Chapter 6

Performance Criteria for Photodetectors

6.1 Introduction

In this chapter we will consider models of detector performance. The performance of detectors may be characterized in terms of their responsiveness in reacting to incident photons. The responsive quantum efficiency may be defined as the number of output events per incident photon. This definition is natural and obvious, but unsatisfactory because it does not account for randomness in the production of output events. A detector that requires many photons to produce an output event may be better than a multiplier tube that produces many photoelectrons per input photon, even though one detector is less "responsive" than the other. The comparison has to include the added randomness of the detection process.

The fundamental principles for detector comparison were formulated by Albert Rose and published a classic1946 paper[1] that laid the foundation for a measure that includes detector randomness. The measure is called Detective Quantum Efficiency (DQE) after R. Clark Jones of Polaroid[2]. The DQE is expressed in terms of the ability of a detector to see a signal against a background of ambient radiation, both impinging on the detector. The

[1] Albert Rose, "A Unified Approach to the Performance of Photographic Film, Television Pickup Tubes, and the Human Eye," *J. SMPE, 47,* 273-294 (1946).

[2] R. Clark Jones, "On the Quantum Efficiency of Photographic Negatives," *Photographic Science and Engineering*, Vol 2, Number 2, August, 1958. (57-65)

quantum fluctuations of the ambient radiation induces randomness in the detector response that is in addition to the randomness that is added by the detector itself. The DQE is essentially a comparison of the two sources of randomness.

These notes will follow the development in Zweig[3] which was written in the form of a tutorial and which illustrates the concept of DQE from three perspectives, the photographic scientist, the engineer and the perceptual psychologist.

Let us first present some examples to introduce the topic. We imagine that a detector is placed in a photon stream with the intention of measuring the photon intensity. The intensity is the parameter q which represents the average value of the photon stream when modeled as a Poisson process. We would like to estimate q or its equivalent, but are prevented from doing so accurately by the random nature of the photon arrivals. The photon stream itself presents both the "signal" q and the interfering noise in terms of the randomness of the process.

The average signal strength is $S = q$ and the root-mean-square (rms) value of the noise is $\sigma = \sqrt{q}$. The signal-to-noise ratio (SNR) is then

$$SNR_{\max} = \frac{q}{\sqrt{q}} = \sqrt{q} \tag{6.1}$$

This is the maximum SNR that can ever be achieved for this measurement. By measuring more photons we get a better estimate of the mean value in accordance with what we have learned from the law of large numbers.

The measurement that we actually make with a real detector will inevitably be corrupted by additional noise. The measured SNR of the detector, SNR_m will never exceed $SNR_{\max}$. We define the DQE as the ratio

$$DQE = \left(\frac{SNR_m}{SNR_{\max}}\right)^2 \tag{6.2}$$

This is a comparison between the theoretical limit and the actual achievement, and will have a value between zero and one. The DQE rating can be used for any kind of detector and therefore is very useful in system assessments. It need not be used just to measure the strength of a photon stream. One can use DQE very widely by properly identifying the appropriate signal

[3] H. J. Zweig, "Performance Criteria for Photodetectors," *Photographic Science and Engineering*, Vol 8, Number 6, Nov-Dec 1964. (305-311).

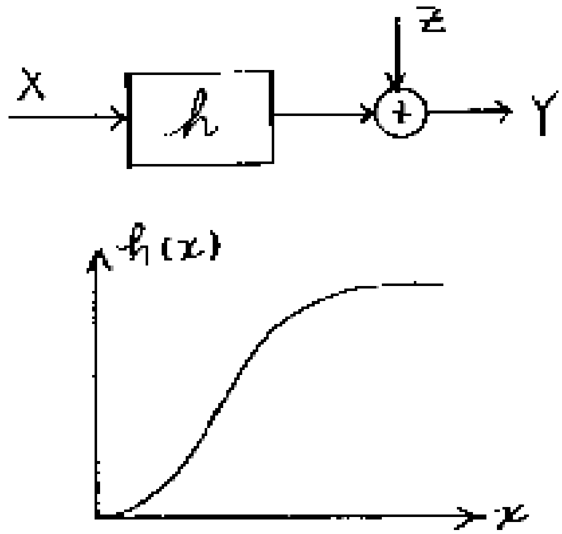

Figure 6.1: A photon detector in which X is the number of arriving photons and Y is the detector response.

and noise components for a given application. Let us develop the model for this measure in more detail.

6.2 Photon Detector Model

Consider a device that can be exposed to a photon beam for a known interval of time. If the size of the aperture is A and the exposure time is τ then the expected number of photons is $q = \lambda A \tau$ where λ is the intensity of the beam in photons per unit area per unit time. Let X be the number of photons that are sampled on a given exposure, and let $Y = h(X) + Z$ be the output of the photon detector when the input is X. The additive term Z is noise that is internal to the detector. We will treat Z as an independent random variable. The value of λ represents the signal and we would like to determine it from the observation Y.

We will now consider the simplest possible examples–namely, a linear detector with additive internal noise. First we will look at the noiseless case and then add detector noise.

Example 6.2.1 ***Noiseless Linear Detector***. *Suppose that the detector has a linear response function $h(x) = gx$ and that $Z = 0.$. Then $Y = gX$. We know that the expected value of the detector output is $E[Y] = gE[X] = g\lambda A\tau$. This means that $\lambda = E[Y]/gA\tau$, so this gives us an indication of how to measure the "signal." Now, we cannot observe $E[Y]$, but we can use the value of Y that is actually observed to compute $\overline{\lambda} = Y/gA\tau$. as the estimate of λ. The randomness in Y is caused by the random nature of the photon stream itself, and not any detector noise. The number of photons that arrive in any experiment is a Poisson random variable with mean value $E[X] = \lambda A\tau$. For a Poisson distribution the variance is also $\sigma_X^2 = \lambda A\tau$. If we could observe X then the SNR would be*

$$SNR_X = \left(\frac{E[X]}{\sigma_X}\right) = \left(\frac{\lambda A\tau}{\sqrt{\lambda A\tau}}\right) = \sqrt{\lambda A\tau} \tag{6.3}$$

Note that SNR_X improves in proportion to $\sqrt{A\tau}$ so that a longer exposure or a larger aperture are better. We presume, however, that we do not have direct access to X. We have to base our calculations on $\overline{\lambda} = Y/gA\tau$. But $E[\overline{\lambda}] = E[Y]/gA\tau. = \lambda$ and $\sigma_{\overline{\lambda}} = \sigma_Y/gA\tau = \sigma_X/A\tau = \sqrt{\lambda/A\tau}$ so that

$$SNR_{\overline{\lambda}} = \left(\frac{E[\overline{\lambda}]}{\sigma_{\overline{\lambda}}}\right) = \left(\frac{\lambda}{\sqrt{\lambda/A\tau}}\right) = \sqrt{\lambda A\tau} \tag{6.4}$$

The quality of the estimate of the intensity done by use of Y to calculate $\overline{\lambda}$ is, in this case, just as good as could be obtained by direct observation of the photon count. This is to be expected because this is a noiseless linear detector, which is the ideal system. It is also possible to calculate the SNR_Y which is the SNR based upon the mean and standard deviation of Y.

$$SNR_Y = \left(\frac{E[Y]}{\sigma_Y}\right) = \left(\frac{g\lambda A\tau}{g\sqrt{\lambda A\tau}}\right) = \sqrt{\lambda A\tau} \tag{6.5}$$

We find that all three measures are equivalent in this linear noiseless case. The DQE is, from (6.2),

$$DQE = \left(\frac{SNR_{\overline{\lambda}}}{SNR_X}\right)^2 = 1 \tag{6.6}$$

so that this is a perfect detector. Note that this result is true for any $g > 0$, so that when the detector is noiseless the DQE does not depend on detector gain. This is not the case for noisy detectors.

Example 6.2.2 ***Linear Detector with Noise*** *Suppose that the detector output is $Y = gX + Z$ where Z is a random variable that is independent of X with $E[Z] = 0$ and $E[Z^2] = \sigma_Z^2$. Then we still have $E[Y] = gE[X] = g\lambda A\tau$. An unbiased estimate of λ is provided by $\overline{\lambda} = Y/gA\tau$. The noise contributes to the variance of the detector output so that $\sigma_Y^2 = g^2\sigma_X^2 + \sigma_Z^2$. Using $\sigma_{\overline{\lambda}} = \sigma_Y/gA\tau$ we then find that the quality ratio is*

$$SNR_{\overline{\lambda}} = \left(\frac{E[\overline{\lambda}]}{\sigma_{\overline{\lambda}}}\right) = \frac{g\lambda A\tau}{\sqrt{g^2\lambda A\tau + \sigma_Z^2}} \tag{6.7}$$

In the event that $\sigma_Z^2 \to 0$ then this case agrees with the noiseless case. The effect of the detector noise is to reduce the SNR. Any noise added by the detector will reduce the ability of the system to measure changes in the photon flux. If we calculate SNR_Y we find

$$SNR_Y = \left(\frac{E[Y]}{\sigma_Y}\right) = \frac{g\lambda A\tau}{\sqrt{g^2\lambda A\tau + \sigma_Z^2}} \tag{6.8}$$

As one would expect, the detector noise affects these two output-related quantities in the same manner. $\overline{\lambda}$ is simply Y reflected back to the input side by inverting the system gain. If one were able to observe the detector input then there would be no noise effect and SNR_X would be the same as in the noiseless case. The DQE is

$$DQE = \left(\frac{SNR_Y}{SNR_X}\right)^2 = \frac{g^2\lambda A\tau}{g^2\lambda A\tau + \sigma_Z^2} \tag{6.9}$$

Note that the DQE now approaches unity when the detector gain is large enough that the first term in the denominator is dominant. Later we will see that high gain in the first stage of a multiple stage system is an especially good thing.

6.3 Basic DQE Analysis

The DQE is a comparison of the variance in the estimate of the signal using an actual measurement system and an ideal measurement system. Let $\overline{\lambda}$ represent the estimate we can make and $\hat{\lambda}$ represent the ideal. Then

$$DQE = \frac{\mathrm{var}(\hat{\lambda})}{\mathrm{var}(\overline{\lambda})} \tag{6.10}$$

Note that the ideal measurement does not yield certainty. It is the estimate that would be made if one could observe X directly, namely,

$$\hat{\lambda} = \frac{X}{A\tau} \tag{6.11}$$

The variance in the ideal estimate is

$$\text{var}(\hat{\lambda}) = \frac{\sigma_x^2}{(A\tau)^2} = \frac{\lambda}{A\tau} \tag{6.12}$$

In the above examples $\overline{\lambda} = Y/gA\tau$ where g was the gain of the system. When $h(x)$ is not a linear function we replace the gain by the slope of $h(x)$ at the operating point.

$$g = \frac{\partial h}{\partial x} \tag{6.13}$$

so that

$$\overline{\lambda} = \frac{Y}{A\tau\left(\frac{\partial h}{\partial x}\right)} \tag{6.14}$$

The variance in $\overline{\lambda}$ is

$$\text{var}(\overline{\lambda}) = \frac{\text{var}(Y)}{(A\tau)^2 g^2} \tag{6.15}$$

and

$$DQE = \frac{\lambda A\tau g^2}{\text{var}(Y)} \tag{6.16}$$

If we note that X is a Poisson process with mean and variance $\lambda A\tau$ then we can write

$$DQE = \frac{g^2 \text{var}(X)}{\text{var}(Y)} \tag{6.17}$$

This form of the expression emphasizes the important point that the DQE is the ratio of the variance that would be observed at the detector output if the detector was noiseless to the variance that is actually present. An equivalent expression in terms of variance ratios at the detector input is found by rearranging

$$DQE = \frac{\text{var}(X)}{\frac{1}{g^2}\text{var}(Y)} \tag{6.18}$$

where now the numerator is the variance in the input stream and the denominator is the variance in the detector output reflected back to the input side by dividing by g^2. Calculations of DQE can be done from either side of the detector.

Example 6.3.1 *Let us calculate the DQE of the linear photon counter that is described by*

$$h(x) = \begin{cases} 0, & x < T \\ x - T + 1, & T \leq x < L + T \\ L, & x \geq L + T \end{cases} \tag{6.19}$$

and let the input level be a random variable X with $E[X] = \lambda A\tau$. The output is $Y = h(X)$. By relating q in the analysis of Section 5.2 to the number of photons arriving at the detector so that $q = \lambda A\tau$, we can make use of the transfer function and gain calculations done there. The expected value of Y is given by

$$E[Y] = \mu_Y(q) = L\left(1 - f_1(q, T, S)e^{-q}\right) \tag{6.20}$$

This is a well-defined function whose value depends only upon the parameters L, T, and S of the detector and the level $q = \lambda A\tau$ of the input photon stream. In a particular measurement we seek to estimate q on the basis of an observation Y, and for this we need an estimator equation. Let $\bar{q} = w(Y)$ be the estimation rule.

The function w should have the property that $E[\bar{q}] = q$ and the variance of the estimate is small. To find the estimator equation we invert the relationship and construct an equation of the form $Y = r(\bar{q})$. This can be expanded about any nearby point q as

$$Y = r(\bar{q}) = r(q) + \frac{dr}{dq}(q - \bar{q}) \tag{6.21}$$

We then note that we want to choose r so that $E[Y] = E[r(\bar{q})] = \mu_Y(q)$. This function can be expanded in a power series for any point q_1 near q.

$$\mu_Y(q_1) = \mu_Y(q) + \frac{d\mu_Y}{dq}(q - q_1) + \cdots \tag{6.22}$$

If we replace $\mu_Y(q_1)$ with the observation Y and q_1 by $\bar{q}$ and drop high order terms we construct a relationship of the form

$$Y = \mu_Y(q) + \frac{d\mu_Y}{dq}(q - \bar{q}) \tag{6.23}$$

Upon comparing equations (6.21) and (6.23) we are led to choose r *to be the function* μ_Y*. For a given* Y *we need to solve the equation* $Y = \mu_Y(\bar{q})$ *for the estimate* $\bar{q}$*. This choice produces an unbiased estimator, which we see by taking expectations above to obtain*

$$0 = E[Y] - \mu_Y(q) = \frac{d\mu_Y}{dq}(q - E[\bar{q}]) \tag{6.24}$$

so that $E[\bar{q}] = q$*. The variance of the estimator about the true value is*

$$E[(q-\bar{q})^2] = \frac{var(Y)}{\left(\frac{d\mu_Y}{dq}\right)^2} \tag{6.25}$$

so that

$$\frac{var(\bar{q})}{var(Y)} = \left(\frac{d\mu_Y}{dq}\right)^2 \tag{6.26}$$

This is the gain that associates the variation at the input with the variation in the output. The derivative is easily calculated from (6.20) as

$$\frac{d\mu_Y}{dq} = Le^{-q}\left(f_1 - \frac{df_1}{dq}\right) \tag{6.27}$$

This is the gradient function that is plotted in Figure 5.4. For values of q *that are in the operating region of the detector it can have a value close to 1.0. The* DQE *is the ratio*

$$DQE = \frac{var(\hat{q})}{var(\bar{q})} = \frac{var(\hat{q})}{\left(\frac{d\mu_Y}{dq}\right)^2 var(Y)} \tag{6.28}$$

The variance of Y *was calculated in Section 5.2. After substitution of (5.23) and related equations we find that the DQE is identical to the comparative noise level of Eq (5.27).*

Estimator Equation

In the above example we related the observation Y to the level of the input as represented by λ or q. In the following let us write $E[Y] = r(\lambda) = \mu_Y(\lambda)$. This is a well-defined relationship that, as we saw above, depends upon the detector parameters. For any particular observation Y we can then form the relationship

$$Y = r(\bar{\lambda}) = r(\lambda) + (\lambda - \bar{\lambda})\frac{dr}{d\lambda} + \cdots \tag{6.29}$$

Since $E[Y] = r(\lambda)$, it is necessary that $E[\bar{\lambda}] = \lambda$, and the estimate of the input intensity based on Y will be unbiased. The variance is calculated by

$$var(Y) = E[(Y - E[Y])^2] = E[(Y - r(\lambda))^2] = E[(\lambda - \bar{\lambda})^2]\left(\frac{dr}{d\lambda}\right)^2 \tag{6.30}$$

Hence, the variance in the estimate is

$$var(\bar{\lambda}) = \frac{var(Y)}{\left(\frac{dr}{d\lambda}\right)^2_{\lambda=\bar{\lambda}}} \tag{6.31}$$

Example 6.3.2 Pure Amplifier. *At the input* $X = \lambda A\tau$ *and* $\hat{\lambda} = X/A\tau$. *The variance in the ideal estimate is* $var\left(\hat{\lambda}\right) = var(X)/A^2\tau^2 = \lambda/A\tau$. *Then*

$$DQE = \frac{var(\hat{\lambda})}{var(\bar{\lambda})} = \frac{\lambda}{A\tau}\frac{\left(\frac{dr}{d\lambda}\right)^2_{\lambda=\bar{\lambda}}}{var(Y)} \tag{6.32}$$

For a pure amplifier $Y = h(X) = QX$ *for some appropriate gain multipier* Q. *The variance of* Y *is* $var(Y) = Q^2 var(X) = Q^2\lambda A\tau$ *and* $r(\lambda) = E[Y] = Q\lambda A\tau$ *so that the gain is* $\frac{dr}{d\lambda} = QA\tau$. *Then*

$$DQE = \frac{\lambda}{A\tau}\frac{(QA\tau)^2}{Q^2\lambda A\tau} = 1 \tag{6.33}$$

No matter how much the amplification, the DQE is unity since no noise is added by the detector.

Example 6.3.3 Each input photon produces an output event with probability η. *If* $X = \lambda A\tau$ *then* $Y = \eta\lambda A\tau$ *and* $r(\lambda) = E[Y] = \eta\lambda A\tau$ *The output process is still Poisson, so that* $var(Y) = E[Y] = \eta\lambda A\tau$. *The gain is* $\frac{dr}{d\lambda} = \eta A\tau$. *The ideal estimate is* $\hat{\lambda} = X/A\tau$ *so that* $var\left(\hat{\lambda}\right) = var(X)/A^2\tau^2 = \lambda/A\tau$. *Then*

$$DQE = \frac{var(\hat{\lambda})}{var(\bar{\lambda})} = \frac{\lambda/A\tau}{\eta\lambda A\tau}(\eta A\tau)^2 = \eta \tag{6.34}$$

6.3.1 Photomultiplier Modeling

In this model we assume that each incoming photon can produce one or more output events. The fact that the number of output events for a given photon is a random variable introduces an added degree of complexity to the model in exchange for greater flexibility. It is an interesting exercise in the application of probability analysis in its own right.

For each input event assume that there is a probability $g(m)$ of producing exactly m output events. The probability that the number of input events is n is given by the probability $f(n)$. The model is reasonably tractable if we assume, following Zweig, that the input and output events are governed by Poisson distributions.

$$f(n) = \frac{q^n e^{-q}}{n!} \tag{6.35}$$

$$g(m) = \frac{Q^m e^{-Q}}{m!} \tag{6.36}$$

This means that there are an average of q photons impinging on the detector and that each of them produces an average of Q output events. We assume that we do not encounter saturation effects, so that Q is not a function of q.

The effect of multiple photon arrivals can be modeled by computing $g(m|n)$, the probability of exactly m output events given exactly n input events, and then averaging over n. If the first photon produces m_1 events then the other $n-1$ photons must produce $m - m_1$ events. Therefore

$$g(m|n) = \sum_{m_1=0}^{m} g(m_1|1)g(m-m_1|n-1) \tag{6.37}$$

By definition, $g(m|1) = g(m)$. We also need to define $g(m|0)$ to carry out the calculation. Set

$$g(m|0 = \begin{cases} 0, & m \neq 0 \\ 1, & m = 0 \end{cases} \tag{6.38}$$

With these definitions we can compute $g(m|n)$ for any n by iteration. The distribution of output events is then

$$h(m) = \sum_{n=0}^{\infty} f(n)g(m|n) \tag{6.39}$$

where $g(m|n)$ is the n-fold convolution above.

The calculation of results is facilitated by the use of generating functions. Define

$$F(s) = \sum_{n=0}^{\infty} f(n)s^n \tag{6.40}$$

$$G(s) = \sum_{m=0}^{\infty} g(m)s^m \tag{6.41}$$

Then

$$G^2(s) = \sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty} g(m_1)g(m-m_1)s^m = \sum_{m=0}^{\infty} g(m|2)s^m \tag{6.42}$$

By repetition we find

$$G^n(s) = \sum_{m=0}^{\infty} g(m|n)s^m \tag{6.43}$$

which is the generating function of $g(m|n)$. This is a very handy computational tool. Then we can show that

$$H(s) = \sum_{n=0}^{\infty} f(n)G^n(s) = F[G(s)] \tag{6.44}$$

For the Poisson distributions given above,

$$F(s) = e^{q(s-1)} \tag{6.45}$$

$$G(s) = e^{Q(s-1)} \tag{6.46}$$

Then

$$H(s) = e^{q(e^{Q(s-1)}-1)} \tag{6.47}$$

The mean value of the output is given by

$$\mu = \left.\frac{dH}{ds}\right|_{s=1} = qQ \tag{6.48}$$

Note that the gain of the system is $\frac{d\mu}{dq} = Q$. The variance of the output can be computed by

$$\sigma^2 = \sum_{n=0}^{\infty} n^2 h(n) - \left(\sum_{n=0}^{\infty} nh(n)\right)^2 \tag{6.49}$$

After a fair amount of algebra, this turns out to be

$$\sigma^2 = qQ(Q+1) \tag{6.50}$$

The DQE is the ratio of the input variance to the output variance reflected back to the input through the system gain.

$$DQE = \frac{q\left(\frac{d\mu}{dq}\right)^2}{\sigma^2} = \frac{qQ^2}{qQ(Q+1)} = \frac{Q}{1+Q} \tag{6.51}$$

6.4 Image Intensification

Intensification processes are used to amplify the effect of photons in a detection system. An intensification process may have several stages, each of which multiplies the number of photons and adds some noise.

One stage of image amplification typically has two active elements. The first element responds to the incoming photons and produces a secondary emission with efficiency η. That is, each input photon has probability η of producing a secondary emission. The secondary emission then interacts with the second element to produce a multiple m of output quanta. The secondary emissions and the output quanta need not be of the same wavelength or even be the same kind of emission. An ideal amplification stage would then produce ηmq output quanta when it received q input quanta. By cascading several stages of amplification one can achieve large multiples in the number of quanta.

Image intensification is limited by the sensitivity of the input stage to arriving quanta and by noise that is introduced in each stage of the amplification process. The noise that is added in the first element is customarily called "background." The output of the first element is

$$n_1 = \eta_1 q + b_1 \tag{6.52}$$

These n_1 emissions from the first stage are then multiplied in the second element. The output of the first stage is then

$$N_1 = m_1 n_1 = m_1(\eta_1 q + b_1) \tag{6.53}$$

Another stage can be added for further intensification. This produces N_2 output emissions. The value of N_2 can be calculated by first finding the number of emissions from the first element

$$n_2 = \eta_2 N_1 + b_2 \tag{6.54}$$

and then multiplying by m_2.

$$N_2 = m_2 n_2 = m_2 (\eta_2 N_1 + b_2) = m_1 m_2 \eta_1 \eta_2 q + m_1 m_2 \eta_2 b_1 + m_2 b_2 \tag{6.55}$$

The first term is the amplification of q and the last two are generated by random processes internal to the intensifier stages. Clearly we can make the output much larger than the input if the combination of multiplication factors and efficiency at each stage satisfy $m_i \eta_i > 1$. However, we see that any internal noise also is amplified. In order to discuss the effect of the amplification and the noise we need a performance measure.

6.4.1 Noise Model

We use the term *noise* for any random phenomenon that tends to obscure a *signal* that we are trying to observe. The most common model of signal and noise is one in which they are assumed to be independent and additive. Suppose that s is a signal and z is a noise and that we can observe the sum

$$x = s + z \tag{6.56}$$

An observation of x will not provide the exact value of s. If we were to make repeated measurements then they would be distributed around the actual value of s by the repeated selections of the random variable z. For a given value of s, a set of repeated measurements would have a histogram that is determined by z with the mean value shifted by s. The probability density function would be

$$f_x(x|s) = f_z(x - s) \tag{6.57}$$

Our ability to measure the value of s from the observation of a value of x is limited by the spread of the noise distribution. Our uncertainty in s is approximated by the standard deviation of z, or $\Delta s \approx \sigma_z$. The relative uncertainty is $\Delta s/s = \sigma_z/s$ which is something we would like to minimize.

We realize that s is also a random variable. If it was not a random variable then it would have a known value and there would be no need to make an observation. Suppose that s can occupy a range of values from s_1 to s_2. Then our ability to resolve the value of s in this interval could be measured

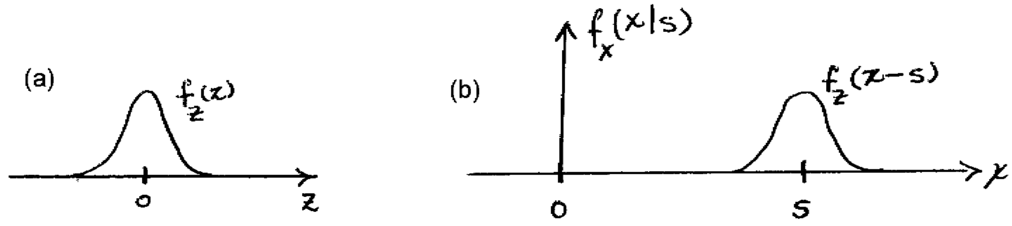

Figure 6.2: Probability density functions related to the sum of signal and noise. (a) noise and (b) conditional pdf of the observation for a given signal level.

by the ratio $\Delta s/(s_2 - s_1) = \sigma_z/(s_2 - s_1)$. A more common measure that is based on the same idea is to use the standard deviation of s rather than the total interval of s. A quality measure of the system would then be related to σ_z/σ_s. The standard deviation is the root-mean-squared value of a quantity. The variance of the same quantity is equal to its mean-squared value, and that is equivalent to the power or energy of many physical quantities. It has become customary to use the signal-to-noise ratio as a quality measure.

6.4.2 Signal-to-Noise Ratio

In terms of the quantities in this discussion,

$$SNR = \frac{\text{Average Signal Power}}{\text{Average Noise Power}} = \frac{\sigma_s^2}{\sigma_z^2} \tag{6.58}$$

Amplification of x does not affect the SNR. If a device produces a gain g in amplitude then its output $y = gx = gs + gz$ has an output SNR of

$$SNR_O = \frac{g^2\sigma_s^2}{g^2\sigma_z^2} = \frac{\sigma_s^2}{\sigma_z^2} = SNR_I \tag{6.59}$$

where the subscripts refer to the output and input of the device. We see that simple amplification produces no change in SNR. That is because it is a relative measure.

No amplification system is perfect. All systems add some noise to the output. The result is

$$y = gx + b \tag{6.60}$$

where we have used the symbol b to stand for "background," a common term in image amplifiers. We need some symbol to stand for the additive noise and n is used for lots of other things. Substitute for x to get an expression that contains the input signal.

$$y = gs + gz + b \tag{6.61}$$

The signal term is gs and it has variance $g^2\sigma_s^2$. The noise now has two terms, $gz + b$. If we assume that these are statistically independent then their variances add so that the output noise variance is $g^2\sigma_z^2 + \sigma_b^2$. The output SNR is now

$$SNR_O = \frac{g^2\sigma_s^2}{g^2\sigma_z^2 + \sigma_b^2} \tag{6.62}$$

The output SNR is reduced by the presence of the internal amplifier noise. Clearly the effect of the internal noise depends upon its size relative to that of the external noise. A useful relative measure can be defined by noting that in the absence of internally generated noise the output and input SNR are equal.

6.4.3 Detective Quantum Efficiency

The DQE of a device can be defined as the ratio

$$DQE = \frac{SNR_O}{SNR_I} \tag{6.63}$$

This ratio is always smaller than unity because the input SNR is always larger than the output SNR. It is the same measure as the comparative noise level that was discussed earlier, but in a different form. We will want to understand how the approaches to constructing the measures are related.

For the amplifier system above the DQE is

$$DQE = \frac{g^2\sigma_s^2}{(g^2\sigma_z^2 + \sigma_b^2)}\frac{g^2\sigma_z^2}{g^2\sigma_s^2} = \frac{g^2\sigma_z^2}{g^2\sigma_z^2 + \sigma_b^2} \tag{6.64}$$

It should be noted that the term on the right does not contain anything related to the signal. It only contains terms that depend upon the external noise, the internal noise and the gain.

The dependency of the DQE on the gain can be seen by constructing a graph. A graph of the function

$$DQE = \frac{(g^2\sigma_z^2/\sigma_b^2)}{(g^2\sigma_z^2/\sigma_b^2) + 1} \tag{6.65}$$

as a function of the gain will have a knee in its graph near the point $g = \sigma_b/\sigma_z$. This happens because the factor $g\sigma_z/\sigma_b$ changes from a number smaller than unity to one greater than unity at that point. This is shown in the graph below, where the variable that is plotted is the normalized variable $r = g\sigma_z/\sigma_b$. What does this graph say about the relative importance of the noise at the input and the internal noise? Does it make sense that the quality measure derived from SNR would not depend upon the signal? If you are given a certain level of input noise, what can you do to get above the knee in the curve? How much value is there in getting far above the knee?

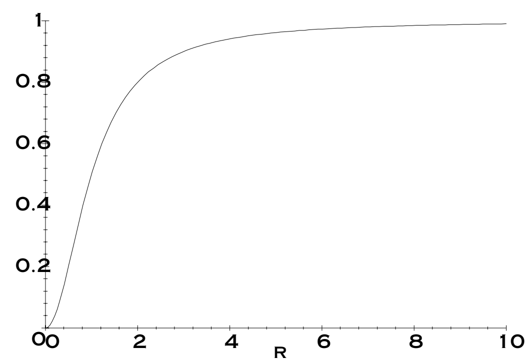

Figure 6.3: The DQE as a function of the quantity $r = g\sigma_z/\sigma_b$.

6.4.4 Application to Image Intensification

To apply the SNR or DQE quality measures to image intensifiers it is necessary to identify the signal, input noise, internal noise and system gain. Let us begin with the output of the first element of the first stage of image intensification. This element can be treated alone and then combined with the second element to describe the whole first stage. We will find a pattern that can then be used to model the effect of additional stages. We will find that each element in the chain can be described in terms of its effect on the overall DQE.

The secondary emissions are described by (6.52) as $n_1 = \eta_1 q + b_1$. The number n_1 actually has one signal quantity and three noise quantities. The signal quantity is $s_1 = E[n_1] = \eta_1 \mu_q$ where $\mu_q = E[q]$. The noise is due to the Poisson fluctuations in the photon stream, the variation in the number of secondary emissions for each incoming photon, and the background noise, b_1. These processes are assumed to be independent. We can write the variation as

$$\Delta n_1 = \eta_1 \Delta q + q \Delta \eta_1 + \Delta b_1 \tag{6.66}$$

The mean-squared fluctuation is the sum of the squares of the three terms since the quantities are statistically independent. Then

$$\overline{\Delta n_1^2} = \eta_1^2 \overline{\Delta q^2} + \overline{q \Delta \eta_1^2} + \overline{\Delta b_1^2} \tag{6.67}$$

Since the photon stream is described by a Poisson distribution, $\overline{\Delta q^2} = \sigma_q^2 = \mu_q$. We will represent the variance of the internal noise by σ_b^2 and deal with specifics later. That leaves the term $\overline{q \Delta \eta_1^2}$. Each photon has a probability η_1 of generating a secondary emission. We can treat each photon as a "trial" and each secondary emission a "success" in a collection of Bernoulli trials. The average number of trials is μ_q and therefore we can calculate $\overline{\mu q \Delta \eta_1^2}$. The statistics will have a binomial distribution with mean value $\mu_q \eta_1$ and variance $\overline{\mu_q \Delta \eta_1^2} = \mu_q \eta_1 (1 - \eta_1)$. (Here we are making use of the fact that a binomial distribution with success probability p has a variance of $np(1-p)$ for n trials.)

After substituting the terms calculated above into (6.67) we find

$$\begin{aligned} \overline{\Delta n_1^2} &= \eta_1^2 \mu_q + \mu_q \eta_1 (1 - \eta_1) + \sigma_b^2 \\ &= \eta_1 \mu_q + \sigma_b^2 \end{aligned} \tag{6.68}$$

This is the mean-squared value of the fluctuations in the stream of secondary emissions after the photons have struck an intensifier screen. The SNR at the output of the first stage is

$$SNR_{O_1} = \frac{(\eta_1 \mu_q)^2}{\eta_1 \mu_q + \sigma_b^2} \tag{6.69}$$

The SNR at the input of the stage is

$$SNR_{I_1} = \frac{\mu_q^2}{\mu_q} = \mu_q \tag{6.70}$$

The DQE is

$$DQE_1 = \frac{SNR_{O_1}}{SNR_{I_1}} = \frac{\eta_1^2 \mu_q}{\eta_1 \mu_q + \sigma_b^2} = \frac{\eta_1}{1 + \sigma_b^2/\eta_1 \mu_q} \tag{6.71}$$

From this we note that the DQE of the first stage will range from 0 to η_1 depending upon the values of the three parameters η_1, μ_q and σ_b^2.

The second element of the first stage of an image intensifier is a multiplier. The output is described by (6.53). The signal at the output of the second element is identified as $s = E[N_1] = m_1 \eta_1 \mu_q$. The noise can be found by looking at the fluctuations.

$$\Delta N_1 = \Delta(m_1 n_1) = m_1 \Delta n_1 + n_1 \Delta m_1(n_1) \tag{6.72}$$

The quantity m_1 is the number of events created in the second element from each emission of the first element. That number is actually a random variable. It may, for example, have a Poisson distribution with parameter μ_{m_1}. The mean-squared value will affect the average result with a variance that is diminished by a factor that is equal to the number of "trials" and therefore is a function of n_1. Assuming that the events are independent, the mean-squared value of ΔN_1 is

$$\overline{\Delta N_1^2} = \mu_{m_1}^2 \overline{\Delta n_1^2} + \overline{(n_1 \Delta m_1(n_1))^2} \tag{6.73}$$

We have already found that $\overline{\Delta n_1^2} = \eta_1 \mu_q + \sigma_b^2$. The second term can be analyzed as follows:

$$\begin{aligned} \overline{\Delta m_1(n_1)^2} &= \frac{\overline{\Delta m_1^2}}{n_1} \\ \overline{(n_1 \Delta m_1(n_1))^2} &= n_1^2 \overline{\Delta m_1(n_1)^2} = n_1 \overline{\Delta m_1^2} \end{aligned} \tag{6.74}$$

Therefore

$$\overline{\Delta N_1^2} = \mu_{m_1}^2 \overline{\Delta n_1^2} + \eta_1 \mu_q \overline{\Delta m_1^2} \tag{6.75}$$

If m_1 is Poisson distributed then $\overline{\Delta m_1^2} = \mu_{m_1}$ and

$$\begin{aligned} \overline{\Delta N_1^2} &= \mu_{m_1}^2 \overline{\Delta n_1^2} + \eta_1 \mu_q \mu_{m_1} \\ &= \mu_{m_1}^2 (\eta_1 \mu_q + \sigma_b^2) + \eta_1 \mu_q \mu_{m_1} \end{aligned} \tag{6.76}$$

The SNR at the output of the second element is

$$SNR_{O_2} = \frac{(\mu_{m_1}\eta_1\mu_q)^2}{\mu_{m_1}^2(\eta_1\mu_q + \sigma_b^2) + \eta_1\mu_q\mu_{m_1}} \tag{6.77}$$

The SNR at the input is given by SNR_{O_1} from the first element. Hence, the DQE is

$$DQE_2 = \frac{SNR_{O_2}}{SNR_{O_1}} = \frac{\mu_{m_1}^2(\eta_1\mu_q + \sigma_b^2)}{\mu_{m_1}^2(\eta_1\mu_q + \sigma_b^2) + \eta_1\mu_q\mu_{m_1}} \tag{6.78}$$

If we divide through by the numerator factor this can be expressed as

$$DQE_2 = \frac{1}{1 + \frac{\eta_1\mu_q}{\mu_{m_1}(\eta_1\mu_q + \sigma_b^2)}}$$

The DQE of the first stage is

$$DQE = \frac{SNR_{O_2}}{SNR_{I_1}} = \frac{SNR_{O_2}}{SNR_{I_1}}\frac{SNR_{O_1}}{SNR_{O_1}} = \frac{SNR_{O_1}}{SNR_{I_1}}\frac{SNR_{O_2}}{SNR_{O_1}} = DQE_1 DQE_2 \tag{6.79}$$

Hence, the overall DQE values multiply and, because each factor is less than unity, can never exceed the value of any factor. When DQE is lost it can never be regained in later stages. For the intensification system,

$$DQE = \frac{\eta_1^2\mu_q}{\eta_1\mu_q + \sigma_b^2}\frac{\mu_{m_1}^2(\eta_1\mu_q + \sigma_b^2)}{\mu_{m_1}^2(\eta_1\mu_q + \sigma_b^2) + \eta_1\mu_q\mu_{m_1}} = \frac{\eta_1^2\mu_{m_1}^2\mu_q}{\mu_{m_1}^2(\eta_1\mu_q + \sigma_b^2) + \eta_1\mu_q\mu_{m_1}}$$