

Averages of Random Variables

Lecture 5

Spring 2002

Averages of Random Variables

Suppose that a random variable U can take on any one of L random values, say u_1, u_2, \dots, u_L . Imagine that we make n independent observations of U and that the value u_k is observed n_k times, $k = 1, 2, \dots, L$. Of course, $n_1 + n_2 + \dots + n_L = n$. The empirical average can be computed by

$$\bar{u} = \frac{1}{n} \sum_{k=1}^L n_k u_k = \sum_{k=1}^L \frac{n_k}{n} u_k$$

The concept of statistical averages extends from this simple concept

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Expected Value

The expected value of a discrete random variable \mathcal{U} is defined by

$$E[\mathcal{U}] = \sum_{\mathcal{U}} u_k P(u_k)$$

Note how this definition compares with the empirical average.

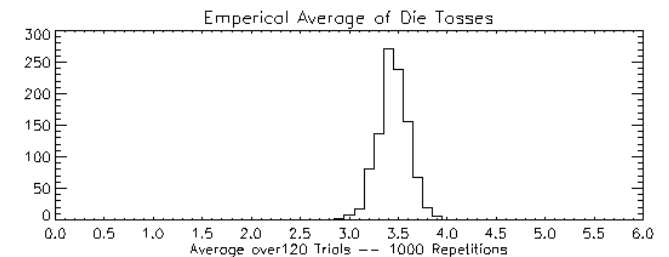
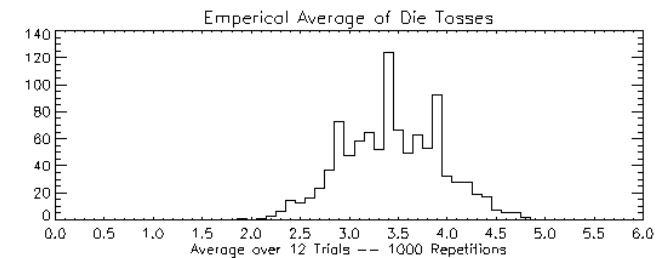
The empirical average \bar{u} is a random variable. It will have a different value with each set of sample values.

The expected value is a single number. It is a parameter of the distribution. How do you think the expected value and the random variable \bar{u} relate?

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Die Tossing Example



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Function of a Discrete Random Variable

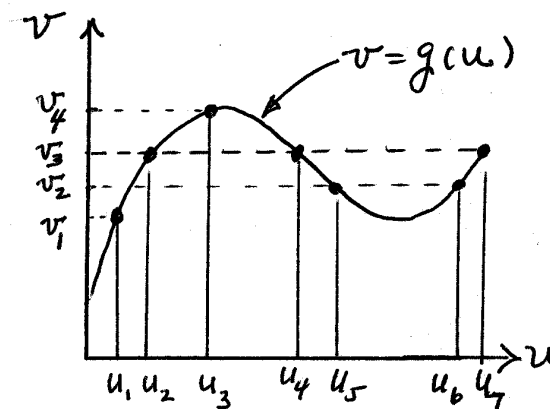
Let U be a discrete random variable and let $V = g(U)$. Suppose that we want to compute the average value of V rather than of U .

Given that U can take on the values u_k with probabilities $P[U = u_k]$, $k = 1, 2, \dots, L$, we can compute both the values and probabilities that can be taken on by V .

Let v_1, v_2, \dots, v_r be the set of values that can be assumed by V and let $P[V = v_j]$, $j = 1, 2, \dots, r$ be the corresponding probabilities. Then

$$E[V] = \sum_{j=1}^r v_j P[V = v_j]$$

We need to determine the values and the probabilities.



$$E[V] = v_1 P[U = u_1] + v_2 (P[U = u_5] + P[U = u_6]) + v_3 (P[U = u_2] + P[U = u_4] + P[U = u_7]) + v_4 P[U = u_3]$$

Expected Value of $V = g(U)$

By making use of the relationship $v = g(u)$ we can now rewrite this in a form that associates $g(u_k)$ with $P[U = u_k]$. The result is the nice compact equation

$$E[V] = \sum_{k=1}^L g(u_k) P[U = u_k]$$

If you were given machine to generate samples of U , how would you go about estimating $E[V]$.

Continuous Random Variables

The expected value of a continuous random variable U is

$$E[U] = \int_{-\infty}^{\infty} u f_U(u) du$$

The expected value of a function of a random variable $V = g(U)$ is

$$E[V] = \int_{-\infty}^{\infty} g(u) f_U(u) du$$

It is not necessary to compute $f_V(v)$

Example

Let U be a random variable with an exponential distribution with parameter a .

$$f_U(u) = \begin{cases} ae^{-au} & \text{for } u \geq 0 \\ 0 & \text{for } u < 0 \end{cases}$$

To find the expected value of U we calculate the integral

$$E[U] = \int_0^{\infty} uae^{-au} du = \frac{1}{a}$$

The parameter a is therefore the reciprocal of the expected value. You should sketch the exponential distribution for a few values of a and convince yourself that this is a reasonable result.

Example: $V = U^2$

Let U have an exponential distribution with parameter a , and let $V = U^2$. Find $E[V] = E[U^2]$.

$$f_U(u) = \begin{cases} ae^{-au} & \text{for } u \geq 0 \\ 0 & \text{for } u < 0 \end{cases}$$

$$E[U^2] = \int_0^{\infty} u^2 ae^{-au} du = \frac{2}{a^2}$$

Mean, Variance and Standard Deviation

The mean value of a random variable U is $\mu = E[U]$.

The variance is $\sigma^2 = E[(U - \mu)^2]$.

The standard deviation is σ .

It is always true that $\sigma^2 = E[U^2] - \mu^2$

For the exponential distribution

$$\sigma^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$$

For an exponential distribution, $\sigma = \mu$.

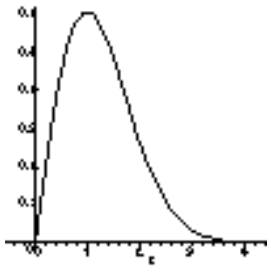
Moments

The k^{th} moment of a random variable U is defined as $E[U^k]$, and can be computed by the following formulas for the discrete and continuous cases, respectively.

$$E[U^k] = \sum_{i=1}^L u_i^k P[U = u_i]$$

$$E[U^k] = \int_{-\infty}^{\infty} u^k f_U(u) du$$

Example: Rayleigh Distribution



$$f_R(r) = \frac{r}{b} e^{-r^2/2b} \quad (r \geq 0)$$

$$\mu_R = E[R] = \int_0^\infty \frac{r^2}{b} e^{-r^2/2b} dr = \sqrt{\frac{\pi b}{2}}$$

$$E[R^2] = \int_0^\infty \frac{r^3}{b} e^{-r^2/2b} dr = 2b$$

$$\text{var}(R) = \sigma_R^2 = 2b - \frac{\pi b}{2} \approx 0.43b$$

The standard deviation is $\sigma_R \approx 0.655\sqrt{b}$. By a little calculus you can show that the peak of the distribution occurs at $r = \sqrt{b}$, so that the standard deviation is about 65% of the location of the peak. As b is increased the curve shifts to the right and broadens in proportion to \sqrt{b} .

Multivariate Functions

Let $\mathbf{U} = [U_1, U_2, \dots, U_n]$ where $U_i, i = 1, \dots, n$ are random variables.

Let $V = g(\mathbf{U})$. Then V is a function of several random variables.

$$E[V] = \int \int \dots \int g(u_1, u_2, \dots, u_n) f_{\mathbf{U}}(u_1, u_2, \dots, u_n) du_1 du_2 \dots du_n$$

Dartboard Example

Consider the problem of throwing darts at a target. Let X and Y be random variables that represent the horizontal and vertical offset of the dart from the center of the target. Note that the center is at $(0,0)$ so that the variables can have both negative and positive values. Suppose that X and Y are statistically independent normal random variables, with

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma^2}\right]$$

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{y^2}{2\sigma^2}\right]$$

Dartboard Example

Because the horizontal and vertical offsets are statistically independent, the joint distribution is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{\sigma^2 2\pi} \exp\left[-\frac{x^2 + y^2}{2\sigma^2}\right]$$

Find the distance the dart hits from the center. Let $R = \sqrt{X^2 + Y^2}$. Substitute $r^2 = x^2 + y^2$ in the probability density function.

The volume above a ring of thickness dr and mean radius r is $2\pi r f_{X,Y}(r) dr$. This must be the same value as achieved under the distribution of $f_R(r)$ over the interval dr . Therefore,

$$f_R(r) dr = 2\pi r f_{X,Y}(r) dr$$

Dartboard Example

$$f_R(r) = \frac{r}{\sigma^2} \exp\left[-\frac{r^2}{2\sigma^2}\right]$$

Replace σ^2 with a parameter b . The result is

$$f_R(r) = \frac{r}{b} \exp\left[-\frac{r^2}{2b}\right]$$

This is the Rayleigh probability density or Rayleigh distribution with parameter b .

What is the PDF on the angle θ ?

Characteristic Functions

Characteristic Function

The characteristic function of a random variable X is the expected value of the function e^{juX} where $j = \sqrt{-1}$.

Continuous Distribution

$$M_X(ju) = E[e^{juX}] = \int_{-\infty}^{\infty} f_X(x)e^{jux} dx$$

Discrete Distribution

$$M_X(ju) = E[e^{juX}] = \sum_i f_X(x_i)e^{jux_i}$$

The similarity of the characteristic function and the Fourier transform is obvious.

A major use of the cf is in generating moments of random variables.

Characteristic Function

Since $f_X(x)$ is nonnegative and e^{jux} has unit magnitude for all values of x and u ,

$$\left| \int_{-\infty}^{\infty} f_X(x)e^{jux} dx \right| \leq \int_{-\infty}^{\infty} |f_X(x)e^{jux}| dx = \int_{-\infty}^{\infty} f_X(x) dx = 1$$

Hence, the characteristic function always exists and

$$|M_X(ju)| \leq M_X(0) = 1$$

If the cf is known, then one can compute the pdf by

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_X(ju)e^{-jux} du$$

Moment Generation

The function e^{juX} can be expanded in a Taylor series:

$$e^{juX} = \sum_{n=0}^{\infty} \frac{(juX)^n}{n!} = \sum_{n=0}^{\infty} \frac{(ju)^n}{n!} X^n$$

Hence,

$$M_X(ju) = \sum_{n=0}^{\infty} \frac{(ju)^n}{n!} E[X^n] = \sum_{n=0}^{\infty} \frac{(ju)^n}{n!} m_n$$

where m_n is the n^{th} moment of X about the origin.

Moment Generation

If we expand the characteristic function in a Taylor series, then m_n can be found from the coefficient of u^n in the expansion. Specifically, if

$$M_X(ju) = \sum_{n=0}^{\infty} k_n u^n$$

then $m_n = n!k_n/j^n$. This can be seen by comparison with

$$M_X(ju) = \sum_{n=0}^{\infty} \frac{(ju)^n}{n!} m_n$$

The moment can also be found by taking derivatives.

$$E[X^m] = \frac{1}{j^m} \left. \frac{d^m M_X}{du^m} \right|_{u=0}$$

Example: Exponential PDF

Consider the exponential pdf given by

$$f_X(x) = be^{-bx} \text{step}(x)$$

The characteristic function is

$$\begin{aligned} M_X(u) &= E[e^{ju}] \\ &= \int_0^{\infty} be^{-(b-ju)x} dx \\ &= \frac{b}{b-ju} = \frac{1}{1-\frac{ju}{b}} \end{aligned}$$

Exponential PDF - Continued

Within the circle $|u| < b$ we can expand the term on the right in a power series by using (for $|t| < 1$)

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots$$

Then the expansion, with $t = ju/b$, is

$$M_X(u) = \sum_{n=0}^{\infty} \left(\frac{ju}{b}\right)^n$$

In this case $k_n = (j/b)^n$ so that

$$m_n = E[X^n] = \frac{n!}{j^n} k_n = \frac{n!}{b^n}$$

Example: Binomial Distribution

The probability of k successes in n trials is governed by the binomial probability distribution

$$b(n, k, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

The cf is

$$\begin{aligned} M_X(ju) &= \sum_{k=0}^n b(n, k, p) e^{juk} = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} e^{juk} \\ &= \sum_{k=0}^n \binom{n}{k} (pe^{ju})^k (1-p)^{n-k} = (pe^{ju} + 1 - p)^n \end{aligned}$$

Here we have made use of the binomial expansion

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Binomial Distribution - continued

The mean value is found by differentiation.

$$m_1 = E[X] = \frac{1}{j} \frac{dM}{du} \Big|_{u=0} = npe^{ju} (pe^{ju} + 1 - p)^{n-1} \Big|_{u=0} = np$$

Similarly,

$$\begin{aligned} m_2 &= E[X^2] = \frac{1}{j^2} \frac{d^2 M}{du^2} \Big|_{u=0} \\ &= n(n-1)p^2 e^{2ju} (pe^{ju} + 1 - p)^{n-2} + npe^{ju} (pe^{ju} + 1 - p)^{n-1} \Big|_{u=0} \\ &= n(n-1)p^2 + np \end{aligned}$$

The variance is easily found by

$$\sigma^2 = \text{var}[X] = E[X^2] - E^2[X] = n(n-1)p^2 + np - n^2p^2 = np(1-p)$$