SIMG-713 SPRING 2002

REVIEW TOPICS FOR FINAL EXAM

Outline of Topics

1. Concept of an experiment that produces outcomes that can be different on each trial.
2. Concept of events as sets of possible experiment outcomes.
3. Probability as a number associated with events.
   (a) Properties of probability
   (b) Joint probabilities
   (c) Conditional probabilities
   (d) Statistical independence
   (e) Mutually exclusive events
4. Random variables
5. Averages over random variables.
   (a) Computation of averages and moments
   (b) Physical interpretation of mean and variance
   (c) Correlation and covariance
6. Averages over functions of a random variable
7. Modeling of discrete events and random arrivals
   (a) Binomial distribution
   (b) Poisson distribution
   (c) Normal distribution.
   (d) Error function and z-score.
   (e) Normal approximation to binomial.
8. Application to detector arrays
9. Transfer function modeling
   (a) Mean value of output as a function of q.
   (b) Variance of output as a function of q.
   (c) Detective Quantum Efficiency
   (d) Detection of input level change - SNR
10. Random Process
    (a) Ensemble of sample functions
    (b) Ensemble averages
       1. Mean value
       2. Autocorrelation and crosscorrelation
       3. Variance
       4. Function of random variable

(c) Stationary vs nonstationary
(d) Wide-sense stationary.

11. Parameter Estimates
    (a) Why is an estimate based on a sample function always a random variable?
    (b) What is an estimator (as opposed to an estimate)?
    (c) What is bias as it relates to an estimator?
    (d) How does the mean and variance of an estimator of the mean change with the number of samples used?

12. Correlation, Covariance and Crosscorrelation
    (a) Definition of covariance.
    (b) Definition of correlation.
    (c) Definition of crosscorrelation.
    (d) Correlation coefficient.
    (e) Bivariate normal distribution.

13. Power spectrum
    (a) Definition as transform of autocorrelation function - Wiener Khintchine Theorem
    (b) Relationship to Fourier transform of a sample function
    (c) Describe the periodogram algorithm for spectral estimation.
    (d) Describe the correlogram algorithm for spectral estimation.

14. Digital Filter Models
    (a) Difference equation
    (b) Block diagram
    (c) Impulse response
    (d) System function
    (e) Relationships between models

15. Special Topics (Not for final exam)
    (a) Z Transform
        1. Definition of z-transform of a sequence
        2. Interpretation of z as a delay operator
        3. Relationship between input, output and system function in z-domain
    (b) What information does the substitution $X(z)$ with $z = e^{i\omega}$ provide?
        1. What range of values of ω is available?
        2. How do we relate to time or frequency for a time-sampled application?
        3. How does $X(\omega)$ for a function $x(t)$ relate to $X(e^{i\omega})$ for a discrete function ?
    (c) Generation of random sequences
    (d) How to simulate a bandlimited random process

Questions from old exams

1. Random variables X and Y have a joint discrete probability function $P_{XY}(X=m,Y=n)$ shown in the table below.

$X\backslash Y$	1	2
1	0.1	0.2
2	0.3	0.4

   (a) Find the mean value of X.
   (b) Find the variance of X.
   (c) Find the correlation coefficient ρ_{XY}.
   (d) Find the expected value of the function $g(X,Y)=(X-Y)(X+Y)$

2. A random number generator can produce a sequence of samples X_n, $n=1,2,\ldots,N$.

   (a) How would you determine an approximation to the probability density function $f_X(x)$? Describe an algorithm that you would use to construct the approximation. List any parameters associated with the algorithm.
   (b) Provide an estimator for the mean value of X and determine the variance of the estimate it provides under the assumption that the samples are statistically independent.

3. A physial analysis of the efficiency of a photon detector has shown that 80% of the incoming photons cause an electron to be produced inside the detector. Construct a model for the detector output Y in terms of the detector input X and derive a formula for the DQE based on this model. List any assumptions you use in constructing the model and deriving the DQE formula.

4. Random variables X and Y take on the values $\{x_1,x_2,x_3,x_4\}$ and $\{y_1,y_2,y_3\}$. The values for $P[X=x_i]$ and
   $P[Y=y_j|X=x_i]$ are given in the table below. Note that some of the information, in the positions labeled A,B,C,D, is missing.

X	x_1	x_2	x_3	x_4
$P[X=x_i]$	0.1	0.2	A	0.3
$P[Y=y_1\|X=x_i]$	B	0.25	0.25	0.25
$P[Y=y_2\|X=x_i]$	0.35	0.45	0.25	C
$P[Y=y_3\|X=x_i]$	0.4	0.3	D	0.25

   (a) Determine the values that should go into locations A,B,C,D in the table.
   (b) Compute $P[X=x_2|Y=y_3]$.

5. We have four boxes. Box 1 contains 2000 components of which 5% are defective. Box 2 contains 500 components of which 40% are defective. Boxes 3 and 4 contain 1000 each with 10% defective. We select *at random* one of the boxes and remove *at random* a single component.

   (a) What is the probability that the selected component is defective?
   (b) We examine the defective component and find it defective. On the basis of this evidence, what is the probability it was drawn from Box 2?

6. Two random processes $x(n)$ and $y(n)$ are related by

$$y(n)=\sum_{k=-\infty}^{n} x(k)\,h(n-k)$$

   where $\sum_{k=0}^{\infty} h^2(k)$ is bounded.

(a) Find an expression for the cross-correlation between the x and y sequences in terms of the impulse response function $h(n)$ under the condition that x is a white-noise sequence with $\mu_x = 0$ and variance σ_x^2.

(b) Find an expression for the variance of the y sequence in terms of the impulse response function $h(n)$ under the condition that x is a white-noise sequence with $\mu_x = 0$ and variance σ_x^2.

7. Let $X(n)$ be a discrete random process that can assume two values 0 and 1, with probabilities 0.2 and 0.8, respectively. The process is ergodic. The values $X(n)$ and $X(m)$ are statistically independent if $m \neq n$. This random process is the input to a digital filter with impulse response

$$h(n) = a^n step(n)$$

where a is a real number with $|a| < 1$. The output of the digital filter is a random process $Y(n)$.

(a) Determine whether $Y(n)$ is a wide-sense stationary random process.

(b) Find the expected value of $Y(n)$ in terms of a.

(c) Find the variance of $Y(n)$ in terms of a

8. A calibrated photon source is designed to produce photons at an average rate of $\lambda_0 = 1000$ photons per second. The sources are tested after they have been manufactured to see if they are acceptable. A source is acceptable if the actual rate λ satisfies $|\lambda - \lambda_0| \leq 5$ photons per second. The acceptance test is to count the number N of photons in $T = 40$ seconds and accept the source if $|N - 40,000| < 200$.

(a) What is the probability that a source will be rejected by the test if it has $\lambda = \lambda_0$? You may want to use the table of values of the normal distribution function that is attached.

(b) What additional information would be needed to compute the fraction of good sources that are incorrectly rejected?

9. A random process $X(n)$ is generated by repeatedly drawing numbers from a source that provides values that are uniformly distributed over $[0, 1]$. Each number is statistically independent of all others in the sequence. A new random sequence is generated by $Y(n) = (2X(n) - 1)^2$. Find the average value of the sequence $Y(n)$.

10. A discrete system is described by the difference equation

$$y(n) + a^2 y(n-2) = x(n)$$

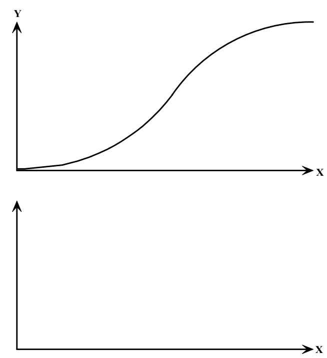
where $x(n)$ is the input sequence, $y(n)$ is the output sequence and a is a real number such that $|a| < 1$. Let $x(n)$ be generated by an infinite sequence of independent tosses of a fair coin, with $Head \rightarrow -1$, $Tail \rightarrow 1$.

(a) What is the autocorrelation function $R_{xx}(n)$?

(b) Find an expression for the spectrum $E[|Y(e^{i\omega})|^2]$.

(c) Determine the normalized radian frequency ω at which the spectrum is maximum. You should not have take derivatives and you can reason your way to the answer even if you can't answer part (b).

11. A certain new detector device has a response function $Y = f(X)$, where Y represents the output when the input is exposed to X photons.. It is known that f is a monotonically increasing function of the kind sketched in the figure below.



(a) Describe an experiment you would do to evaluate the DQE of the detector as a function of the input intensity.

(b) In the space provided below the sketch of f, provide a sketch of DQE as a function of input intensity.

12. Let X be an exponential random variable with probability density

$$f_X(x) = 2e^{-2x}\text{step}(x)$$

What is the probability density function of $Y = e^X$?