

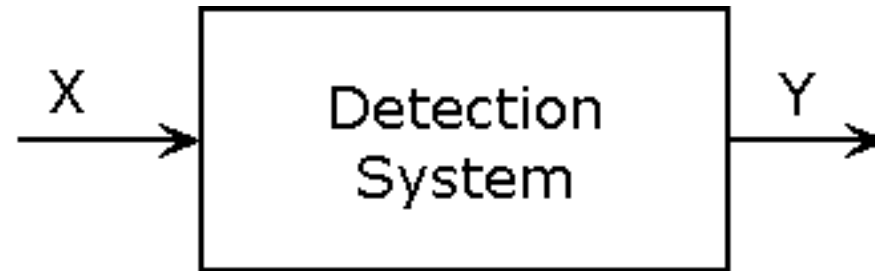
# *DQE of Image Intensifier*

Lecture 10

Spring 2002

# Detector

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An input photon stream  $X$  is presented to the detector.

$X$  is a random variable with a Poisson distribution.

$$P(X = k) = \frac{q^k}{k!} e^{-q} \text{ where } q = \lambda A \tau$$

The output is the count produced by an image intensifier.

# Intensifier System

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An intensifier system can produce more output events than there are incoming photons. However, many systems do not simply multiply the number of arrivals by some factor,  $Q$ . Instead, each input photon generates a random number of output events, where the average number of events per photon is  $Q$ .

How do we compute the DQE for such a system?

This is a very difficult problem in general, but it can be done if we assume that each incoming photon acts independently of all others.

This is also a good exercise in modeling and probability calculation.

# Intensifier Model

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Let  $f(n)$  be the probability distribution on the number of arriving photons. We will assume that  $f(n)$  is Poisson with mean value  $\lambda$ .

Suppose that each incoming photon can generate  $m$  output events, where  $m = 0, 1, 2, \dots$  is a random variable. We will assume that this distribution of  $m$  is also Poisson, with mean value  $Q$ .

The reason for this assumption is that it makes the computations tractable.

**Strategy:** Compute the distribution on the number of events produced

$$h(m) = \sum_{n=0}^{\infty} f(n)g(m|n)$$

Compute DQE by the SNR method using  $\mu_f, \sigma_f, \mu_h, \sigma_h$ .

## Intensifier (continued)

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We need to compute the probabilities  $g(m|n)$ . We know that

$$g(m|1) = g(m) = \frac{Q^m e^{-Q}}{m!}$$

We also assume that there is no output if there is no input.

$$g(m|0) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m > 0 \end{cases}$$

We can compute  $g(m|n)$  for  $n = 2, 3, 4, \dots$  by an iteration.

## Intensifier (continued)

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**Case n=2:** Let  $m = m_1 + m_2$ , where  $m_1$  the number produced by the first photon and  $m_2$  is the number produced by the second.

$$g(m|2) = \sum_{m_2=0}^2 g(m_2|1)g(m - m_2|1)$$

**Case n=3:** Let  $m = m_1 + m_2$ , where  $m_1$  the number produced by the first *two* photons and  $m_2$  is the number produced by the third.

$$g(m|3) = \sum_{m_2=0}^3 g(m_2|1)g(m - m_2|2)$$

We see a sequence of convolutions being created.

## Intensifier (continued)

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**General case  $n=k$ :** Let  $m = m_1 + m_2$ , where  $m_1$  the number produced by the first  $k - 1$  photons and  $m_2$  is the number produced by the last photon.

$$g(m|k) = \sum_{m_2=0}^k g(m_2|1)g(m - m_2|k - 1)$$

The probability of getting exactly  $m$  output events is

$$h(m) = \sum_{n=0}^{\infty} f(n)g(m|n)$$

To do these calculations it is useful to employ a generating function (equivalent to the characteristic function).

# Generating Functions

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Let

$$F(s) = \sum_{n=0}^{\infty} f(n)s^n = \sum_{n=0}^{\infty} \frac{\lambda^n s^n}{n!} e^{-\lambda} = e^{-\lambda} e^{\lambda s}$$

$$G(s) = \sum_{m=0}^{\infty} g(m)s^m = \sum_{m=0}^{\infty} \frac{Q^m s^m}{m!} e^{-Q} = e^{-Q} e^{Qs}$$

Then

$$\begin{aligned} G^2(s) &= \sum_{m_2=0}^{\infty} \sum_{m_1=0}^{\infty} g(m_2)g(m_1)s^{m_1+m_2} \\ &= \sum_{m_2=0}^{\infty} \sum_{m=m_2}^{\infty} g(m_2)g(m-m_2)s^m \end{aligned}$$

Because  $g(k) = 0$  for  $k < 0$ , we can extend the lower limit to  $m = 0$ .  
Change the order of summation to obtain

# Generating Functions (continued)

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$$G^2(s) = \sum_{m=0}^{\infty} g(m|2)s^m$$

By repeating this technique,

$$G^n(s) = \sum_{m=0}^{\infty} g(m|n)s^m$$

We can now compute  $H(s)$ .

$$\begin{aligned} H(s) &= \sum_{m=0}^{\infty} h(m)s^m = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(n)g(m|n)s^m \\ &= \sum_{n=0}^{\infty} f(n) \sum_{m=0}^{\infty} g(m|n)s^m = \sum_{n=0}^{\infty} f(n)G^n(s) \end{aligned}$$

# Completing Strategy Step 1

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The last equation reduces to

$$H(s) = F[G(s)]$$

Recall that

$$F(s) = e^{-\lambda} e^{\lambda s}$$

$$G(s) = e^{-Q} e^{Qs}$$

Hence,

$$H(s) = F[G(s)] = e^{-\lambda} e^{\lambda G(s)} = e^{-\lambda} e^{\lambda e^{-Q} e^{Qs}}$$

We can now compute the necessary moments by differentiation.

# Computing the Moments

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$$\begin{aligned}\mu_h &= \sum_{n=0}^{\infty} nh(n) = \sum_{n=0}^{\infty} nh(n)s^{n-1} \Big|_{s=0} \\ &= \frac{dH(s)}{ds} \Big|_{s=0}\end{aligned}$$

We need to carry out the differentiation.

$$\frac{dH(s)}{ds} = \frac{d}{ds} F[G(s)] = F'(G)G'(s)$$

$$F'(G) = \frac{d}{dG} (e^{-\lambda} e^{\lambda G}) = \lambda e^{-\lambda} e^{\lambda G}$$

$$G'(s) = \frac{d}{ds} (e^{-Q} e^{Qs}) = Q e^{-Q} e^{Qs}$$

# Computing the Moments

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$$H'(s) = \lambda Q e^{-\lambda} (e^{-Q} e^{Qs}) e^{\lambda(e^{-Q} e^{Qs})}$$

$$\mu_h = \lambda Q e^{-\lambda} (e^{-Q} e^{Qs}) e^{\lambda(e^{-Q} e^{Qs})} \Big|_{s=1} = \lambda Q$$

This represents the output signal. The corresponding input signal is  $\lambda$ , so that the gain is  $Q$ , which is as expected.

We find the output noise by computing the second moment

$$\begin{aligned} \sigma_h^2 &= \sum_{n=1}^{\infty} n^2 h(n) - \left( \sum_{n=1}^{\infty} n h(n) \right)^2 \\ &= \sum_{n=1}^{\infty} (n^2 - n) h(n) + \sum_{n=1}^{\infty} n h(n) - \left( \sum_{n=1}^{\infty} n h(n) \right)^2 \\ &= H''(s) \Big|_{s=1} + H'(s) \Big|_{s=1} - \left( H'(s) \Big|_{s=1} \right)^2 \end{aligned}$$

# Getting that Second Moment

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We need to compute

$$H''(s) = \frac{d}{ds}F'(G)G'(s) = F''(G)(G'(s))^2 + F'(G)G''(s)$$

$$F'(G) = \lambda e^{-\lambda} e^{\lambda G} \Rightarrow F''(G) = \lambda^2 e^{-\lambda} e^{\lambda G}$$

$$G'(s) = Q e^{-Q} e^{Qs} \Rightarrow G''(s) = Q^2 e^{-Q} e^{Qs}$$

Therefore,

$$H''(1) = F''(G(1))(G'(1))^2 + F'(G(1))G''(1) = \lambda^2 Q^2 + \lambda Q^2$$

We can now compute the output variance.

$$\begin{aligned}\sigma^2 &= (\lambda^2 Q^2 + \lambda Q^2) + \lambda Q - (\lambda Q)^2 \\ &= \lambda Q(Q + 1)\end{aligned}$$

## Finally, the Intensifier DQE

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$$\begin{aligned} DQE &= \frac{(\text{SNR}_{out})^2}{(\text{SNR}_{in})^2} \\ &= \frac{\mu_h^2/\sigma_h^2}{\mu_f^2/\sigma_f^2} \\ &= \frac{(\lambda Q)^2/(\lambda Q(1+Q))}{\lambda^2/\lambda} \\ &= \frac{Q}{1+Q} \end{aligned}$$

We see that no matter how large the intensification factor, the DQE cannot exceed 1.0, although that value is approached with large  $Q$ .

## Example: HW 4 Problems 7-10

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A certain photomultiplier emits  $N$  electrons when it is struck by one photon. The number  $N$  is a random variable with the probability distribution given by the table below.

$N$	0	1	2	3
$P(N)$	.1	.2	.3	.4

The detector is actually struck by  $X$  photons per second, with  $E[X] = 4$ , which leads to the production of  $Y$  electrons. What is  $E[Y]$ , the expected number of electron emissions per second?

# Discussion

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$E[Y]$  can be calculated in two ways:

$$E[Y] = \sum_{y \in S_y} y P[Y = y] = \sum_{x \in S_x} h(x) P[X = x]$$

The second form is easiest if we can determine the transfer function,  $h(x)$ . Unfortunately, this is difficult for this problem. Therefore, we proceed by finding  $P(Y)$ . This can be done by

$$P[Y = y] = \sum_{x \in S_x} P[x = X] P[y|x]$$

We know that  $X$  has a Poisson distribution. We will assume that the detector produces  $y = 0$  electrons when there are  $x = 0$  photons. Thus,  $P[y|0] = \delta(y)$ . When  $x = 1$  the probability distribution on  $Y$  is given by  $P(N)$  above.

$Y$	0	1	2	3
$P(Y 1)$	.1	.2	.3	.4

# Calculation Plan

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We will assume that two or more photons act independently.

For  $x \geq 2$  we have  $Y = Y_1 + Y_2 + \dots + Y_x$  where  $Y_i$  is the number of electrons generated by the  $i^{\text{th}}$  photon acting independently.

Each of the terms in the sum has the probabilities in the table above. The probability distribution for the sum, which yields  $P(y|x)$ , is found by convolving the  $P(Y|1) = P(N)$  distribution with itself  $x$  times.

Once  $P(Y = y)$  has been computed, one can easily compute  $E[Y]$  by

$$P[Y = y] = \sum_{x \in S_x} P[x = X] P[y|x]$$

An IDL program that computes the various probabilities is listed below.

# IDL Program

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```
FUNCTION PROB7,q,pp,py,pe
;Set the number of values to compute for p[X=k]. The number needed
;depends on q. A reasonable number is about 3 sigma above the mean.
nv=ceil(q+3*sqrt(q))

;Set the electron distribution per photon. This is the essence of the
;model for the detector.
pn=[0.1,0.2,0.3,0.4]

;Construct an array that will hold the p(y|x) values, where x is the
;row index and y is the column index. The maximum number of y values
;is 3*nv+1.
py=fltarr(3*nv+1,nv+1)
```

## IDL Program (cont)

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```
;When x=0 the only possibility is y=0. Hence,  $p(0|0)=1$ .  
py[0,0]=1
```

```
;When x=1,  $p(y|1)$  corresponds to the vector pe.  
py[0:3,1]=pn
```

```
;Fill in the rest of the rows by convolving.  
FOR k=2,nv DO $  
    py[0:3*k,k]=convolve(pn,py[0:3*(k-1),k-1])
```

```
;Compute the distribution on photon arrivals assuming  
;average rate q. All probabilities  $p(x)$  represented by  
;the vector pp can be computed at once.
```

```
k=findgen(nv+1)  
pp=q^k*exp(-q)/factorial(k)
```

## IDL Program (cont)

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```
;Multiply p(x)p(y|x) and sum over x to get p(y) for each
;value of y. This is easily done by array multiplication
pe=pp##py

;Compute the mean value by multiplying y*p(y) and summing.
;Again, easily done with array multiplication.
av=findgen(3*nv+1)##transpose(pe)
return,av
END
```

The average is computed by

```
muy4=prob7(4,pp,py,pe)
```

Some of the results are presented below.

# Results: Input Photon Distribution

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The input photon distribution is given by the probability vector  $p_p$ . This is listed in the table below.

$X$	0	1	2	3	4	5
$P(X)$	0.0183	0.0733	0.1465	0.1954	0.1954	0.1563
$X$	6	7	8	9	10	
$P(X)$	0.1042	0.0595	0.0298	0.0132	0.0053	

## Results: $P(Y|X)$

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A portion of the results are shown in the table below.

$y x$	0.0000	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000
0	1.0000	0.1000	0.0100	0.0010	0.0001	0.0000	0.0000
1	0.0000	0.2000	0.0400	0.0060	0.0008	0.0001	0.0000
2	0.0000	0.3000	0.1000	0.0210	0.0036	0.0006	0.0001
3	0.0000	0.4000	0.2000	0.0560	0.0120	0.0022	0.0004
4	0.0000	0.0000	0.2500	0.1110	0.0310	0.0069	0.0013
5	0.0000	0.0000	0.2400	0.1740	0.0648	0.0177	0.0040
6	0.0000	0.0000	0.1600	0.2190	0.1124	0.0383	0.0103
7	0.0000	0.0000	0.0000	0.2040	0.1608	0.0704	0.0228
8	0.0000	0.0000	0.0000	0.1440	0.1905	0.1109	0.0437
9	0.0000	0.0000	0.0000	0.0640	0.1840	0.1497	0.0736
10	0.0000	0.0000	0.0000	0.0000	0.1376	0.1720	0.1086

## Results: $P(X, Y)$

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The joint probabilities are calculated by  $P(X, Y) = P(X)P(Y|X)$ . Each column of the previous array is multiplied by the corresponding value of  $P(X)$ . The value of  $P(Y)$  in the last column is found by summing the rows.

$Y X$	0.0000	1.0000	2.0000	3.0000	4.0000	5.0000	$P(Y)$
0	0.0183	0.0073	0.0015	0.0002	0.0000	0.0000	0.0273
1	0.0000	0.0147	0.0059	0.0012	0.0002	0.0000	0.0219
2	0.0000	0.0220	0.0147	0.0041	0.0007	0.0001	0.0415
3	0.0000	0.0293	0.0293	0.0109	0.0023	0.0003	0.0723
4	0.0000	0.0000	0.0366	0.0217	0.0061	0.0011	0.0656
5	0.0000	0.0000	0.0352	0.0340	0.0127	0.0028	0.0851
6	0.0000	0.0000	0.0234	0.0428	0.0220	0.0060	0.0954
7	0.0000	0.0000	0.0000	0.0399	0.0314	0.0110	0.0851
8	0.0000	0.0000	0.0000	0.0281	0.0372	0.0173	0.0882
9	0.0000	0.0000	0.0000	0.0125	0.0359	0.0234	0.0814
10	0.0000	0.0000	0.0000	0.0000	0.0269	0.0269	0.0685

## Mean Value $E[Y]$ when $E[X] = 4$

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The mean value is calculated by multiplying the values of  $Y$  (first column) with the values of  $P(Y)$  (last column) and summing.

The result is  $E[Y] = 7.935$

The mean-squared value  $E[Y^2]$  is calculated by summing  $Y^2P(Y)$ .  
The result is  $E[Y^2] = 82.47$

$$\text{var}[Y] = E[Y^2] - E^2[Y] = 82.47 - 7.935^2 = 19.5$$

## Results for $E[X] = 5$

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The computations are done in the manner described above.

The mean response is  $E[Y] = 9.945$

The mean-squared response is  $E[Y^2] = 123.49$

The variance is  $\text{var}[Y] = E[Y^2] - E^2[Y] = 123.49 - 9.945^2 = 24.58$

The gain is the difference in mean values:  $g = 9.945 - 7.935 = 2.01$ .

We can calculate  $DQE$  at both input levels and compare:

$$DQE = \frac{g^2 \text{var}[X]}{\text{var}[Y]} \Big|_{E[X]=4} = \frac{(2.01)^2 \times 4}{19.5} = 0.83$$

$$DQE = \frac{g^2 \text{var}[X]}{\text{var}[Y]} \Big|_{E[X]=5} = \frac{(2.01)^2 \times 5}{24.58} = 0.82$$