Averages of Random Variables

Lecture 5

Spring 2002

Averages of Random Variables

Suppose that a random variable U can take on any one of L random values, say $u_{1,}u_2, \ldots u_L$. Imagine that we make n independent observations of U and that the value u_k is observed n_k times, $k = 1, 2, \ldots, L$. Of course, $n_1 + n_2 + \cdots + n_L = n$. The emperical average can be computed by

$$\bar{\mathcal{U}} = \frac{1}{n} \sum_{k=1}^{L} n_k u_k = \sum_{k=1}^{L} \frac{n_k}{n} u_k$$

The concept of statistical averages extends from this simple concept

Expected Value

The expected value of a discrete random variable $\mathcal{U}$ is defined by

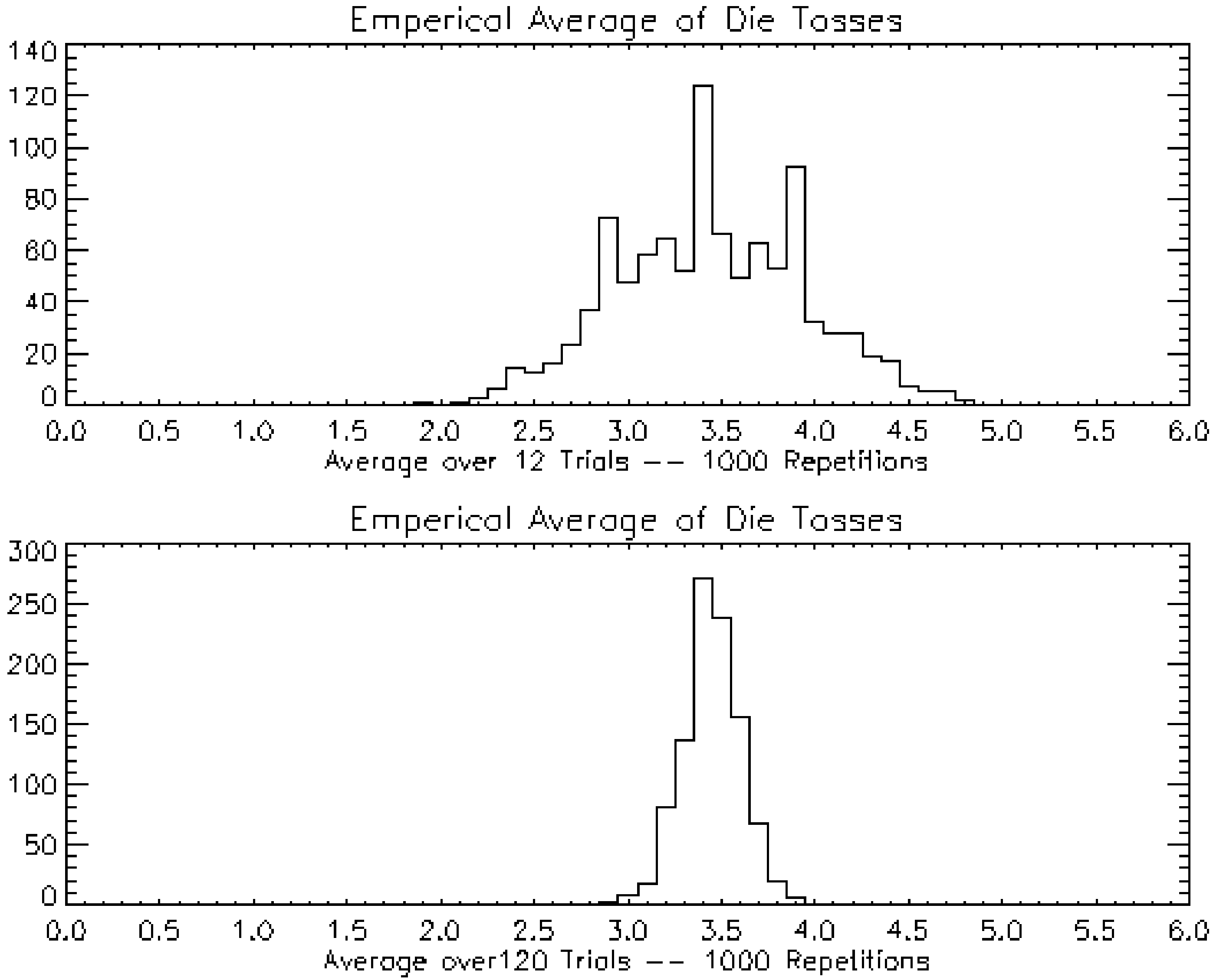
$$E[\mathcal{U}] = \sum_{\mathcal{U}} u_k P(u_k)$$

Note how this definition compares with the empirical average.

The empirical average $\bar{\mathcal{U}}$ is a random variable. It will have a different value with each set of sample values.

The expected value is a single number. It is a parameter of the distribution. How do you think the expected value and the random variable $\bar{\mathcal{U}}$ relate?

Die Tossing Example



Function of a Discrete Random Variable

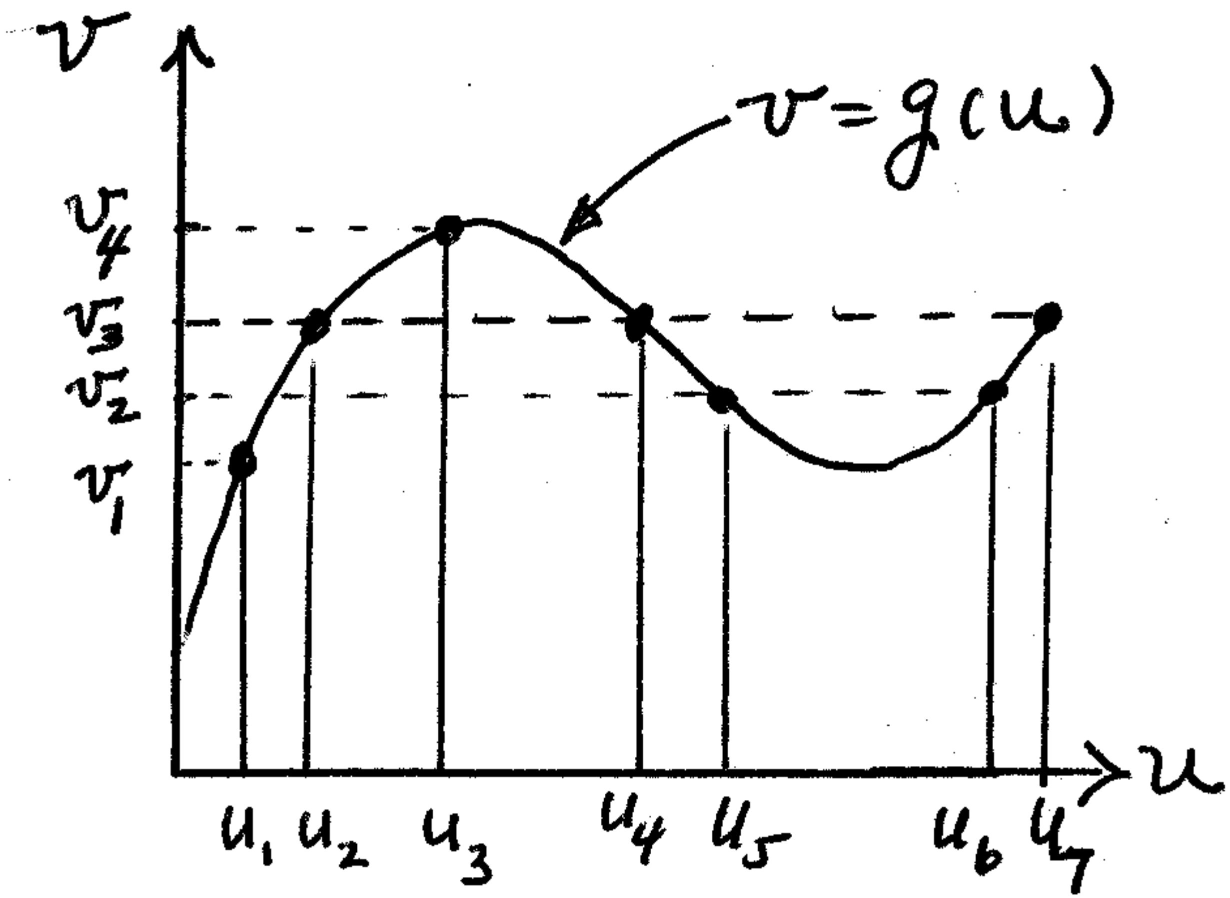
Let U be a discrete random variable and let $V = g(U)$ Suppose that we want to compute the average value of V rather than of U.

Given that U can take on the values u_k with probabilities $P[U = u_k]$, $k = 1, 2, \ldots, L$, we can compute both the values and probabilities that can be taken on by V.

Let $v_1, v_2, \ldots, v_r$ be the set of values that can be assumed by V and let $P[V = v_j]$, $j = 1, 2, \ldots r$ be the corresponding probabilities. Then

$$E[V] = \sum_{j=1}^{r} v_j P[V = v_j]$$

We need to determine the values and the probabilities.



$$
\begin{aligned}
E[V] \;=\; & v_1 P[U = u_1] + v_2 (P[U = u_5] + P[U = u_6]) \\
& + v_3 (P[U = u_2] + P[U = u_4] + P[U = u_7]) + v_4 P[U = u_3]
\end{aligned}
$$

Expected Value of $V = g(U)$

By making use of the relationship $v = g(u)$ we can now rewrite this in a form that associates $g(u_k)$ with $P[U = u_k]$. The result is the nice compact equation

$$E[V] = \sum_{k=1}^{L} g(u_k) P[U = u_k]$$

If you were given machine to generate samples of U, how would you go about estimating $E[V]$.

Continuous Random Variables

The expected value of a continuous random variable U is

$$E[U] = \int_{-\infty}^{\infty} u f_U(u) du$$

The expected value of a function of a random variable $V = g(U)$ is

$$E[V] = \int_{-\infty}^{\infty} g(u) f_U(u) du$$

It is not necessary to compute $f_V(v)$

Example

Let U be a random variable with an exponential distribution with parameter a.

$$f_U(u) = \begin{cases} ae^{-au} & \text{for} \quad u \geq 0 \\ 0 & \text{for} \quad u < 0 \end{cases}$$

To find the expected value of U we calculate the integral

$$E[U] = \int_0^\infty uae^{-au} du = \frac{1}{a}$$

The parameter a is therefore the reciprocal of the expected value. You should sketch the exponential distribution for a few values of a and convince yourself that this is a reasonable result.

Example: $V = U^2$

Let U have an exponential distribution with parameter a, and let $V = U^2$. Find $E[V] = E[U^2]$.

$$f_U(u) = \begin{cases} ae^{-au} & \text{for} \quad u \geq 0 \\ 0 & \text{for} \quad u < 0 \end{cases}$$

$$E[U^2] = \int_0^\infty u^2 ae^{-au} du = \frac{2}{a^2}$$

Mean, Variance and Standard Deviation

The mean value of a random variable U is $\mu = E[U]$.

The variance is $\sigma^2 = E[(U - \mu)^2]$.

The standard deviation is σ.

It is always true that $\sigma^2 = E[U^2] - \mu^2$

For the exponential distribution

$$\sigma^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$$

For an exponential distribution, $\sigma = \mu$.

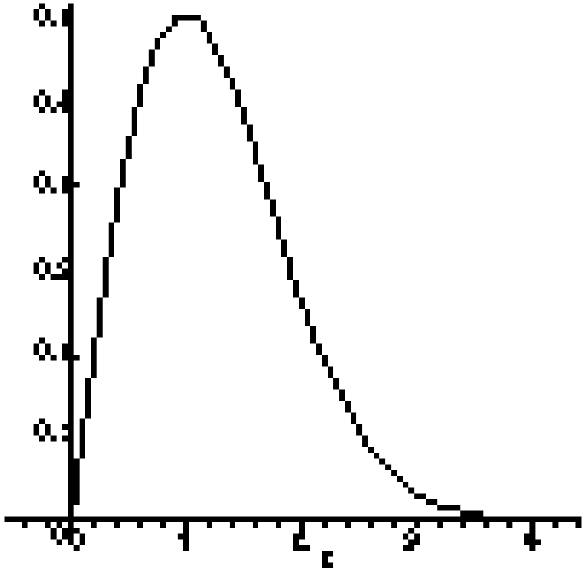
Moments

The k^{th} moment of a random variable U is defined as $E[U^k]$, and can be computed by the following formulas for the discrete and continuous cases, respectively.

$$E[U^k] = \sum_{i=1}^{L} u_i^k P[U = u_i]$$

$$E[U^k] = \int_{-\infty}^{\infty} u^k f_U(u) du$$

Example: Rayleigh Distribution



$$f_R(r) = \frac{r}{b} e^{-r^2/2b} \quad (r \geq 0)$$

$$\mu_R = E[R] = \int_0^\infty \frac{r^2}{b} e^{-r^2/2b} dr = \sqrt{\frac{\pi b}{2}}$$

$$E[R^2] = \int_0^\infty \frac{r^3}{b} e^{-r^2/2b} dr = 2b$$

$$\text{var}(R) = \sigma_R^2 = 2b - \frac{\pi b}{2} \approx 0.43b$$

The standard deviation is $\sigma_R \approx 0.655\sqrt{b}$. By a little calculus you can show that the peak of the distribution occurs at $r = \sqrt{b}$, so that the standard deviation is about 65% of the location of the peak. As b is increased the curve shifts to the right and broadens in proportion to $\sqrt{b}$.

Multivariate Functions

Let $\mathbf{U} = [U_1, U_2, \ldots, U_n]$ where U_i, $i = 1, \ldots, n$ are random variables.

Let $V = g(U)$. Then V is a function of several random variables.

$$E[V] = \int \int \cdots \int g(u_1, u_2, \ldots, u_n) f_{\mathbf{U}}(u_1, u_2, \ldots u_n) du_1 du_2 ... du_n$$

Dartboard Example

Consider the problem of throwing darts at a target. Let X and Y be random variables that represent the horizontal and vertical offset of the dart from the center of the target. Note that the center is at (0,0) so that the variables can have both negative and positive values. Suppose that X and Y are statistically independent normal random variables, with

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\text{exp}\left[-\frac{x^2}{2\sigma^2}\right]$$

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}}\text{exp}\left[-\frac{y^2}{2\sigma^2}\right]$$

Dartboard Example

Because the horizontal and vertical offsets are statistically independent, the joint distribution is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{\sigma^2 2\pi}\text{exp}\left[-\frac{x^2+y^2}{2\sigma^2}\right]$$

Find the distance the dart hits from the center. Let $R = \sqrt{X^2+Y^2}$ Substitute $r^2 = x^2+y^2$ in the probability density function.

The volume above a ring of thickness dr and mean radius r is $2\pi r f_{X,Y}(r)dr$. This must be the same value as achieved under the distribution of $f_R(r)$ over the interval dr. Therefore,

$$f_R(r)dr = 2\pi r f_{X,Y}(r)dr$$

Dartboard Example

$$f_R(r) = \frac{r}{\sigma^2}\text{exp}\left[-\frac{r^2}{2\sigma^2}\right]$$

Replace σ^2 with a parameter b. The result is

$$f_R(r) = \frac{r}{b}\text{exp}\left[-\frac{r^2}{2b}\right]$$

This is the Rayleigh probability density or Rayleigh distribution with parameter b.

What is the PDF on the angle θ?

Characteristic Functions

Characteristic Function

The characteristic function of a random variable X is the expected value of the function e^{juX} where $j = \sqrt{-1}$.

Continuous Distribution

$$M_X(ju) = E\left[e^{juX}\right] = \int_{-\infty}^{\infty} f_X(x) e^{jux} dx$$

Discrete Distribution

$$M_X(ju) = E\left[e^{juX}\right] = \sum_i f_X(x_i) e^{jux_i}$$

The similarity of the characteristic function and the Fourier transform is obvious.

A major use of the cf is in generating moments of random variables.

Characteristic Function

Since $f_X(x)$ is nonnegative and e^{jux} has unit magnitude for all values of x and u,

$$\left|\int_{-\infty}^{\infty} f_X(x)e^{jux}dx\right| \leq \int_{-\infty}^{\infty} \left|f_X(x)e^{jux}\right| dx = \int_{-\infty}^{\infty} f_X(x)dx = 1$$

Hence, the characteristic function always exists and

$$|M_X(ju)| \leq M_X(0) = 1$$

If the cf is known, then one can compute the pdf by

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M_X(ju)e^{-jux}du$$

Moment Generation

The function e^{jux} can be expanded in a Taylor series:

$$e^{juX} = \sum_{n=0}^{\infty} \frac{(juX)^n}{n!} = \sum_{n=0}^{\infty} \frac{(ju)^n}{n!} X^n$$

Hence,

$$M_X(ju) = \sum_{n=0}^{\infty} \frac{(ju)^n}{n!} E[X^n] = \sum_{n=0}^{\infty} \frac{(ju)^n}{n!} m_n$$

where m_n is the n^{th} moment of X about the origin.

Moment Generation

If we expand the characteristic function in a Taylor series, then m_n can be found from the coefficient of u^n in the expansion. Specifically, if

$$M_X(ju) = \sum_{n=0}^{\infty} k_n u^n$$

then $m_n = n!k_n/j^n$. This can be seen by comparison with

$$M_X(ju) = \sum_{n=0}^{\infty} \frac{(ju)^n}{n!} m_n$$

The moment can also be found by taking derivatives.

$$E[X^m] = \frac{1}{j^m} \left. \frac{d^m M_X}{du^m} \right|_{u=0}$$

Example: Exponential PDF

Consider the exponential pdf given by

$$f_X(x) = be^{-bx}\mathsf{step}(x)$$

The characteristic function is

$$\begin{aligned} M_X(u) &= E\left[e^{ju}\right] \\ &= \int_0^{\infty} be^{-(b-ju)x}dx \\ &= \frac{b}{b-ju} = \frac{1}{1-\frac{ju}{b}} \end{aligned}$$

Exponential PDF - Continued

Within the circle $|u| < b$ we can expand the term on the right in a power series by using (for $|t| < 1$)

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$$

Then the expansion, with $t = ju/b$, is

$$M_X(u) = \sum_{n=0}^{\infty} \left(\frac{ju}{b}\right)^n$$

In this case $k_n = (j/b)^n$ so that

$$m_n = E\left[X^n\right] = \frac{n!}{j^n} k_n = \frac{n!}{b^n}$$

Example: Binomial Distribution

The probability of k successes in n trials is governed by the binomial probability distribution

$$b(n,k,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

The cf is

$$\begin{aligned} M_X(ju) &= \sum_{k=0}^{n} b(n,k,p) e^{juk} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{juk} \\ &= \sum_{k=0}^{n} \binom{n}{k} (pe^{ju})^k (1-p)^{n-k} = (pe^{ju} + 1 - p)^n \end{aligned}$$

Here we have made use of the binomial expansion

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Binomial Distribution - continued

The mean value is found by differentiation.

$$m_1 = E[X] = \frac{1}{j}\frac{dM}{du}\bigg|_{u=0} = npe^{ju}(pe^{ju} + 1 - p)^{n-1}\Big|_{u=0} = np$$

Similarly,

$$\begin{aligned} m_2 &= E[X^2] = \frac{1}{j^2}\frac{d^2M}{du^2}\bigg|_{u=0} \\ &= n(n-1)p^2e^{ju}(pe^{ju} + 1 - p)^{n-2} + npe^{ju}(pe^{ju} + 1 - p)^{n-1}\Big|_{u=0} \\ &= n(n-1)p^2 + np \end{aligned}$$

The variance is easily found by

$$\sigma^2 = \mathrm{var}[X] = E[X^2] - E^2[X] = n(n-1)p^2 + np - n^2p^2 = np(1-p)$$