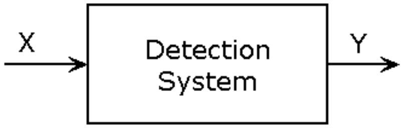
DQE of Image Intensifier

Lecture 10

Spring 2002

Detector


X

Detection System

Y

An input photon stream X is presented to the detector.

X is a random variable with a Poisson distribution.

$$P(X=k)=\frac{q^k}{k!}e^{-q} \text{ where } q=\lambda A\tau$$

The output is the count produced by an image intensifier.

Intensifier System

An intensifier system can produce more output events than there are incoming photons. However, many systems do not simply multiply the number of arrivals by some factor, Q. Instead, each input photon generates a random number of output events, where the average number of events per photon is Q.

How do we compute the DQE for such a system?

This is a very difficult problem in general, but it can be done if we assume that each incoming photon acts independently of all others.

This is also a good exercise in modeling and probability calculation.

Intensifier Model

Let $f(n)$ be the probability distribution on the number of arriving photons. We will assume that $f(n)$ is Poisson with mean value λ.

Suppose that each incoming photon can generate m output events, where $m=0,1,2,\ldots$ is a random variable. We will assume that this distribution of m is also Poisson, with mean value Q.

The reason for this assumption is that it makes the computations tractable.

Strategy: Compute the distribution on the number of events produced

$$h(m)=\sum_{n=0}^{\infty} f(n)g(m|n)$$

Compute DQE by the SNR method using μ_f, σ_f, μ_h, σ_h.

Intensifier (continued)

We need to compute the probabilities $g(m|n)$. We know that

$$g(m|1) = g(m) = \frac{Q^m e^{-Q}}{m!}$$

We also assume that there is no output if there is no input.

$$g(m|0) = \begin{cases} 1 & \text{if} \quad m = 0 \\ 0 & \text{if} \quad m > 0 \end{cases}$$

We can compute $g(m|n)$ for $n = 2, 3, 4, \ldots$ by an iteration.

Intensifier (continued)

Case n=2: Let $m = m_1 + m_2$, where m_1 the number produced by the first photon and m_2 is the number produced by the second.

$$g(m|2) = \sum_{m_2=0}^{2} g(m_2|1)g(m - m_2|1)$$

Case n=3: Let $m = m_1 + m_2$, where m_1 the number produced by the first *two* photons and m_2 is the number produced by the third.

$$g(m|3) = \sum_{m_2=0}^{3} g(m_2|1)g(m - m_2|2)$$

We see a sequence of convolutions being created.

Intensifier (continued)

General case n=k: Let $m = m_1 + m_2$, where m_1 the number produced by the first $k-1$ photons and m_2 is the number produced by the last photon.

$$g(m|k) = \sum_{m_2=0}^{k} g(m_2|1)g(m - m_2|k-1)$$

The probability of getting exactly m output events is

$$h(m) = \sum_{n=0}^{\infty} f(n)g(m|n)$$

To do these calculations it is useful to employ a generating function (equivalent to the characteristic function).

Generating Functions

Let

$$F(s) = \sum_{n=0}^{\infty} f(n)s^n = \sum_{n=0}^{\infty} \frac{\lambda^n s^n}{n!} e^{-\lambda} = e^{-\lambda} e^{\lambda s}$$

$$G(s) = \sum_{m=0}^{\infty} g(m)s^m = \sum_{m=0}^{\infty} \frac{Q^m s^m}{m!} e^{-Q} = e^{-Q} e^{Qs}$$

Then

$$\begin{aligned} G^2(s) &= \sum_{m_2=0}^{\infty} \sum_{m_1=0}^{\infty} g(m_2)g(m_1)s^{m_1+m_2} \\ &= \sum_{m_2=0}^{\infty} \sum_{m=m_2}^{\infty} g(m_2)g(m-m_2)s^m \end{aligned}$$

Because $g(k) = 0$ for $k < 0$, we can extend the lower limit to $m = 0$. Change the order of summation to obtain

Generating Functions (continued)

$$G^2(s) = \sum_{m=0}^{\infty} g(m|2)s^m$$

By repeating this technique,

$$G^n(s) = \sum_{m=0}^{\infty} g(m|n)s^m$$

We can now compute $H(s)$.

$$\begin{aligned} H(s) &= \sum_{m=0}^{\infty} h(m)s^m = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(n)g(m|n)s^m \\ &= \sum_{n=0}^{\infty} f(n) \sum_{m=0}^{\infty} g(m|n)s^m = \sum_{n=0}^{\infty} f(n)G^n(s) \end{aligned}$$

Completing Strategy Step 1

The last equation reduces to

$$H(s) = F[G(s)]$$

Recall that

$$\begin{aligned} F(s) &= e^{-\lambda}e^{\lambda s} \\ G(s) &= e^{-Q}e^{Qs} \end{aligned}$$

Hence,

$$H(s) = F[G(s)] = e^{-\lambda}e^{\lambda G(s)} = e^{-\lambda}e^{\lambda e^{-Q}e^{Qs}}$$

We can now compute the necessary moments by differentiation.

Computing the Moments

$$\begin{aligned} \mu_h &= \sum_{n=0}^{\infty} nh(n) = \sum_{n=0}^{\infty} nh(n)s^{n-1}\Bigg|_{s=0} \\ &= \frac{dH(s)}{ds}\Bigg|_{s=0} \end{aligned}$$

We need to carry out the differentiation.

$$\frac{dH(s)}{ds} = \frac{d}{ds}F[G(s)] = F'(G)G'(s)$$

$$F'(G) = \frac{d}{dG}\left(e^{-\lambda}e^{\lambda G}\right) = \lambda e^{-\lambda}e^{\lambda G}$$

$$G'(s) = \frac{d}{ds}\left(e^{-Q}e^{Qs}\right) = Qe^{-Q}e^{Qs}$$

Computing the Moments

$$H'(s) = \lambda Q e^{-\lambda}\left(e^{-Q}e^{Qs}\right)e^{\lambda\left(e^{-Q}e^{Qs}\right)}$$

$$\mu_h = \lambda Q e^{-\lambda}\left(e^{-Q}e^{Qs}\right)e^{\lambda\left(e^{-Q}e^{Qs}\right)}\Bigg|_{s=1} = \lambda Q$$

This represents the output signal. The corresponding input signal is λ, so that the gain is Q, which is as expected.

We find the output noise by computing the second moment

$$\begin{aligned} \sigma_h^2 &= \sum_{n=1}^{\infty} n^2h(n) - \left(\sum_{n=1}^{\infty} nh(n)\right)^2 \\ &= \sum_{n=1}^{\infty}(n^2-n)h(n) + \sum_{n=1}^{\infty} nh(n) - \left(\sum_{n=1}^{\infty} nh(n)\right)^2 \\ &= H''(s)\Big|_{s=1} + H'(s)\Big|_{s=1} - \left(H'(s)\Big|_{s=1}\right)^2 \end{aligned}$$

Getting that Second Moment

We need to compute

$$\begin{aligned} H''(s) &= \frac{d}{ds}F'(G)G'(s) = F''(G)(G'(s))^2 + F'(G)G''(s) \\ F'(G) &= \lambda e^{-\lambda}e^{\lambda G} \Rightarrow F''(G) = \lambda^2 e^{-\lambda}e^{\lambda G} \\ G'(s) &= Qe^{-Q}e^{Qs} \Rightarrow G''(s) = Q^2 e^{-Q}e^{Qs} \end{aligned}$$

Therefore,

$$H''(1) = F''(G(1))(G'(1))^2 + F'(G(1))G''(1) = \lambda^2 Q^2 + \lambda Q^2$$

We can now compute the output variance.

$$\begin{aligned} \sigma^2 &= \left(\lambda^2 Q^2 + \lambda Q^2\right) + \lambda Q - (\lambda Q)^2 \\ &= \lambda Q(Q+1) \end{aligned}$$

Finally, the Intensifier DQE

$$\begin{aligned} DQE &= \frac{(\mathrm{SNR}_{out})^2}{(\mathrm{SNR}_{in})^2} \\ &= \frac{\mu_h^2/\sigma_h^2}{\mu_f^2/\sigma_f^2} \\ &= \frac{(\lambda Q)^2/(\lambda Q(1+Q))}{\lambda^2/\lambda} \\ &= \frac{Q}{1+Q} \end{aligned}$$

We see that no matter how large the intensification factor, the DQE cannot exceed 1.0, although that value is approached with large Q.

Example: HW 4 Problems 7-10

A certain photomultiplier emits N electrons when it is struck by one photon. The number N is a random variable with the probability distribution given by the table below.

N	0	1	2	3
$P(N)$	.1	.2	.3	.4

The detector is actually struck by X photons per second, with $E[X] = 4$, which leads to the production of Y electrons. What is $E[Y]$, the expected number of electron emissions per second?

Discussion

$E[Y]$ can be calculated in two ways:

$$E[Y] = \sum_{y \in S_y} yP[Y=y] = \sum_{x \in S_x} h(x)P[X=x]$$

The second form is easiest if we can determine the transfer function, $h(x)$. Unfortunately, this is difficult for this problem. Therefore, we proceed by finding $P(Y)$. This can be done by

$$P[Y=y] = \sum_{x \in S_x} P[x=X]P[y|x]$$

We know that X has a Poisson distribution. We will assume that the detector produces $y = 0$ electrons when there are $x = 0$ photons. Thus, $P[y|0] = \delta(y)$. When $x = 1$ the probability distribution on Y is given by $P(N)$ above.

Y	0	1	2	3
$P(Y\|1)$	.1	.2	.3	.4

Calculation Plan

We will assume that two or more photons act independently.

For $x \geq 2$ we have $Y = Y_1 + Y_2 + \cdots + Y_x$ where Y_i is the number of electrons generated by the i^{th} photon acting independently.

Each of the terms in the sum has the probabilities in the table above. The probability distribution for the sum, which yields $P(y|x)$, is found by convolving the $P(Y|1) = P(N)$ distribution with itself x times.

Once $P(Y = y)$ has been computed, one can easily compute $E[Y]$ by

$$P[Y = y] = \sum_{x \in S_x} P[x = X]\, P[y|x]$$

An IDL program that computes the various probabilities is listed below.

IDL Program

```
FUNCTION PROB7,q,pp,py,pe
;Set the number of values to compute for p[X=k]. The number needed
;depends on q. A reasonable number is about 3 sigma above the mean.
nv=ceil(q+3*sqrt(q))

;Set the electron distribution per photon. This is the essence of the
;model for the detector.
pn=[0.1,0.2,0.3,0.4]

;Construct an array that will hold the p(y|x) values, where x is the
;row index and y is the column index. The maximum number of y values
;is 3*nv+1.
py=fltarr(3*nv+1,nv+1)
```

IDL Program (cont)

```
;When x=0 the only possibility is y=0. Hence,p(0|0)=1.
py[0,0]=1

;When x=1, p(y|1) corresponds to the vector pe.
py[0:3,1]=pn

;Fill in the rest of the rows by convolving.
FOR k=2,nv DO $
    py[0:3*k,k]=convolve(pn,py[0:3*(k-1),k-1])

;Compute the distribution on photon arrivals assuming
;average rate q. All probabilities p(x) represented by
;the vector pp can be computed at once.
k=findgen(nv+1)
pp=q^k*exp(-q)/factorial(k)
```

IDL Program (cont)

```
;Multiply p(x)p(y|x) and sum over x to get p(y) for each
;value of y. This is easily done by array multiplication
pe=pp##py

;Compute the mean value by multiplying y*p(y) and summing.
;Again, easily done with array multiplication.
av=findgen(3*nv+1)##transpose(pe)
return,av
END
```

The average is computed by

```
muy4=prob7(4,pp,py,pe)
```

Some of the results are presented below.

Results: Input Photon Distribution

The input photon distribution is given by the probability vector `pp`. This is listed in the table below.

X	0	1	2	3	4	5
$P(X)$	0.0183	0.0733	0.1465	0.1954	0.1954	0.1563
X	6	7	8	9	10	
$P(X)$	0.1042	0.0595	0.0298	0.0132	0.0053	

Results: $P(Y|X)$

A portion of the results are shown in the table below.

$y\|x$	0.0000	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000
0	1.0000	0.1000	0.0100	0.0010	0.0001	0.0000	0.0000
1	0.0000	0.2000	0.0400	0.0060	0.0008	0.0001	0.0000
2	0.0000	0.3000	0.1000	0.0210	0.0036	0.0006	0.0001
3	0.0000	0.4000	0.2000	0.0560	0.0120	0.0022	0.0004
4	0.0000	0.0000	0.2500	0.1110	0.0310	0.0069	0.0013
5	0.0000	0.0000	0.2400	0.1740	0.0648	0.0177	0.0040
6	0.0000	0.0000	0.1600	0.2190	0.1124	0.0383	0.0103
7	0.0000	0.0000	0.0000	0.2040	0.1608	0.0704	0.0228
8	0.0000	0.0000	0.0000	0.1440	0.1905	0.1109	0.0437
9	0.0000	0.0000	0.0000	0.0640	0.1840	0.1497	0.0736
10	0.0000	0.0000	0.0000	0.0000	0.1376	0.1720	0.1086

Results: $P(X,Y)$

The joint probabilities are calculated by $P(X,Y) = P(X)P(Y|X)$. Each column of the previous array is multiplied by the corresponding value of $P(X)$. The value of $P(Y)$ in the last column is found by summing the rows.

$Y\|X$	0.0000	1.0000	2.0000	3.0000	4.0000	5.0000	$P(Y)$
0	0.0183	0.0073	0.0015	0.0002	0.0000	0.0000	0.0273
1	0.0000	0.0147	0.0059	0.0012	0.0002	0.0000	0.0219
2	0.0000	0.0220	0.0147	0.0041	0.0007	0.0001	0.0415
3	0.0000	0.0293	0.0293	0.0109	0.0023	0.0003	0.0723
4	0.0000	0.0000	0.0366	0.0217	0.0061	0.0011	0.0656
5	0.0000	0.0000	0.0352	0.0340	0.0127	0.0028	0.0851
6	0.0000	0.0000	0.0234	0.0428	0.0220	0.0060	0.0954
7	0.0000	0.0000	0.0000	0.0399	0.0314	0.0110	0.0851
8	0.0000	0.0000	0.0000	0.0281	0.0372	0.0173	0.0882
9	0.0000	0.0000	0.0000	0.0125	0.0359	0.0234	0.0814
10	0.0000	0.0000	0.0000	0.0000	0.0269	0.0269	0.0685

Mean Value $E[Y]$ when $E[X] = 4$

The mean value is calculated by multiplying the values of Y (first column) with the values of $P(Y)$ (last column) and summing.

The result is $E[Y] = 7.935$

The mean-squared value $E[Y^2]$ is calculated by summing $Y^2P(Y)$. The result is $E[Y^2] = 82.47$

$\text{var}[Y] = E[Y^2] - E^2[Y] = 82.47 - 7.935^2 = 19.5$

Results for $E[X] = 5$

The computations are done in the manner described above.

The mean response is $E[Y] = 9.945$
The mean-squared response is $E[Y^2] = 123.49$
The variance is $\text{var}[Y] = E[Y^2] - E^2[Y] = 123.49 - 9.945^2 = 24.58$

The gain is the difference in mean values: $g = 9.945 - 7.935 = 2.01$.
We can calculate DQE at both input levels and compare:

$$DQE = \left.\frac{g^2\texttt{var}[X]}{\texttt{var}[Y]}\right|_{E[X]=4} = \frac{(2.01)^2 \times 4}{19.5} = 0.83$$

$$DQE = \left.\frac{g^2\texttt{var}[X]}{\texttt{var}[Y]}\right|_{E[X]=5} = \frac{(2.01)^2 \times 5}{24.58} = 0.82$$