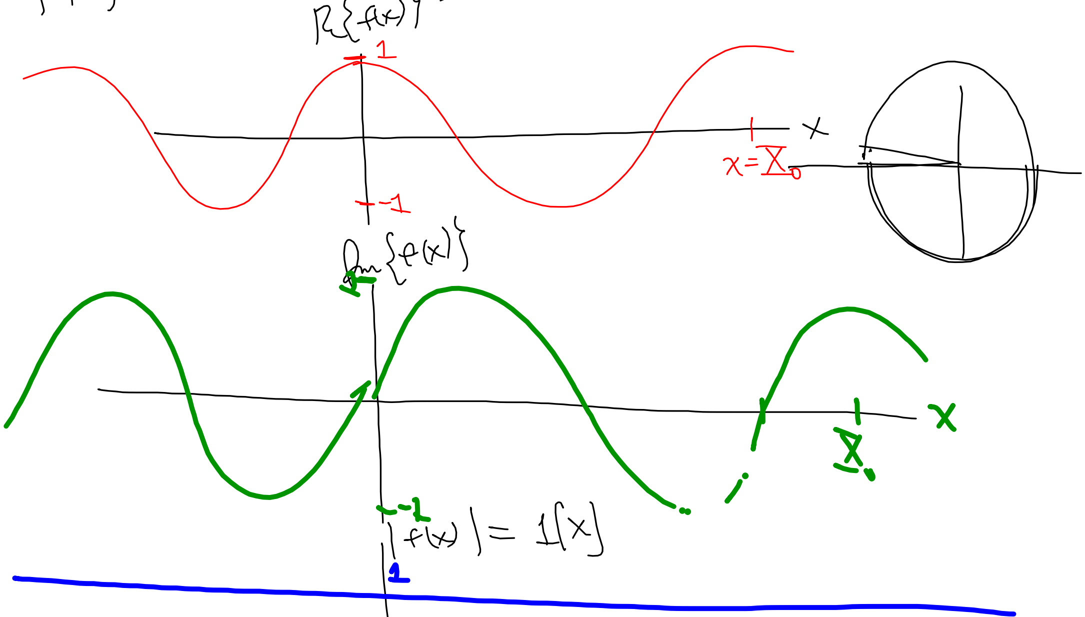
COMPLEX

$$= |f(x)|\, e^{i\Phi\{f(x)\}}$$

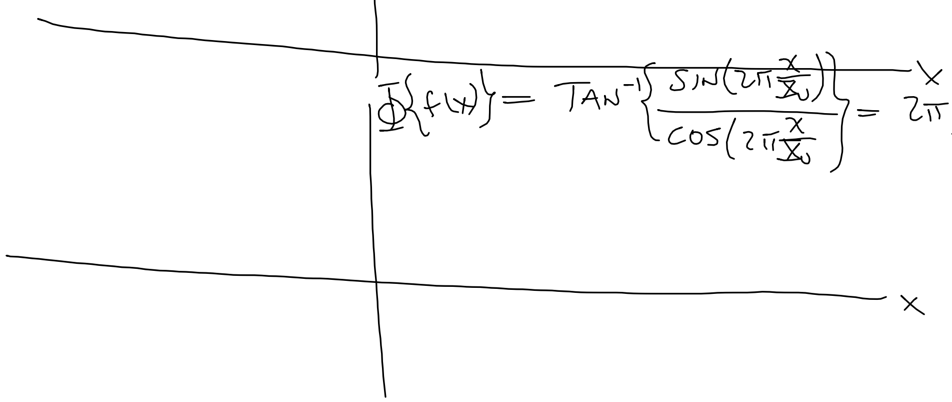
$$|f(x)| = \sqrt{\left(\mathrm{Re}\{f(x)\}\right)^2 + \left(\mathrm{Im}\{f(x)\}\right)^2}$$

$$\Phi\{f(x)\} = \tan^{-1}\left\{\frac{\mathrm{Im}\{f(x)\}}{\mathrm{Re}\{f(x)\}}\right\}$$

$$f(x) = \cos\left(2\pi \frac{x}{X_0}\right) + i \sin\left(2\pi \frac{x}{X_0}\right)$$



$$\Phi\{f(x)\} = \tan^{-1}\left\{\frac{\sin\left(2\pi \frac{x}{X_0}\right)}{\cos\left(2\pi \frac{x}{X_0}\right)}\right\} = 2\pi \frac{x}{X_0} \propto x$$



$$\xi\left[f(x)\right] \longrightarrow \text{EVALUATE } \Phi\left\{f(x)\right\} = \tan^{-1}\left(\frac{\mathrm{Im}\{f(x)\}}{\mathrm{Re}\{f(x)\}}\right)$$

RATE OF CHANGE OF PHASE

$$\frac{d\Phi}{dx} \quad \left(\frac{\text{RADIANS}}{\text{LENGTH}}\right) \Longrightarrow \text{ANGULAR SPATIAL FREQUENCY}$$

$$\frac{1}{2\pi \frac{\text{RAD}}{\text{CYCLE}}} \cdot \frac{d\phi}{dx} \quad \left(\frac{\text{cycles}}{\text{LENGTH}}\right) \quad \text{e.g.,} \quad \frac{\text{cycles}}{\text{mm}}$$

$$\boxed{\xi(x) = \frac{1}{2\pi}\frac{d\phi}{dx}}$$

$$f(x) = \cos\left(2\pi \frac{x}{X_0}\right) + i\sin\left(2\pi \frac{x}{X_0}\right)$$

$$\Phi(x) = 2\pi \frac{x}{X_0}$$

$$\frac{d\phi}{dx} = \frac{2\pi}{X_0}$$

$$\frac{1}{2\pi}\frac{d\phi}{dx} = \frac{1}{X_0}\frac{\text{cycles}}{\text{LENGTH}} = \xi(x)$$

$$f(x) = 1(x)\, e^{+i\pi \frac{x^2}{\alpha_0^2}} = \cos\left[\pi \frac{x^2}{\alpha_0^2}\right] + i \sin\left(\frac{\pi x^2}{\alpha_0^2}\right)$$

$$|f(x)| = 1(x)$$

$$\Phi\{f(x)\} = \pi \frac{x^2}{\alpha_0^2} = \pi\left(\frac{x}{\alpha_0}\right)^2 \text{ RADIAN}$$

$$\propto x^2$$

α_0 = LENGTH

QUADRATIC PHASE

VECTORS

COMPLEX NUMBERS

DEMOIVRE'S THEOROM

$$z^n = \left(|z| e^{i\phi}\right)^n = |z|^n \left(e^{i\phi}\right)^n$$

$$= |z|^n e^{i(n\phi)}$$

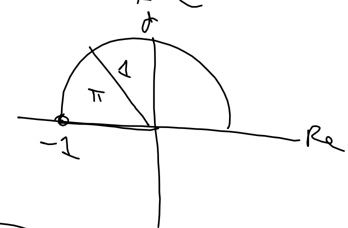
$$z^{\frac{1}{n}} = |z|^{\frac{1}{n}} e^{i\left(\frac{\phi}{n}\right)}$$

$$= |z|^{\frac{1}{n}} e^{i\left(\frac{\phi + 2\pi l}{n}\right)} = |z|^{\frac{1}{n}} e^{i\frac{2\pi l}{n}} e^{i\left(\frac{\phi}{n}\right)}$$

$$1^{\frac{1}{2}} = \left(1\, e^{i\cdot 0}\right)^{\frac{1}{2}} \Rightarrow z_0 = 1\, e^{i\cdot\frac{0}{2}} = 1$$

$$z_1 = 1\, e^{i\frac{(0+2\pi)}{2}}$$

$$= 1\, e^{i\pi} = -1 + i\cdot 0$$

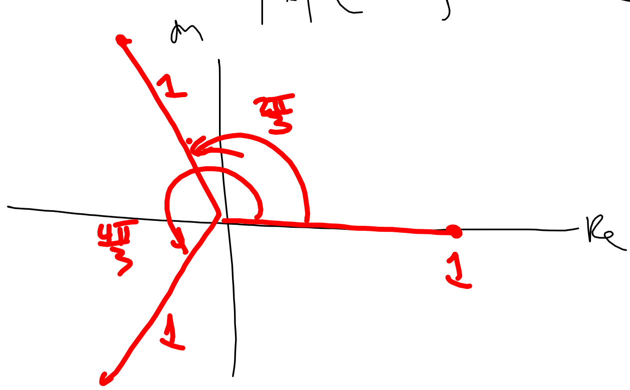


$$1^{1/3}$$

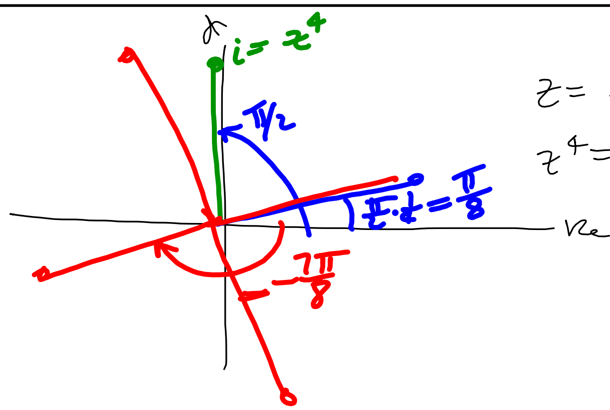
$$1 = |1|\, e^{i\cdot 0} \; ; \; 1^{1/3} = 1\, e^{i\cdot 0/3} = 1$$

$$|1|\, e^{i\cdot 2\pi} \; ; \; 1^{1/3} = 1\, e^{i\frac{2\pi}{3}}$$

$$|1|\, e^{i\cdot 4\pi} \; ; \; 1^{1/3} = 1\, e^{i\frac{4\pi}{3}}$$



$i^{\frac{1}{4}}$



$z = 1\, e^{i\frac{\pi}{8}}$

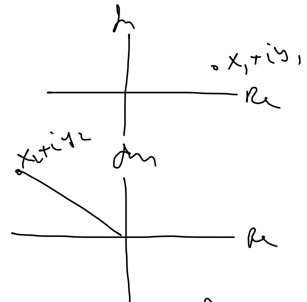
$z^4 = 1^4 \left(e^{i\frac{\pi}{8}}\right)^4 = e^{i\frac{\pi}{2}}$

$= i$

VECTORS COMPLEX NUMBERS

VECTORS WITH COMPLEX-VALUED COEFFICIENTS (COMPONENTS)

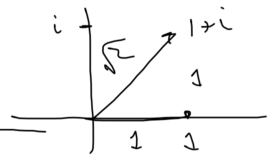
$$\underset{\sim}{x} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix}$$



$|\underset{\sim}{x}|$ IS REAL VALUED, ≥ 0

$$\begin{pmatrix} 0+0i \\ 0+0i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underset{\sim}{0}$$

$$\boxed{\underset{\sim}{x} \circ \underset{\sim}{x}} = |\underset{\sim}{x}|^2$$



$$\underset{\sim}{x} \equiv \begin{pmatrix} 1+i \\ 0+0i \end{pmatrix} ; \ \underset{\sim}{x} = \sqrt{2}$$

$$\underset{\sim}{x} \circ \underset{\sim}{x} = (1+i)(1+i) + 0 \cdot 0$$

$$\cancel{1 + i^2 + i + i}$$

$$\cancel{1 + (-1) + 2i = 2i \neq \text{Real}}$$

$$\sqrt{(1+i)(1+i)^* + 0 \cdot 0^*} = \sqrt{(1+i)(1-i) + 0}$$

$$= \sqrt{2}$$

$$\underset{\sim}{x} \circ \underset{\sim}{x}$$

SCALAR PRODUCT OF 2 ARBITRARY VECTORS WITH COMPLEX COMPONENTS

$$\underset{\sim}{a} \circ \underset{\sim}{x} = \begin{cases} \sum_{n=1}^{N} \underset{\sim}{a}_n^* \underset{\sim}{x}_n & \longleftarrow \text{PHYSICS} \\ \sum_{n} (\underset{\sim}{a})_n (\underset{\sim}{x}^*)_n & \longleftarrow \text{MATH} \end{cases}$$

REF

$$\underset{\sim}{a} \circ \underset{\sim}{x} \equiv \sum_{n=1}^{N} \left(\underset{\sim}{a}^*\right)_n \left(\underset{\sim}{x}\right)_n = b$$

SCALAR

$$\begin{pmatrix} \overline{\quad a_n \quad} \\ \underset{\sim}{A} \end{pmatrix} \begin{pmatrix} \underset{\sim}{x} \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{N-1} \end{pmatrix}$$

CONSTANT $\rightarrow \xi < 0$

OSCILLAT $\rightarrow \xi = \frac{1}{2}$