Poisson and Normal Distributions

Lectures 7

Spring 2002

Poisson Distribution

- The Poisson* distribution can be derived as a limiting form of the binomial distribution in which n is increased without limit as the product $\lambda = np$ is kept constant.
- This corresponds to conducting a very large number of Bernoulli trials with the probability p of success on any one trial being very small.
- The Poisson distribution can also be derived directly in a manner that shows how it can be used as a model of real situations. In this sense, it stands alone and is independent of the binomial distribution.

* Siméon D. Poisson, (1781-1840).

Conceptual Model

Imagine that you are able to observe the arrival of photons at a detector. Your detector is designed to count the number of photons that arrive in an interval $\Delta\tau$.

The Poisson distribution is based on three basic assumptions, listed on the next slide.

The generality of the assumptions permits the model to be used in analyzing many kinds of systems.

Poisson Assumptions

1. The probability of one photon arriving in $\Delta\tau$ is proportional to $\Delta\tau$ when $\Delta\tau$ is very small.

   $$P(1;\Delta\tau) = a\Delta\tau \quad \text{for small } \Delta\tau$$

   where a is a constant whose value is not yet determined.

2. The probability that more than one photon arrives in $\Delta\tau$ is negligible when $\Delta\tau$ is very small.

   $$P(0;\Delta\tau) + P(1;\Delta\tau) = 1 \quad \text{for small } \Delta\tau$$

3. the number of photons that arrive in one interval is independent of the number of photons that arrive in any other non-overlapping interval.

Derivation: Step 1

Find the probability that 0 photons arrive in an interval τ.

The probability that a zero photons arrive in τ is equal to the probability that zero photons arrive in $\tau - \Delta\tau$ and no photons arrive in $\Delta\tau$. Since the intervals do not overlap, the events are independent and

$$P(0;\tau) = P(0;\tau - \Delta\tau)P(0;\Delta\tau)$$

If we use assumptions 1 and 2 and rearrange we find

$$\frac{P(0;\tau) - P(0;\tau - \Delta\tau)}{\Delta\tau} = -aP(0;\tau - \Delta\tau)$$

Step 1 (continued)

This leads to the differential equation

$$\frac{dP(0;\tau)}{d\tau} = -aP(0;\tau)$$

The solution is

$$P(0;\tau) = Ce^{-a\tau}$$

When we apply the boundary condition $\lim_{\tau \to 0} P(0;\tau) = 1$ we find $C = 1$, so that

$$P(0;\tau) = e^{-a\tau}$$

Derivation: Step 2

Consider next the probability that k photons arrive in interval $\tau + \Delta\tau$. There are only two possibilities.

1. k arrive in τ and 0 arrive in $\Delta\tau$
2. $k-1$ arrive in τ and 1 arrives in $\Delta\tau$.

Since these events are mutually exclusive,

$$P(k;\tau + \Delta\tau) = P(k;\tau)P(0;\Delta\tau) + P(k-1;\tau)P(1;\Delta\tau)$$

Now substitute for $P(0;\Delta\tau)$ and $P(1;\Delta\tau)$ and rearrange.

$$\frac{P(k;\tau + \Delta\tau) - P(k;\tau)}{\Delta\tau} + aP(k;\tau) = aP(k-1;\tau)$$

Step 2 (continued)

In the limit we have the differential equation

$$\frac{dP(k;\tau)}{d\tau} + aP(k;\tau) = aP(k-1;\tau)$$

This is a recursive equation that ties $P(k;\tau)$ to $P(k-1;\tau)$. To solve it we need to convert it into something we can integrate. Multiply through by $e^{a\tau}$:

$$e^{a\tau}\frac{dP(k;\tau)}{d\tau} + ae^{a\tau}P(k;\tau) = ae^{a\tau}P(k-1;\tau)$$

The term on the left can be expressed as a total derivative

$$\frac{d}{d\tau}\left(e^{a\tau}P(k;\tau)\right) = ae^{a\tau}P(k-1;\tau)$$

Step 2 (continued)

$$\frac{d}{d\tau}\left(e^{a\tau}P(k;\tau)\right) = ae^{a\tau}P(k-1;\tau)$$

Integrate with respect to τ

$$e^{a\tau}P(k;\tau) = \int_0^\tau ae^{at}P(k-1;t)dt + C$$

Note that $P(k;0) = 0$ so that the constant of integration is $C = 0$.

$$P(k;\tau) = ae^{-a\tau}\int_0^\tau e^{at}P(k-1;t)dt$$

Derivation: Step 3

Apply a recursion starting with $k = 1$ to obtain $P(1;\tau)$.

$$\begin{aligned} P(1;\tau) &= ae^{-a\tau}\int_0^\tau e^{at}P(0;t)dt \\ &= ae^{-a\tau}\int_0^\tau e^{at}e^{-at}dt \\ &= ae^{-a\tau}\int_0^\tau dt \\ &= a\tau e^{-a\tau} \end{aligned}$$

Then do the recursion again to obtain $P(2;\tau)$, and so on.

Ultimately conclude that the Poisson distribution can be expressed as

$$P(k;\tau) = \frac{(a\tau)^k e^{-a\tau}}{k!}$$

Expected Value of k

The expected number of photons in τ can be obtained by finding the first moment.

$$E[k] = \sum_{k=0}^{\infty} \frac{k(a\tau)^k e^{-a\tau}}{k!} = a\tau$$

If τ corresponds to time in seconds then a corresponds to the average rate of photon arrival in photons per second.

The quantity $a\tau$ corresponds to the parameter $\lambda = np$ of the binomial distribution.

Average Number of Events

The expected number of photons in the n intervals is $np = \lambda = a\tau$. Hence, $\lambda = a\tau$.

When the rate a is given and the interval τ is fixed, it is common to write the Poisson distribution as

$$P(k;\lambda) = \frac{(\lambda)^k e^{-\lambda}}{k!}$$

CF of Poisson Distribution

The characteristic function is

$$\begin{aligned} M(iu) &= E[e^{iu}] \\ &= \sum_{k=0}^{\infty} P[k,\lambda]e^{iuk} \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} e^{iuk} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\left(\lambda e^{iu}\right)^k}{k!} \\ &= e^{\lambda\left(e^{iu}-1\right)} \end{aligned}$$

Variance of the Poisson Distribution

$$\text{var}(k) = E[k^2] - E^2[k] = a\tau$$

The calculation is left as an exercise.

Note that the mean and the variance of a Poisson distribution are equal to each other.

Normal Distribution

The function defined by

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

is called the normal probability density function.

The normal probability distribution function is

$$\mathcal{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt$$

Error Function

The distribution function is often presented in a slightly different form This form, which is called the "error function" is

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du$$

This form of the error function is built into the IDL language as the function ERRORF. In the exercises you are asked to show that the distribuition function and the error function are related by

$$\mathcal{Q}(x) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2}}\right)$$

Relationship to the Binomial Distribution

Let S_n be the number of successes in n Bernoulli trials. The probability that the number of successes is between two values, a and b,

$$P[a \le S_n \le b] = \sum_{r=a}^{b} b[n, r, p]$$

The following theorem states that this probability can be computed by use of the normal distribution.

(DeMoivre-Laplace limit theorem) Let $m = np$ and $\sigma = \sqrt{np(1-p)}$. For fixed values of parameters z_1 and z_2, as $n \to \infty$,

$$P[m + z_1\sigma \le S_n \le m + z_2\sigma] \to \mathcal{Q}(z_2) - \mathcal{Q}(z_1)$$

Z-Score

The parameters z_1 and z_2 are distances from the mean measured in units of σ. If we define a normalized random variable

$$Z_n = \frac{S_n - np}{\sqrt{np(1-p)}}$$

we have the equivalent probability relationship

$$P[z_1 \le Z_n \le z_2] \to \mathcal{Q}(z_2) - \mathcal{Q}(z_1)$$

In terms of the error function is

$$P[z_1 \le Z_n \le z_2] \to \frac{1}{2}\text{erf}\left(\frac{z_2}{\sqrt{2}}\right) - \frac{1}{2}\text{erf}\left(\frac{z_2}{\sqrt{2}}\right)$$

Comparison of Distributions

A comparison of the binomial, Poisson and normal probability functions for $n = 1000$ and $p = 0.1$, 0.3, 0.5. The normal and Poisson functions agree well for all of the values of p, and agree with the binomial function for $p = 0.1$.