Linear Filtering of Random Processes

Lecture 13

Spring 2002

Wide-Sense Stationary

A stochastic process $X(t)$ is wss if its mean is constant

$$E[X(t)] = \mu$$

and its autocorrelation depends only on $\tau = t_1 - t_2$

$$R_{xx}(t_1, t_2) = E[X(t_1)X^*(t_2)]$$

$$E[X(t+\tau)X^*(t)] = R_{xx}(\tau)$$

Note that $R_{xx}(-\tau) = R^*_{xx}(\tau)$ and

$$R_{xx}(0) = E[|X(t)|^2]$$

Example

We found that the random telegraph signal has the autocorrelation function

$$R_{xx}(\tau) = e^{-c|\tau|}$$

We can use the autocorrelation function to find the second moment of linear combinations such as $Y(t) = aX(t) + bX(t - t_0)$.

$$\begin{aligned}
R_{yy}(0) &= E[Y^2(t)] = E[(aX(t) + bX(t - t_0))^2] \\
&= a^2 E[X^2(t)] + 2abE[X(t)X(t - t_0)] + b^2 E[X^2(t - t_0)] \\
&= a^2 R_{xx}(0) + 2abR_{xx}(t_0) + b^2 R_{xx}(0) \\
&= (a^2 + b^2)R_{xx}(0) + 2abRxx(t_0) \\
&= (a^2 + b^2)R_{xx}(0) + 2abe^{-ct_0}
\end{aligned}$$

Example (continued)

We can also compute the autocorrelation $R_{yy}(\tau)$ for $\tau \neq 0$.

$$\begin{aligned}
R_{yy}(\tau) &= E[Y(t+\tau)Y^*(t)] \\
&= E[(aX(t+\tau) + bX(t+\tau-t_0))\,(aX(t) + bX(t-t_0))] \\
&= a^2E[X(t+\tau)X(t)] + abE[X(t+\tau)X(t-t_0)] \\
&+ abE[X(t+\tau-t_0)X(t)] + b^2E[X(t+\tau-t_0)X(t-t_0)] \\
&= a^2R_{xx}(\tau) + abR_{xx}(\tau+t_0) + abR_{xx}(\tau-t_0) + b^2R_{xx}(\tau) \\
&= (a^2+b^2)R_{xx}(\tau) + abR_{xx}(\tau+t_0) + abRxx(\tau-t_0)
\end{aligned}$$

Linear Filtering of Random Processes

The above example combines weighted values of $X(t)$ and $X(t-t_0)$ to form $Y(t)$. Statistical parameters $E[Y]$, $E[Y^2]$, var(Y) and $R_{yy}(\tau)$ are readily computed from knowledge of $E[X]$ and $R_{xx}(\tau)$.

The techniques can be extended to linear combinations of more than two samples of $X(t)$.

$$Y(t) = \sum_{k=0}^{n-1} h_k X(t - t_k)$$

This is an example of linear filtering with a discrete filter with weights

$$\mathbf{h} = [h_0, h_1, \ldots, h_{n-1}]$$

The corresponding relationship for continuous time processing is

$$Y(t) = \int_{-\infty}^{\infty} h(s)X(t-s)ds = \int_{-\infty}^{\infty} X(s)h(t-s)ds$$

Filtering Random Processes

Let $X(t,e)$ be a random process. For the moment we show the outcome e of the underlying random experiment.

Let $Y(t,e) = \mathcal{L}[X(t,e)]$ be the output of a linear system when $X(t,e)$ is the input. Clearly, $Y(t,e)$ is an ensemble of functions selected by e, and is a random process.

What can we say about Y when we have a statistical description of X and a description of the system?

Note that $\mathcal{L}$ does not need to exhibit random behavior for Y to be random.

Time Invariance

We will work with time-invariant (or shift-invariant) systems. The system is time-invariant if the response to a time-shifted input is just the time-shifted output.

$$Y(t+\tau) = \mathcal{L}[X(t+\tau)]$$

The output of a time-invariant linear system can be represented by convolution of the input with the impulse response, $h(t)$.

$$Y(t) = \int_{-\infty}^{\infty} X(t-s)h(s)ds$$

Mean Value

The following result holds for any linear system, whether or not it is time invariant or the input is stationary.

$$E[\mathcal{L}X(t)] = \mathcal{L}E[X(t)] = \mathcal{L}[\mu(t)]$$

When the process is stationary we find $\mu_y = \mathcal{L}[\mu_x]$, which is just the response to a constant of value μ_x.

Output Autocorrelation

The autocorrelation function of the output is

$$R_{yy}(t_1, t_2) = E[y(t_1)y^*(t_2)]$$

We are particularly interested in the autocorrelation function $R_{yy}(\tau)$ of the output of a linear system when its input is a wss random process.

When the input is wss and the system is time invariant the output is also wss.

The autocorrelation function can be found for a process that is not wss and then specialized to the wss case without doing much additional work. We will follow that path.

Crosscorrelation Theorem

Let $x(t)$ and $y(t)$ be random processes that are related by

$$y(t) = \int_{-\infty}^{\infty} x(t-s)h(s)ds$$

Then

$$R_{xy}(t_1, t_2) = \int_{-\infty}^{\infty} R_{xx}(t_1, t_2 - \beta)h(\beta)d\beta$$

and

$$R_{yy}(t_1, t_2) = \int_{-\infty}^{\infty} R_{xy}(t_1 - \alpha, t_2)h(\alpha)d\alpha$$

Therefore,

$$R_{yy}(t_1, t_2) = \iint_{-\infty}^{\infty} R_{xx}(t_1 - \alpha, t_2 - \beta)h(\alpha)h(\beta)d\alpha d\beta$$

Crosscorrelation Theorem

Proof Multiply the first equation by $x(t_1)$ and take the expected value.

$$E[x(t_1)y(t)] = \int_{-\infty}^{\infty} E[x(t_1)x(t-s)]h(s)ds = \int_{-\infty}^{\infty} R_{xx}(t_1, t-s)h(s)ds$$

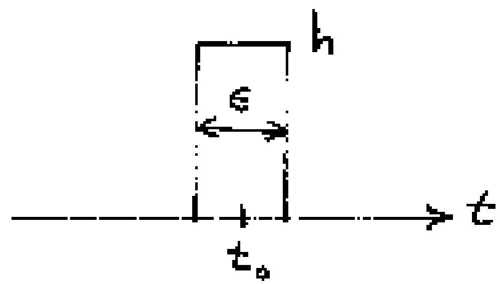
This proves the first result. To prove the second, multiply the first equation by $y(t_2)$ and take the expected value.

$$E[y(t)y(t_2)] = \int_{-\infty}^{\infty} E[x(t-s)y(t_2)]h(s)ds = \int_{-\infty}^{\infty} R_{xy}(t-s, t_2)h(s)ds$$

This proves the second and third equations. Now substitute the second equation into the third to prove the last.

Example: Autocorrelation for Photon Arrivals

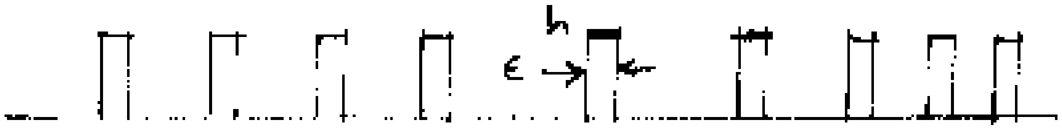
Assume that each photon that arrives at a detector generates an impulse of current. We want to model this process so that we can use it as the excitation $X(t)$ to detection systems. Assume that the photons arrive at a rate λ photons/second and that each photon generates a pulse of height h and width ϵ.



To compute the autocorrelation function we must find

$$R_{xx}(\tau) = E[X(t+\tau)X(t)]$$

Photon Pulses (continued)



Let us first assume that $\tau > \epsilon$. Then it is impossible for the instants t and $t+\tau$ to fall within the same pulse.

$$
\begin{aligned}
E[X(t+\tau)X(t)] &= \sum_{x_1}\sum_{x_2} x_1 x_2 P(X_1 = x_1, X_2 = x_2) \\
&= 0 \cdot 0P(0,0) + 0 \cdot hP(0,h) + h \cdot 0P(h,0) + h^2 P(h,h) \\
&= h^2 P(X_1 = h) P(x_2 = h)
\end{aligned}
$$

The probability that the pulse waveform will be at level h at any instant is $\lambda\epsilon$, which is the fraction of the time occupied by pulses. Hence,

$$E[X(t+\tau)X(t)] = (h\lambda\epsilon)^2 \texttt{ for } |\tau| > \epsilon$$

Photon Pulses (continued)

Now consider the case $|\tau| < \epsilon$. Then, by the Poisson assumption, there cannot be two pulses so close together so that $X(t) = h$ and $X(t+\tau) = h$ only if t and $t + |\tau|$ fall within the same pulse.

$P(X_1 = h, X_2 = h) = P(X_1 = h)P(X_2 = h|X_1 = h) = \lambda\epsilon P(X_2 = h|X_1 = h)$

The probability that $t + |\tau|$ also hits the pulse is

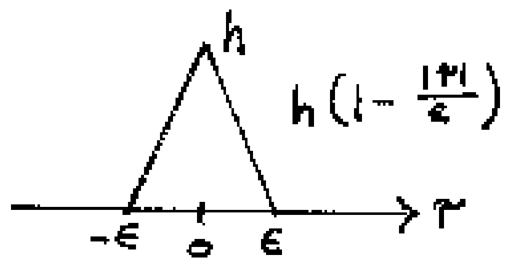
$$P(X_2 = h|X_1 = h) = 1 - |\tau|/\epsilon$$

Hence,

$$E[X(t+\tau)X(t)] = h^2\alpha\epsilon\left(1 - \frac{|\tau|}{\epsilon}\right) \quad \texttt{for} \ |\tau| \leq \epsilon$$

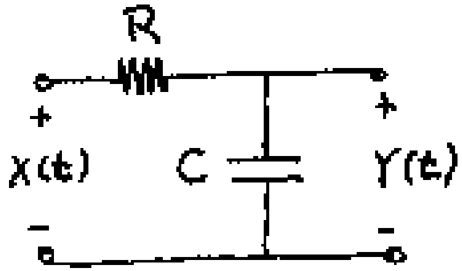
If we now let $\epsilon \rightarrow 0$ and keep $h\epsilon = 1$ the triangle becomes an impulse of area $h\epsilon$ and we have

$$R_{xx}(\tau) = \lambda\delta(\tau) + \lambda^2$$



Detector Response to Poisson Pulses

It is common for a physical detector to have internal resistance and capacitance. A series RC circuit has impulse response



$$h(t) = \frac{1}{RC} e^{-t/RC} \texttt{step}(t)$$

The autocorrelation function of the detector output is

$$\begin{aligned} R_{yy}(t_1, t_2) &= \iint_{-\infty}^{\infty} R_{xx}(t_1 - \alpha, t_2 - \beta) h(\alpha) h(\beta) d\alpha d\beta \\ &= \frac{1}{(RC)^2} \iint_0^{\infty} \left(\lambda \delta(\tau + \alpha - \beta) + \lambda^2 \right) e^{-(\alpha+\beta)/RC} d\alpha d\beta \\ &= \frac{\lambda}{(RC)^2} \int_0^{\infty} e^{-(\tau+2\alpha)/RC} d\alpha + \lambda^2 \left(\int \frac{1}{RC} e^{-u/RC} du \right)^2 \\ &= \frac{\lambda}{2RC} e^{-\tau/RC} + \lambda^2 \texttt{ with } \tau \geq 0 \end{aligned}$$

White Noise

We will say that a random process $w(t)$ is white noise if its values $w(t_i)$ and $w(t_j)$ are uncorrelated for every t_i and $t_j \neq t_i$. That is,

$$C_w(t_i, t_j) = E[w(t_i)w^*(t_j)] = 0 \text{ for } t_i \neq t_j$$

The autocovariance must be of the form

$$C_w(t_i, t_j) = q(t_i)\delta(t_i - t_j) \text{ where } q(t_i) = E[|w(t_i)|^2] \geq 0$$

is the mean-squared value at time t_i. Unless specifically stated to be otherwise, it is assumed that the mean value of white noise is zero. In that case, $R_w(t_i, t_j) = C_w(t_i, t_j)$

Examples: A coin tossing sequence (discrete). Thermal resistor noise (continuous).

White Noise

Suppose that $w(t)$ is white noise and that

$$y(t) = \int_0^t w(s)ds$$

Then

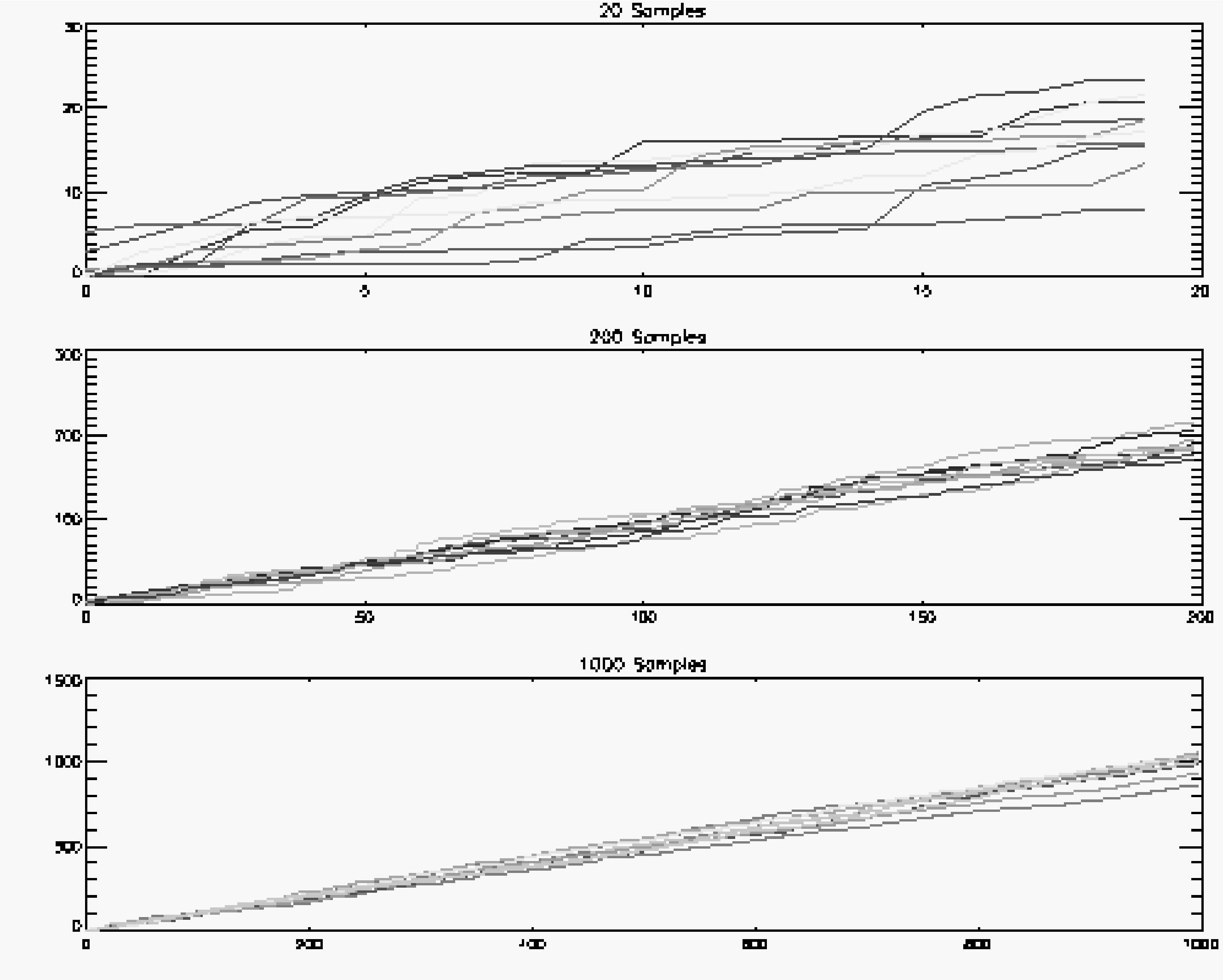
$$\begin{aligned} E[y^2(t)] &= \iint_0^t E[w(u)w(v)]dudv \\ &= \iint_0^t q(u)\delta(u-v)dudv \\ &= \int_0^t q(v)dv \end{aligned}$$

If the noise is stationary then

$$E[Y(t)] = \int_0^t \mu_w ds = \mu_w t$$

$$E[Y^2(t)] = qt$$

Plots of $y(t)$ for $t = 20, 200, 1000$



Filtered White Noise

Find the response of a linear filter with impulse response $h(t)$ to white noise.

$$R_{yy}(t_1, t_2) = \iint_{-\infty}^{\infty} R_{xx}(t_1 - \alpha, t_2 - \beta) h(\alpha) h(\beta) d\alpha d\beta$$

with $x(t) = w(t)$ we have $R_{xx}(t_1, t_2) = q\delta(t_1 - t_2)$. Then, letting $\tau = t_1 - t_2$ we have

$$\begin{aligned} R_{yy}(t_1, t_2) &= \iint_{-\infty}^{\infty} q\delta(\tau - \alpha + \beta) h(\alpha) h(\beta) d\alpha d\beta \\ &= q \int_{-\infty}^{\infty} h(\alpha) h(\alpha - \tau) d\alpha \end{aligned}$$

Because $\delta(-\tau) = \delta(\tau)$, this result is symmetric in τ.

Example

Pass white noise through a filter with the exponential impulse response $h(t) = Ae^{-bt}\text{step}(t)$.

$$\begin{aligned} R_{yy}(\tau) &= A^2 q \int_{-\infty}^{\infty} e^{-b\alpha} e^{-b(\alpha-\tau)} \text{step}(\alpha)\text{step}(\alpha-\tau) d\alpha \\ &= A^2 q e^{-b\tau} \int_{\tau}^{\infty} e^{-2b\alpha} d\alpha \\ &= \frac{A^2 q}{2b} e^{-b\tau} \text{ for } \tau \geq 0 \end{aligned}$$

Because the result is symmetric in τ,

$$R_{yy}(\tau) = \frac{A^2 q}{2b} e^{-b|\tau|}$$

Interestingly, this has the same form as the autocorrelation function of the random telegraph signal and, with the exception of the constant term, also for the Poisson pulse sequence.

Practical Calculations

Suppose that you are given a set of samples of a random waveform. Represent the samples with a vector $\mathbf{x} = [x_0, x_1, \ldots, x_{N-1}]$. It is assumed that the samples are taken at some sampling frequency $f_s = 1/T_s$ and are representative of the entire random process. That is, the process is ergodic and the set of samples is large enough.

Sample Mean: The mean value can be approximated by

$$\bar{X} = \frac{1}{N} \sum_{i=0}^{N-1} x_i$$

This computation can be represented by a vector inner (dot) product. Let $\mathbf{1} = [1, 1, \ldots, 1]$ be a vector of ones of the appropriate length. Then

$$\bar{X} = \frac{\langle \mathbf{x}, \mathbf{1} \rangle}{N}$$

Practical Calculations

Mean-squared value: In a similar manner, the mean-squared value can be approximated by

$$\overline{X^2} = \frac{1}{N} \sum_{i=0}^{N-1} x_i^2 = \frac{\langle \mathbf{x}, \mathbf{x} \rangle}{N}$$

Variance: An estimate of the variance is

$$S^2 = \frac{1}{N-1} \sum_{j=0}^{N-1} \left(x_j - \bar{X} \right)^2$$

It can be shown that

$$E[S^2] = \sigma^2$$

and is therefore an unbiased estimator of the variance.