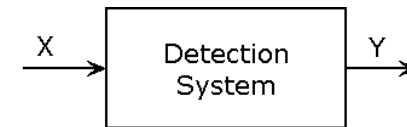


## Detective Quantum Efficiency

Lecture 9

Spring 2002

## Detector



An input photon stream  $X$  is presented to the detector.

$X$  is a random variable with a Poisson distribution.

$$P(X = k) = \frac{q^k}{k!} e^{-q} \text{ where } q = \lambda A \tau$$

The response is an event stream  $Y$ .

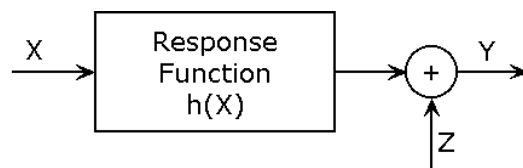
$$Y = h(X) + Z$$

**Goal:** Determine the intensity  $\lambda$  of the input photon stream.

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## Detector Model



$$Y = h(X) + Z$$

$h(x)$  is a known detector response function

$Z$  is an independent random variable with a known probability distribution.

### Questions:

1. How can we estimate  $q$  or  $\lambda$  based on the observation  $Y$ ?
2. How do we describe the quality of the estimate?

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## Estimator

An estimator is a rule to calculate the value of a quantity of interest from one or more observations of a random variable.

An estimate is a function of a random variable, and is therefore itself a random variable.

Let  $\tilde{\lambda}$  be an estimate of  $\lambda$ . We would like to have  $E[\tilde{\lambda}] = \lambda$ . This is an *unbiased estimator* of  $\lambda$ .

We would also like to have  $\text{var}[\tilde{\lambda}]$  be as small as possible.

The quality of the detector is measured by comparing the quality of the estimate that could be made from  $X$  to the quality of the estimate that can be made from  $Y$ .

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## Detective Quantum Efficiency

DQE can be defined in a number of equivalent forms. The basic definition that we will use is–

**Definition:** The DQE for a detector is the ratio of the variance of an estimate of  $\hat{\lambda}$  based on the detector input  $X$  to the variance of an estimate of  $\bar{\lambda}$  based on the detector output  $Y$ .

$$DQE = \frac{\text{var}[\hat{\lambda}]}{\text{var}[\bar{\lambda}]}$$

The DQE will always be in the range 0 to 1.

## Estimate based on $X$

Pretend that  $X$  is available and construct an unbiased estimate of  $\lambda$ .

The mean value of the input Poisson distribution is

$$E[X] = q = \lambda A\tau \Rightarrow \lambda = \frac{E[X]}{A\tau}$$

Given an observation  $X$ , construct an estimate

$$\hat{\lambda} = \frac{X}{A\tau}$$

This estimator is unbiased, since

$$E[\hat{\lambda}] = \frac{E[X]}{A\tau} = \frac{q}{A\tau} = \lambda$$

The variance of the estimator is

$$\text{var}[\hat{\lambda}] = \frac{\text{var}[X]}{A^2\tau^2} = \frac{q}{A^2\tau^2} = \frac{\lambda}{A\tau}$$

## Estimate based on $Y$

To estimate  $q$  or  $\lambda$  based on  $Y$  we need an equation of the form

$$\bar{q} = w(Y)$$

To find this function, begin with the inverse function and then invert.

From a previous lecture,

$$E[Y] = \mu_Y(q) = L(1 - f_1(q, T, S)e^{-q})$$

Given any chosen “operating point”  $q_0$  we can write an expression for  $\mu_Y(q)$  in terms of  $\mu_Y(q_0)$ .

$$\mu_Y(q) = \mu_Y(q_0) + \left. \frac{d\mu_Y}{dq} \right|_{q_0} (q - q_0) + \dots$$

## Estimate (continued)

This gives us a hint at the estimator equation. Replace  $\mu_Y(q)$  by  $Y$  and  $q$  by the estimate  $\bar{q}$  and drop higher order terms.

$$Y = \mu_Y(q_0) + \left. \frac{d\mu_Y}{dq} \right|_{q_0} (\bar{q} - q_0)$$

The derivative is the detector gain at the operating point. Solve for the estimate as a function of  $Y$

$$\bar{q} = q_0 + \frac{Y - \mu_Y(q_0)}{g} = w(Y)$$

The function  $\bar{q} = w(Y)$  is an unbiased estimator of  $q$  near  $q = q_0$ .

An unbiased estimate of  $\lambda$  is

$$\bar{\lambda} = \frac{\bar{q}}{A\tau} = \frac{w(Y)}{A\tau g}$$

## Estimator (continued)

The variance of  $\bar{\lambda}$  is

$$\text{var}[\bar{\lambda}] = \frac{\text{var}[Y]}{(A\tau g)^2}$$

From the results of the last lecture, the variance of a piecewise linear detector is

$$\text{var}[Y] = L^2 [(1 - f_3 e^{-q}) - (1 - f_1 e^{-q})^2]$$

The gain is given by

$$g(q_0) = \left. \frac{d\mu_Y}{dq} \right|_{q=q_0} = L e^{-q_0} f_2(L, q_0)$$

where

$$f_2(L, q) = \frac{1}{L} \sum_{k=0}^{L-1} \frac{q^k}{k!}$$

## DQE

The DQE of a piecewise linear detector can be calculated by

$$\begin{aligned} DQE &= \frac{\text{var}[\bar{\lambda}]}{\text{var}[\bar{\lambda}]} \\ &= \frac{(A\tau g)^2 \lambda / A\tau}{L^2 [(1 - f_3 e^{-q}) - (1 - f_1 e^{-q})^2]} \\ &= \frac{g^2 q}{L^2 [(1 - f_3 e^{-q}) - (1 - f_1 e^{-q})^2]} \end{aligned}$$

This is the same result as that obtained in the last lecture for the *comparative noise level* when the variance of  $Z$  is zero.

## Example—Linear Amplifier

Consider a system in which  $Y = gX + Z$ , where  $X$  is a Poisson r.v. with mean  $q = \lambda A\tau$ ,  $g$  is known, and  $Z$  is an independent r.v. with zero mean and variance  $\sigma_Z^2$ .

An estimate of  $\lambda$  based on  $X$  would be

$$\hat{\lambda} = \frac{X}{A\tau}$$

The estimate has variance

$$\text{var}[\hat{\lambda}] = \frac{\text{var}[X]}{A^2 \tau^2} = \frac{\lambda}{A\tau}$$

An estimate based on  $Y$  is

$$\bar{\lambda} = \frac{Y}{A\tau g}$$

## Linear Amplifier (continued)

This estimate has variance

$$\text{var}[\bar{\lambda}] = \frac{\text{var}[Y]}{A^2 \tau^2 g^2} = \frac{g^2 \sigma_X^2 + \sigma_Z^2}{A^2 \tau^2 g^2} = \frac{\lambda}{A\tau} + \frac{\sigma_Z^2}{A^2 \tau^2 g^2}$$

The DQE is

$$\begin{aligned} DQE &= \frac{\text{var}[\hat{\lambda}]}{\text{var}[\bar{\lambda}]} = \frac{\frac{\lambda}{A\tau}}{\frac{\lambda}{A\tau} + \frac{\sigma_Z^2}{A^2 \tau^2 g^2}} \\ &= \frac{A\tau \lambda g^2}{A\tau \lambda g^2 + \sigma_Z^2} = \frac{A\tau \lambda}{A\tau \lambda + \frac{\sigma_Z^2}{g^2}} = \frac{1}{1 + \frac{\sigma_Z^2}{g^2 q}} \end{aligned}$$

The DQE can be viewed from either the input or output perspective.

The DQE for a linear amplifier is close to unity when  $\sigma_Z/g$  is small compared to  $q = A\tau\lambda$ . Increasing  $g$  improves the DQE.

## Example–Secondary Emission

Each input photon produces an output event with probability  $\eta$ . If  $q_X = \lambda A\tau$  then  $q_Y = \eta\lambda A\tau$  and  $E[Y] = \eta\lambda A\tau$

The output process is still Poisson, so that

$$\text{var}(Y) = E[Y] = \eta\lambda A\tau$$

The gain is  $g = \eta A\tau$ .

The ideal estimate is  $\hat{\lambda} = X/A\tau$  so that

$$\text{var}[\hat{\lambda}] = \text{var}[X]/A^2\tau^2 = \lambda/A\tau$$

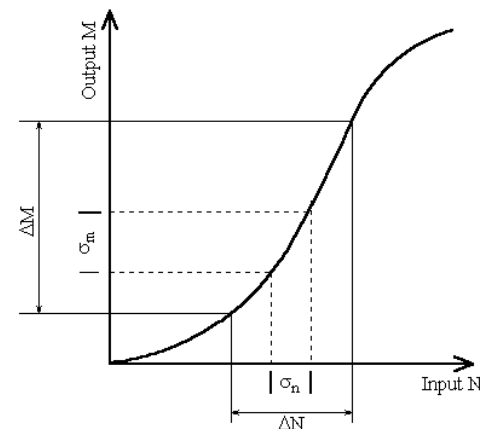
Then

$$DQE = \frac{\text{var}(\hat{\lambda})}{\text{var}(\lambda)} = \frac{\lambda/A\tau}{\eta\lambda A\tau} (\eta A\tau)^2 = \eta$$

## DQE and SNR

DQE may also be defined as the ratio of the (power) SNR at the detector output to the maximum possible SNR.

$$\begin{aligned} \text{SNR}_{in} &= \frac{\Delta N}{\sigma_n} \\ \text{SNR}_{out} &= \frac{\Delta M}{\sigma_m} \\ DQE &= \frac{(\text{SNR}_{out})^2}{(\text{SNR}_{in})^2} \\ &= \frac{(\Delta M)^2/\sigma_m^2}{(\Delta N)^2/\sigma_n^2} \end{aligned}$$



## Linear Amplifier Revisited

Input Signal= $E[N] = q$

Input Noise= $\sigma_n = \sqrt{q}$

Output Signal= $E[M] = gq$

Output Noise= $\sigma_m = \sqrt{g^2q + \sigma_z^2}$

$$\begin{aligned} DQE &= \frac{(\Delta M)^2/\sigma_m^2}{(\Delta N)^2/\sigma_n^2} \\ &= \frac{(gq)^2/(g^2q + \sigma_z^2)}{(q/\sqrt{q})^2} \\ &= \frac{g^2q}{g^2q + \sigma_z^2} = \frac{1}{1 + \frac{\sigma_z^2}{g^2q}} \end{aligned}$$

which is equivalent to our previous result.

## Noisy Detector

A photon detector has a dark noise whose variance  $\sigma_D^2 = 0.1$  counts/second is proportional to the exposure time and a readout noise whose variance  $\sigma_R^2 = 10$  counts is at a fixed level. The detector area is  $A = 0.3$  cm<sup>2</sup> and the detector conversion efficiency is  $\epsilon = 0.75$  counts/photon.

Find the DQE for when the detector is placed in a flux of  $\Phi = 3.3$  photons/cm<sup>2</sup>/s and exposed for  $T = 10$  minutes.

## Input Signal & Noise

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The input signal is the expected number of photons to the detector.  
This is

$$S_{in} = q = \Phi AT = (3.3)(0.3)(10)(60) = 594 \text{ photons}$$

The standard deviation of the photon stream is the input noise.

$$N_{in} = \sqrt{q} = 24.4 \text{ photons}$$

The input SNR is

$$\left(\frac{S}{N}\right)_{in} = \frac{594}{24.4} = 24.4$$

## Output Signal & Noise

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The output signal is the input signal times the conversion efficiency

$$S_{out} = q\epsilon = 5940.75 = 445 \text{ counts}$$

The output noise has three contributors—the Poisson noise in the detected counts, the dark noise and the readout noise.

$$\sigma_{N_{out}}^2 = q\epsilon + \sigma_D^2 T + \sigma_R^2 = 445 + 0.1 \times 600 + 10^2 = 615$$

The standard deviation of the output noise is

$$\sigma_{N_{out}} = 24.8 \text{ counts}$$

The output SNR is then

$$\left(\frac{S}{N}\right)_{out} = \frac{445}{24.8} = 18$$

## DQE of Noisy Detector

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The DQE can be computed by

$$DQE = \left[\frac{(S/N)_{out}}{(S/N)_{in}}\right]^2 = \left(\frac{18}{24.4}\right)^2 = 0.54$$