Photon Detection

Lectures 8

Spring 2002

Photon Detection

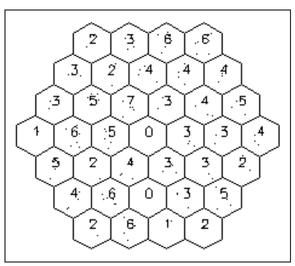
- The goal is a simple model for photon detection.
- The detector model will have an onset threshold and a saturation level.
- The photon stream will be treated as a Poisson process.
- The ability to detect and measure changes in the photon stream will be analyzed.
- The performance measures – detective quantum efficiency (DQE) and noise equivalent power (NEP) will be introduced.

Array of Photon Detectors

We will use a simple model with a close-packed array of N detectors, each with area area A, and identical response functions. This simple model will be adequate to demonstrate the basic principles. The expected number of photons at cell k is

$$q_k = I_k A_k T$$

where A_k is the area of cell k, I_k is the intensity of the photon beam at cell k, and T is the exposure time.

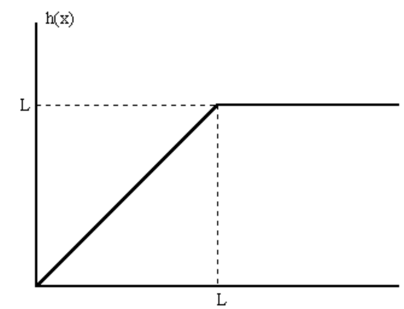


Response Function

Let X be the number of photons that arrive at a detector. Let the detector count all photons up to a level L. The response function is is $Y = h(X)$, where

$$h(x) = \begin{cases} x, & x \leq L-1 \\ L, & x \geq L \end{cases}$$

We will later add an onset threshold to this basic function.



Expected Detector Response

The response Y of a detector in the array is a random variable that is measured every T seconds.

The number of photons that arrive at a detector element a Poisson random variable X with mean and standard deviation $q = IAT$.

The expected value of the response is

$$\mu_Y = E[Y] = \sum_{k=0}^{\infty} P[X = k]h(k)$$

Insert the model of $h(k)$.

$$\mu_Y = \sum_{k=0}^{L-1} \frac{kq^k e^{-q}}{k!} + \sum_{k=L}^{\infty} L\frac{q^k e^{-q}}{k!}$$

Expected Detector Response

$$\mu_Y = \sum_{k=0}^{L-1} \frac{kq^k e^{-q}}{k!} + \sum_{k=L}^{\infty} L\frac{q^k e^{-q}}{k!}$$

Expand and rearrange.

$$\begin{aligned} \mu_Y &= \textstyle\sum_{k=1}^{\infty} \frac{q^k e^{-q}}{k!} + \sum_{k=2}^{\infty} \frac{q^k e^{-q}}{k!} + \sum_{k=3}^{\infty} \frac{q^k e^{-q}}{k!} + \\ & \cdots + \textstyle\sum_{k=L}^{\infty} \frac{q^k e^{-q}}{k!} \end{aligned}$$

There are L terms. Term r is of the form

$$\begin{aligned} \sum_{k=r}^{\infty} \frac{q^k e^{-q}}{k!} &= \sum_{k=0}^{\infty} \frac{q^k e^{-q}}{k!} - \sum_{k=0}^{r-1} \frac{q^k e^{-q}}{k!} \\ &= 1 - \sum_{k=0}^{r-1} \frac{q^k e^{-q}}{k!} \end{aligned}$$

Expected Detector Response

$$\begin{aligned} \mu_Y &= (1 - e^{q}) + \left(1 - \textstyle\sum_{k=0}^{1} \frac{q^k e^{-q}}{k!}\right) + \left(1 - \textstyle\sum_{k=0}^{2} \frac{q^k e^{-q}}{k!}\right) + \\ & \cdots + \left(1 - \textstyle\sum_{k=0}^{L-1} \frac{q^k e^{-q}}{k!}\right) \end{aligned}$$

Gather terms and rearrange.

$$\mu_Y = L - e^{-q}\left(1 + \sum_{k=0}^{1} \frac{q^k}{k!} + \sum_{k=0}^{2} \frac{q^k}{k!} + \cdots + \sum_{k=0}^{L-1} \frac{q^k}{k!}\right)$$

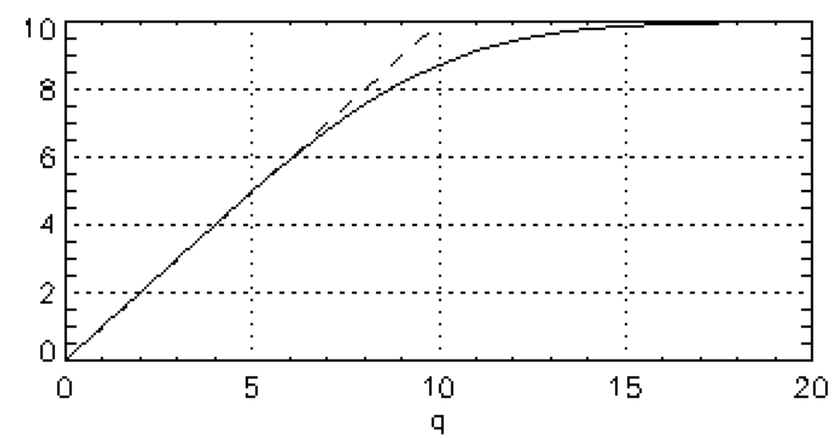
Define a function

$$f_1(L,q) = \frac{1}{L}\left(1 + \sum_{k=0}^{1} \frac{q^k}{k!} + \sum_{k=0}^{2} \frac{q^k}{k!} + \cdots + \sum_{k=0}^{L-1} \frac{q^k}{k!}\right)$$

$$\mu_Y = L\left(1 - f_1(q,L)e^{-q}\right)$$

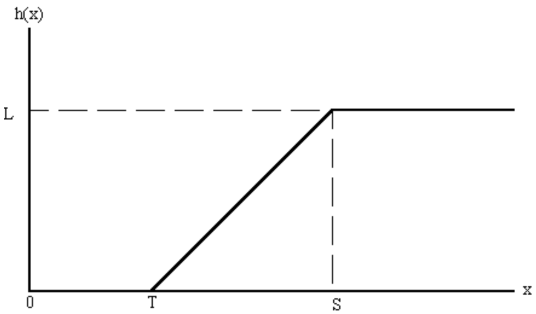
Expected Detector Response

The response μ_Y as a function of q for a saturation level of $L = 10$.



Threshold=T & Saturation Level: $S = L + T - 1$

$$h(x) = \begin{cases} 0, & x < T \\ x - T + 1, & T \le x < L + T \\ L, & x \ge L + T \end{cases}$$



Calculation of μ_Y

The equation for the average count now becomes

$$\mu_Y = \sum_{k=0}^{\infty} P[X = k]h(k)$$

$$\mu_Y = \left(\sum_{k=T}^{L+T-2} (k - T + 1) \frac{q^k e^{-q}}{k!} \right) + \left(\sum_{k=L+T-1}^{\infty} L \frac{q^k e^{-q}}{k!} \right)$$

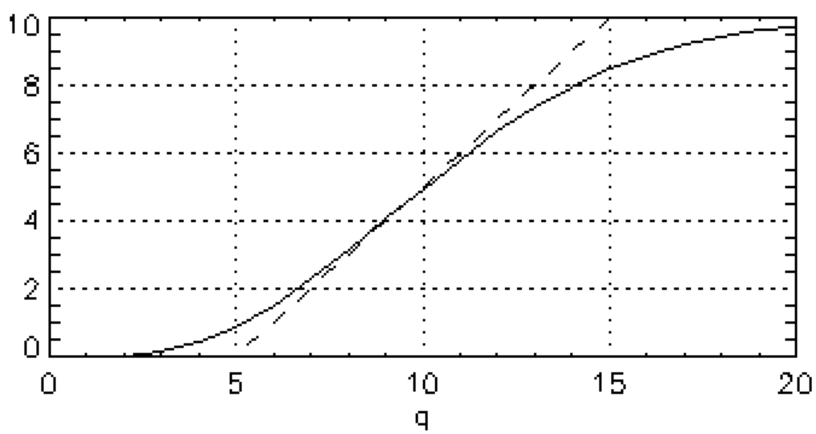
Modify the function f_1 so that

$$f_1(L, q) = \frac{1}{L} \left(\sum_{k=0}^{T-1} \frac{q^k}{k!} + \sum_{k=0}^{T} \frac{q^k}{k!} + \sum_{k=0}^{T+1} \frac{q^k}{k!} + \cdots + \sum_{k=0}^{L+T-1} \frac{q^k}{k!} \right)$$

Response function has the same form:

$$\mu_Y = L \left(1 - f_1(q, L) e^{-q} \right)$$

Detector Response μ_Y with $T = 5$, $L = 10$



Contrast

Contrast is determined by the variation in the output when the input changes. The change must be measured at an operating point q_0:

$$\mu_Y(q_0 + \Delta q) = \mu_Y(q_0) + g(q_0)\Delta q$$

and $g(q_0)$ is the gradient at the operating point.

$$g(q_0) = \left. \frac{d\mu_Y}{dq} \right|_{q=q_0} = L e^{-q_0} \left(f_1 - \frac{df_1}{dq} \right)_{q=q_0}$$

Contrast

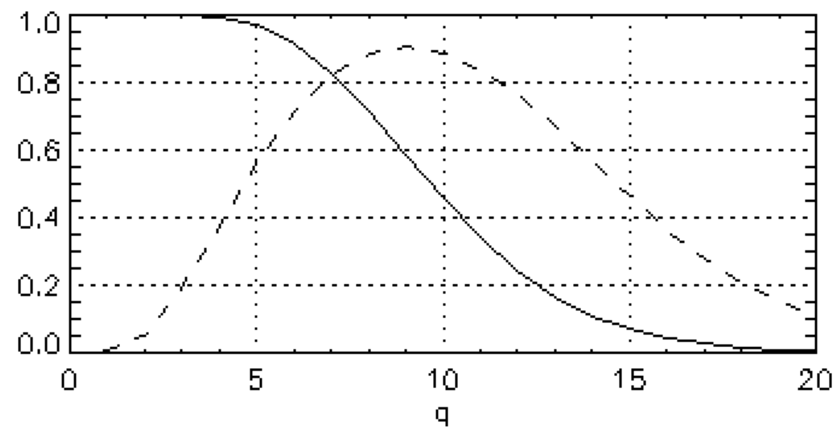
$$f_1(L,q) = \frac{1}{L}\left(\sum_{k=0}^{T-1}\frac{q^k}{k!} + \sum_{k=0}^{T}\frac{q^k}{k!} + \sum_{k=0}^{T+1}\frac{q^k}{k!} + \cdots + \sum_{k=0}^{L+T-1}\frac{q^k}{k!}\right)$$

$$\frac{df_1(L,q)}{dq} = \frac{1}{L}\left(\sum_{k=0}^{T-2}\frac{q^k}{k!} + \sum_{k=0}^{T-1}\frac{q^k}{k!} + \sum_{k=0}^{T}\frac{q^k}{k!} + \cdots + \sum_{k=0}^{L+T-2}\frac{q^k}{k!}\right)$$

The derivative of $q^k/k!$ is $q^{k-1}/(k-1)!$. Effectively, each term is reduced in length by one.

If we now subtract the derivative from f_1 all that remains is the last term in each sum.

$$f_2(L,q) = \left(f_1 - \frac{df_1}{dq}\right) = \frac{1}{L}\sum_{k=0}^{L-1}\frac{q^k}{k!}$$



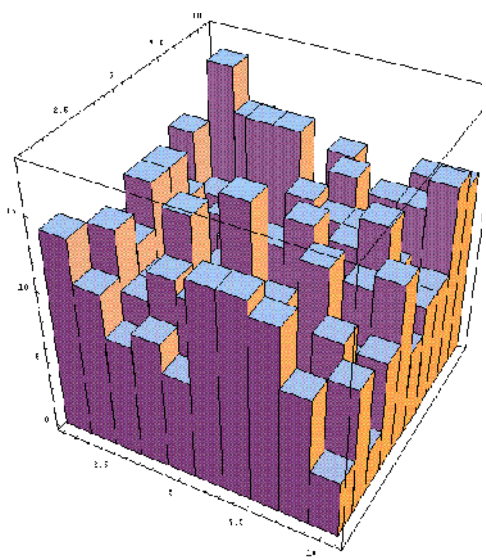
Solid curve: $L = 10$, $T = 0$; dashed curve: $L = 10$, $T = 5$. These correspond to the response curves in the earlier figures.

Comparative Noise Level

The cells in a region of the detector that is under uniform quantum exposure will exhibit statistical differences in their responses because different numbers of quanta fall upon them.

μ_Y represents the mean detector response.

The mean-squared value can be found by a similar analysis.



Mean-squared Detector Response

$$\begin{aligned} E[Y^2] &= \sum_{k=0}^{\infty} h^2(k)P[X=k] \\ &= \sum_{k=0}^{L-1} k^2\frac{q^k e^{-q}}{k!} + \sum_{k=L}^{\infty} L^2\frac{q^k e^{-q}}{k!} \end{aligned}$$

This can be expressed in terms of a function $f_3(L,q)$ as

$$E[Y^2] = L^2\left(1 - f_3 e^{-q}\right)$$

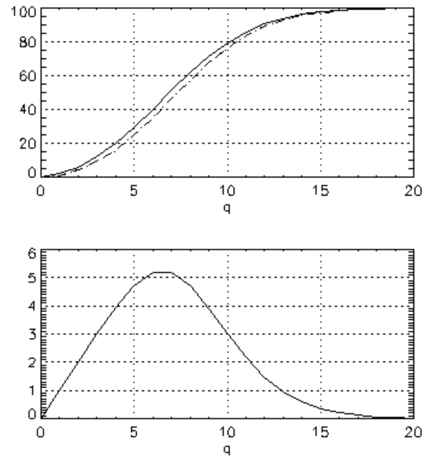
where

$$\begin{aligned} f_3(L,q) &= \frac{1}{L^2}\left[1 + 3\sum_{k=0}^{1}\frac{q^k}{k!} + 5\sum_{k=0}^{2}\frac{q^k}{k!} + \cdots + (2L-1)\sum_{k=0}^{L-1}\frac{q^k}{k!}\right] \\ &= \frac{1}{L^2}\sum_{n=0}^{L-1}(2n+1)\sum_{k=0}^{n}\frac{q^k}{k!} \end{aligned}$$

Variance of the Detector Response

$$\sigma_Y^2 = E[Y^2] - \mu_Y^2$$

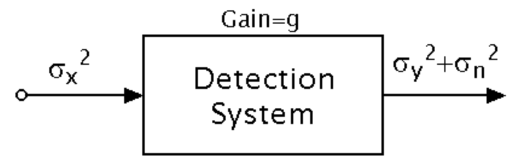
$$= L^2\Big[(1 - f_3 e^{-q}) - (1 - f_1 e^{-}$$



The upper figure shows $E[Y^2]$ (solid) and μ_Y^2 (dashed) for a quantum detector with $L = 10$ and $T = 0$. The lower curve shows the variance, which is the difference between the two curves in the upper figure.

Comparative Noise Level

The action of the detector and internally produced noise contribute to the variance of the detector output.



The detector noise performance can be described by the comparative noise level.

$$\varepsilon = \frac{g^2\sigma_X^2}{\sigma_Y^2 + \sigma_n^2}$$

Comparative Noise Level

The variance of the input is $\sigma_X^2 = q$. The gain is the response gradient

$$g(q) = Le^{-q} f_2(L, q)$$

The variance of the output in the absence of internal noise is

$$\sigma_Y^2 = L^2\left[(1 - f_3 e^{-q}) - (1 - f_1 e^{-q})^2\right]$$

The comparative noise level is

$$\varepsilon = \frac{g^2\sigma_X^2}{\sigma_Y^2 + \sigma_n^2}$$

$$= \frac{q\left(Le^{-q} f_2(L, q)\right)^2}{L^2\left[(1 - f_3 e^{-q}) - (1 - f_1 e^{-q})^2\right] + \sigma_n^2}$$

The second law of thermodynamics requires that the output noise be at least as large as the noise that is actually present at the input, so that the comparative noise level is always less than unity.

Example