

## *Random Processes IV*

Lecture 14

Spring 2002

## Correlated Random Processes

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Correlated random processes can be described by:

1. The autocorrelation function
2. The power spectrum
3. A filter with a white noise input

The three descriptions are related. Each provides a different perspective on the random process.

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## Generating a Correlated Random Process

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We will provide a means to generate correlated random processes. If you can provide a generator, you have certainly provided one description of the rp.

Later we will look at how to determine the parameters of a filter model that generates a given rp.

We will only work here with WSS random processes.

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## Difference Equation/Digital Filter Model

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A digital filter can be described in several ways.

- Difference equation
- Block diagram
- Impulse response
- System function

The descriptions are equivalent but each provides a different insight. These descriptions are related to those of a random process when the filter input is white noise.

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## Difference Equation Model

Let the input sequence to a digital filter be denoted by  $x(n)$  and the corresponding response by  $y(n)$ . The current output value can depend upon the current input value, past input values and past output values.

$$\begin{aligned} y(n) + a_1y(n-1) + a_2y(n-2) + \dots + a_p y(n-p) \\ = b_0x(n) + b_1x(n-1) + \dots + b_q x(n-q) \end{aligned}$$

The difference equation is completely described by the coefficients  $\{1, a_1, \dots, a_p\}$  and  $\{b_0, b_1, \dots, b_p\}$ . We will always assume (without loss of generality) that  $a_0 = 1$ .

## Difference Equation Model

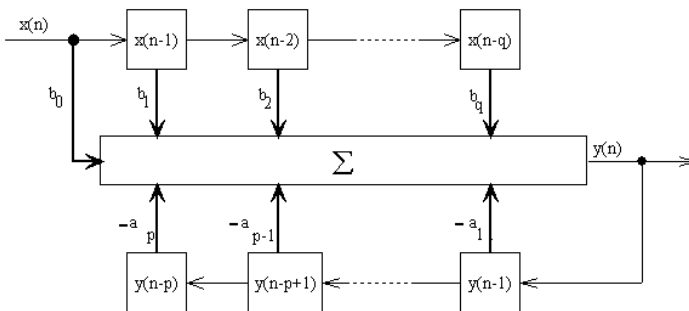
The difference equation can be solved for  $y_n$  in terms of the current and past inputs and past outputs.

$$\begin{aligned} y(n) = b_0x(n) + b_1x(n-1) + \dots + b_q x(n-q) \\ - a_1y(n-1) - a_2y(n-2) - \dots - a_p y(n-p) \end{aligned}$$

The properties of the output sequence are determined by

- The properties of the input sequence
- The filter coefficients

## Block Diagram



## Example: Single-pole Filter

Let

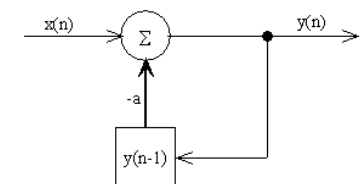
$$x(n) = [\dots, 0, 1, 0, 0, 0, 0, \dots]$$

and choose the origin so that  $x(0) = 1$ .

$$y(n) = x(n) - ay(n-1)$$

Assume initial condition  $y(-1) = 0$

$n$	0	1	2	3	4
$x(n)$	1	0	0	0	0
$y(n)$	1	$-a$	$a^2$	$-a^3$	$a^4$



## Impulse Response

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The impulse response of a digital filter is the sequence that is generated when the input sequence is  $x(n) = \delta(n)$ , where

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

The impulse response of the single-pole filter is

$$y(n) = h(n) = (-a)^n \text{ for } n \geq 0$$

The output is clearly bounded if and only if  $|a| \leq 1$ .

We will see later when we discuss the system function that this system has a "single pole" whose location is determined by  $a$ , and which is stable if the "pole" is inside the unit circle.

## Impulse Response

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If the impulse response of a discrete system is  $h(n)$  then the response to any input sequence  $x(n)$  can be computed by the convolution

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$$

For the previous example

$$\begin{aligned} y(n) &= \sum_{m=-\infty}^{\infty} x(m)(-a)^{n-m} \text{step}(n-m) \\ &= \sum_{k=0}^{\infty} (-a)^k x(n-k) \end{aligned}$$

This provides an exponential weighting of the past inputs.

## System Function Model

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The system function  $H(z)$  describes the response of the system to an exponential input sequence,  $x(n) = z^n$ .

$z$  may be any complex number, often expressed in the form  $z = re^{i\omega}$

Assume that the response is of the form  $y(n) = H(z)z^n$ . Note that  $H(z) = y(0)$  is a number once  $z$  has been specified.

Substitute both sequences into the difference equation and eliminate common  $z^n$  terms.

$$(1 + a_1z^{-1} + a_2z^{-2} + \dots + a_pz^{-p})z^n H(z) = (b_0 + b_1z^{-1} + \dots + b_qz^{-q})z^n$$

## System Function

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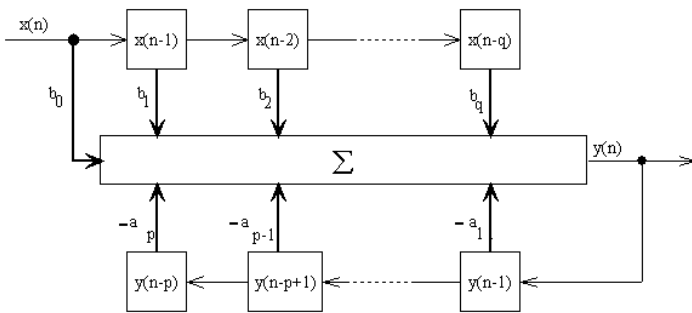
$$H(z) = \frac{b_0 + b_1z^{-1} + \dots + b_qz^{-q}}{1 + a_1z^{-1} + a_2z^{-2} + \dots + a_pz^{-p}}$$

The close relationship between the system function, the difference equation and the block diagram is evident.

The system function is defined wherever the denominator is not zero.

The roots of the denominator that are not cancelled by roots of the numerator are called the "poles" of the system function.

## System Models



$$y(n) = b_0x(n) + b_1x(n-1) + \dots + b_qx(n-q) - a_1y(n-1) - a_2y(n-2) - \dots - a_p y(n-p)$$

$$H(z) = \frac{b_0 + b_1z^{-1} + \dots + b_qz^{-q}}{1 + a_1z^{-1} + a_2z^{-2} + \dots + a_pz^{-p}}$$

## Z-Transform

The  $z$ -transform is defined for discrete sequences. It is closely related to the Fourier transform for a discrete sequence, and contains information about the frequencies contained in the sequence.

The definition is simply

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

The inverse  $z$ -transform can be defined, but is not necessary in this analysis.

We will illustrate the computation of the  $z$ -transform for some example functions.

## Example: Exponential Sequence

Let

$$x(n) = A^n \text{step}(n)$$

where  $A$  is a constant. The sequence is decreasing in magnitude if  $|A| < 1$

$$X(z) = \sum_{n=0}^{\infty} A^n z^{-n} = \sum_{n=0}^{\infty} (A/z)^n$$

This is just a geometric series which converges provided  $|A/z| < 1$ . Hence,

$$X(z) = \frac{1}{1 - A/z} = \frac{z}{z - A}$$

The denominator has a root (pole) at  $z = A$ . The sequence converges if the pole is inside the unit circle in the  $z$ -plane.

## Delay Operator

Suppose that the  $z$ -transform of  $x(n)$  is  $X(z)$ . Then, the  $z$ -transform of  $x(n-k)$  is  $X(z)z^{-k}$ .

$$\begin{aligned} X(z)z^{-k} &= \sum_{n=-\infty}^{\infty} x(n)z^{-(n+k)} \\ &= \sum_{n=-\infty}^{\infty} x(m-k)z^{-m} \end{aligned}$$

We can therefore refer to  $z^{-1}$  as the *delay operator*.

## Difference Equation and System Function

Take the  $z$ -transform of all terms in the difference equation by making use of the delay operator. Because all operations are linear,

$$\begin{aligned} Y(z) + a_1 Y(z)z^{-1} + a_2 Y(z)z^{-2} + \dots + a_p Y(z)z^{-p} \\ = b_0 X(z) + b_1 X(z)z^{-1} + \dots + b_q X(z)z^{-q} \end{aligned}$$

This can be rearranged as a ratio of polynomials in  $z$  that are identical to the system equation.

$$\frac{Y(z)}{X(z)} = H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_q z^{-q}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_p z^{-p}}$$

This leads to the system function equation

$$Y(z) = H(z)X(z)$$

## Difference Equation and System Function

It is natural to define the polynomials

$$\begin{aligned} A(z) &= 1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_p z^{-p} \\ B(z) &= b_0 + b_1 z^{-1} + \dots + b_q z^{-q} \end{aligned}$$

These are the  $z$ -transforms of the denominator and numerator polynomials in the system function and also polynomials formed from the difference equation coefficients.

The preceding results then show the following useful relationships:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)}$$

$$A(z)Y(z) = B(z)X(z)$$

## System Function and Impulse Response

In terms of the impulse response,

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$$

Take the  $z$ -transform of both sides by multiplying by  $z^{-n}$  and summing over  $n$ .

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n)z^{-n} \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(m)h(n-m)z^{-n} \\ &= \left( \sum_{m=-\infty}^{\infty} x(m)z^{-m} \right) \left( \sum_{k=-\infty}^{\infty} h(k)z^{-k} \right) \\ &= X(z)H(z) \end{aligned}$$

## System Function and Impulse Response

The system function is related to the impulse response by

$$H(z) = \sum_{k=-\infty}^{\infty} h(k)z^{-k}$$

For the single-pole filter,

$$h(k) = (-a)^k \text{step}(k)$$

so that

$$H(z) = \sum_{k=0}^{\infty} (-a)^k z^{-k} = \frac{1}{1 + az^{-1}}$$

The root at  $z = -a$  is evident.

## Frequency Response

Let  $x(n) = e^{i\omega n}$  be an input sequence. The output computed using the convolution equation is

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)e^{i\omega(n-m)} = e^{i\omega n} \sum_{m=-\infty}^{\infty} h(m)e^{-i\omega m}$$

The summation is just  $H(z)$  with  $z = e^{i\omega}$ . Hence, an exponential input sequence produces the exponential output sequence

$$y(n) = H(e^{i\omega})e^{i\omega n}$$

where

$$H(e^{i\omega}) = \sum_{m=-\infty}^{\infty} h(m)e^{-i\omega m}$$

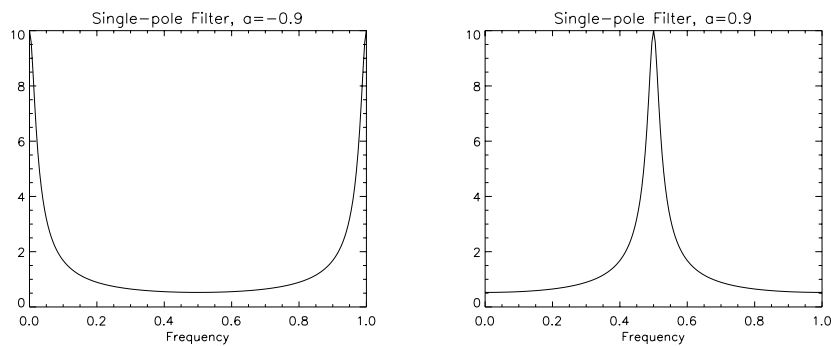
## Frequency Response

The system response  $H(e^{i\omega})$  provides the same kind of information about a discrete system that  $H(i\omega)$  provides about a continuous system.

For any  $\omega$ ,  $e^{i(\omega+2\pi)} = e^{i\omega}$ . Hence,  $H(e^{i\omega})$  is periodic with period  $2\pi$ . For the single-pole filter,

$$H(e^{i\omega}) = \frac{1}{1 + ae^{-i\omega}}$$

## Response of Single-pole Filter



## Damped Sinusoid Filter

A damped sinusoid filter has response

$$h(n) = e^{-n\tau} \sin(n\omega_0)$$

1. Impulse response, calculated & experimental
2. Response to white noise input sequence
3. Autocorrelation function calculated & experimental
4. Power spectrum and filter frequency response

