Functions of Random Variables

Lecture 4

Spring 2002

Function of a Random Variable

Let U be an random variable and $V = g(U)$. Then V is also a rv since, for any outcome e, $V(e) = g(U(e))$.

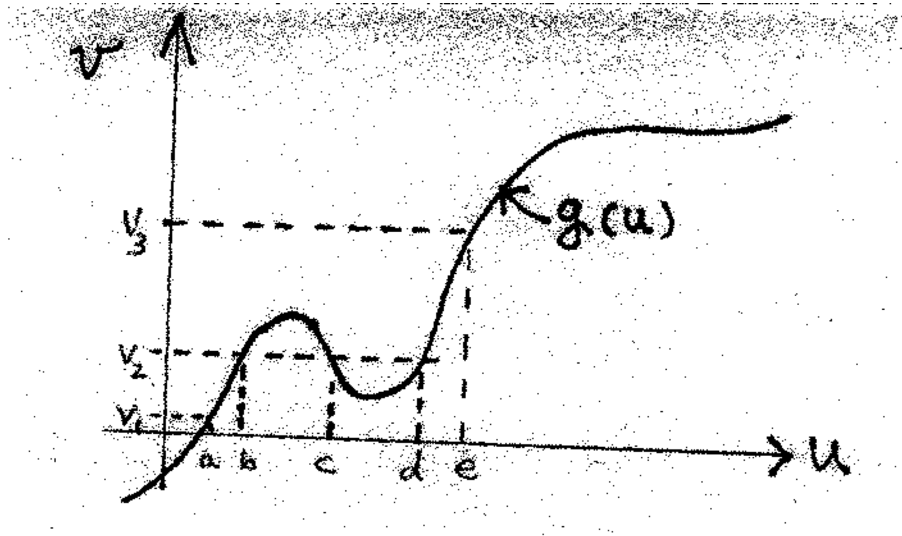
There are many applications in which we know $F_U(u)$ and we wish to calculate $F_V(v)$ and $f_V(v)$.

The distribution function must satisfy

$$F_V(v) = P[V \leq v] = P[g(U) \leq v]$$

To calculate this probability from $F_U(u)$ we need to find all of the intervals on the u axis such that $g(u) \leq v$.

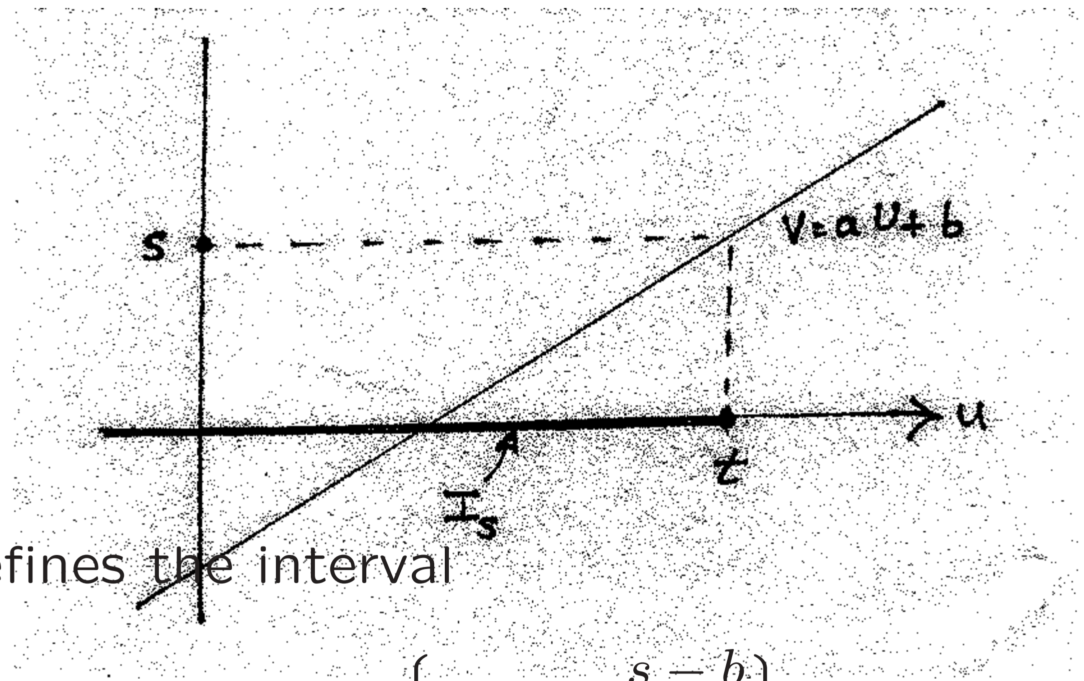
Function of a Random Variable



$$v \le v_1 \quad \text{if} \quad u \le a$$
$$v \le v_2 \quad \text{if} \quad u \le b \text{ or } c \le u \le d$$
$$v \le v_3 \quad \text{if} \quad u \le e$$

For any number s, values of u such that $g(u) \le s$ fall in a set of intervals $\mathcal{I}_s$.

Example: $V = aU + b$



For any s, $t = \frac{s-b}{a}$ defines the interval

$$\mathcal{I}_s = \{u : u \leq t\} = \left\{u : u \leq \frac{s-b}{a}\right\}$$

For any probability distribution function $F_U(u)$ we then find

$$F_V(v) = P\left[U \leq \frac{v-b}{a}\right] = F_U\left[\frac{v-b}{a}\right]$$

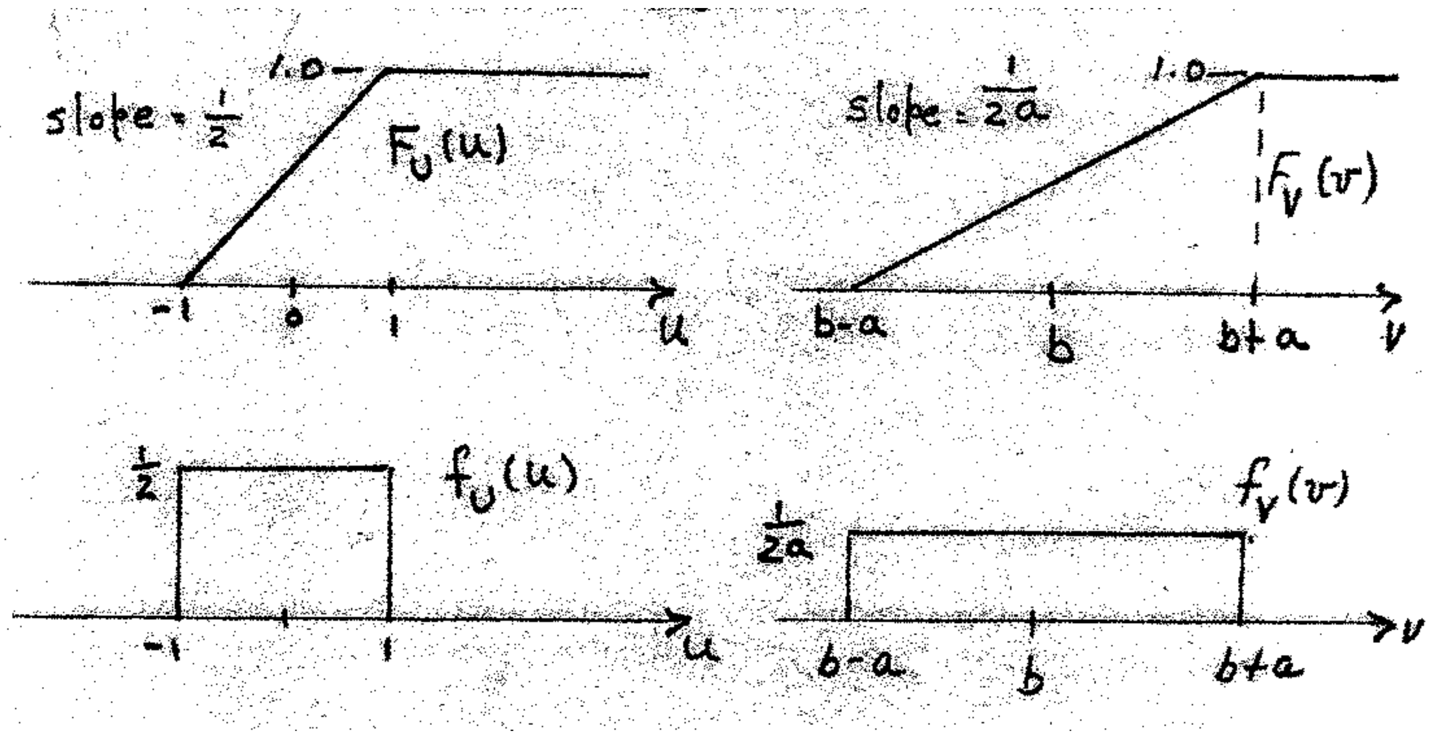
Example: $V = aU + b$

Suppose U has a uniform distribution on the interval $-1 \le u \le 1$. Then

$$F_U(u) = \begin{cases} 0 & \text{for} \quad u \le -1 \\ \frac{1}{2} + \frac{u}{2} & \text{for} \quad -1 \le u \le 1 \\ 1 & \text{for} \quad u \ge 1 \end{cases}$$

$$F_V(v) = F_U\left[\frac{v-b}{a}\right] = \begin{cases} 0 & \text{for} \quad v \le b - a \\ \frac{1}{2} + \frac{v-b}{2} & \text{for} \quad b - a \le v \le b + a \\ 1 & \text{for} \quad v \ge b + a \end{cases}$$

Example: $V = aU + b$



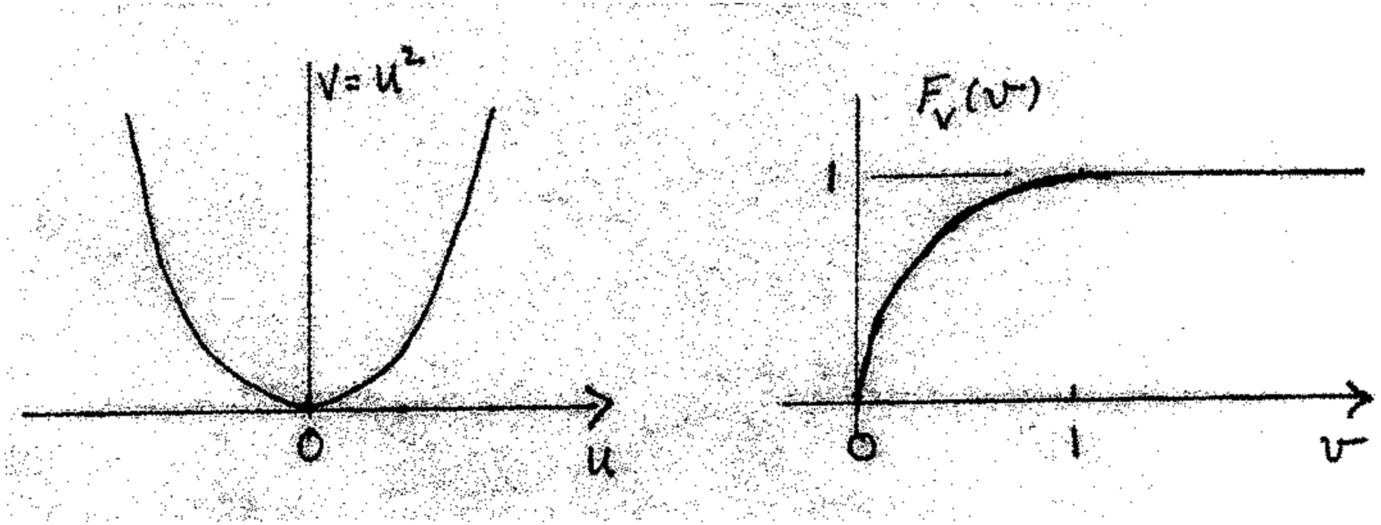
Example: $V = U^2$

$$\mathcal{I}_s = \{u : -\sqrt{s} \leq u \leq \sqrt{s} \text{ for } s \geq 0\}$$

$$P\left[V \leq v\right] = P\left[U \in \mathcal{I}_v\right] = P\left[\sqrt{v} \leq U \leq \sqrt{v}\right] \text{ for } v \geq 0$$

$$P\left[V \leq v\right] = \begin{cases} 0 & \text{for } V \leq 0 \\ \frac{2\sqrt{v}}{2} = \sqrt{v} & \text{for } 0 \leq v \leq 1 \\ 1 & \text{for } v \geq 1 \end{cases}$$

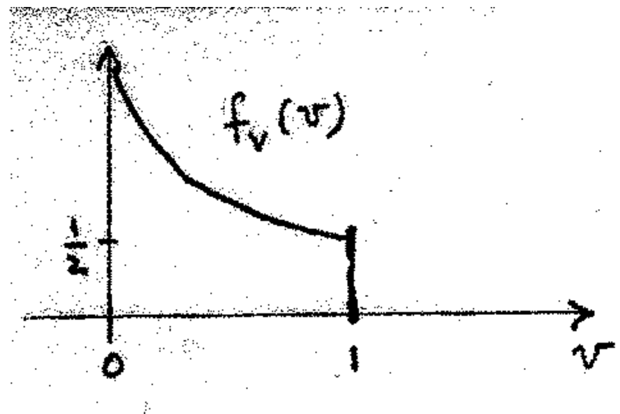
Example: $V = U^2$ with same $F_U(u)$



Probability density function

$$f_V(v) = \frac{dF_V}{dv} = \frac{1}{2\sqrt{v}}$$

for $0 < v \leq 1$



Probability Density Function

The probability density function can be computed by

$$f_V(v) = \frac{dF_V}{dv}$$

This requires first computing $F_V(v)$ as in the last example.

It is often convenient to compute $f_V(v)$ directly from $f_U(u)$ and the function $V = g(U)$.

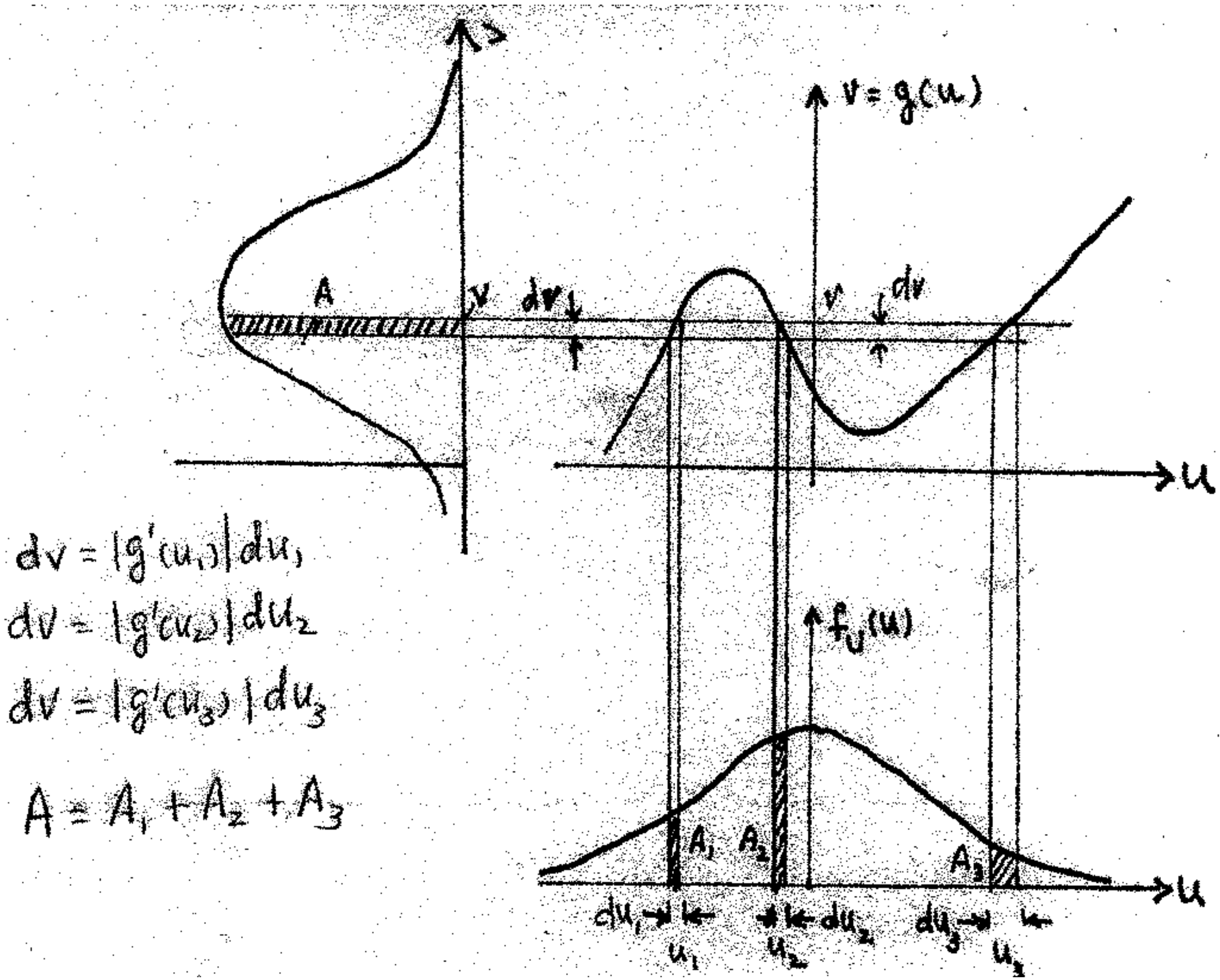
For any specific value of y find the solutions u_n such that

$$v = g(u_1) = \cdots = g(u_n) = \cdots$$

We will show that

$$f_V(v) = \frac{f_U(u_1)}{|g'(u_1)|} + \cdots + \frac{f_U(u_n)}{|g'(u_n)|} + \cdots$$

Probability Density Function



Probability Density Function

The areas under the curves must be equal.

$$A = A_1 + A_2 + A_3$$

$$f_V(v)dv = f_U(u_1)du_1 + f_U(u_2)du_2 + f_U(u_3)du_3$$

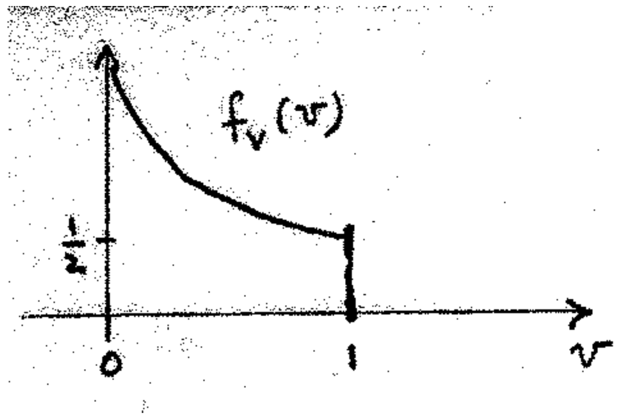
$$f_V(v)dv = f_U(u_1)\frac{dv}{|g'(u_1)|} + f_U(u_2)\frac{dv}{|g'(u_2)|} + f_U(u_3)\frac{dv}{|g'(u_3)|}$$

$$f_V(v) = \frac{f_U(u_1)}{|g'(u_1)|} + \frac{f_U(u_2)}{|g'(u_2)|} + \frac{f_U(u_3)}{|g'(u_3)|}$$

In general, the summation is over all the roots of $v = g(u)$ for any particular v. If there are no roots, then $f_V(v) = 0$ for that v.

Example: $V = U^2$ with rectangular $f_U(u)$

If $v < 0$ there are no roots $\Rightarrow f_V(v) = 0$.
If $v \geq 0 \Rightarrow u_1 = -\sqrt{v}$, $u_2 = \sqrt{v}$
$g'(u) = 2u \Rightarrow g'(u_1) = -2\sqrt{v}$ and $g'(u_2) = 2\sqrt{v}$



$$f_V(v) = \frac{f_U(-\sqrt{v})}{2\sqrt{v}} + \frac{f_U(\sqrt{v})}{2\sqrt{v}}$$

$$f_U(u) = \frac{1}{2}\text{Rect}(u/2)$$

$$f_V(v) = \frac{1}{2\sqrt{v}} \quad \text{for} \quad 0 < v \leq 1$$

Sums of Random Variables

Let U and V be random variables, and let $W = U + V$.

Given $F_{U,V}(u,v)$ and the pdf $f_{U,V}(u,v)$ find $F_W(w)$ and $f_W(w)$.

$$F_W(w) = P[U + V \leq w]$$

$$F_W(w) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{w-u} f_{U,V}(u,v) dv \right] du$$

$$f_W(w) = \frac{dF_W}{dw}$$

Leibnitz' Rule

If $a(t)$, $b(t)$ and $r(s,t)$ are all differentiable with respect to t then

$$\frac{d}{dt}\int_{a(t)}^{b(t)} r(s,t)ds = r[b(t),t]b'(t) - r[a(t),t]a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} r(s,t)ds$$

Sums of Random Variables

$$
\begin{aligned}
f_W(w) \; &= \tfrac{d}{dw} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{w-u} f_{U,V}(u,v) dv \right] du \\
&= \int_{-\infty}^{\infty} \left[\tfrac{d}{dw} \int_{-\infty}^{w-u} f_{U,V}(u,v) dv \right] du \\
&= \int_{-\infty}^{\infty} f_{U,V}(u, w-u) du
\end{aligned}
$$

If U and V are statistically independent random variables then

$$
f_W(w) = \int_{-\infty}^{\infty} f_U(u) f_V(w-u) du
$$

Here we recognize an old friend, the convolution integral.

Averages of Random Variables

Suppose that a random variable U can take on any one of L random values, say $u_{1,}u_2, \ldots u_L$. Imagine that we make n independent observations of U and that the value u_k is observed n_k times, $k = 1, 2, \ldots, L$. Of course, $n_1 + n_2 + \cdots + n_L = n$. The emperical average can be computed by

$$\overline{u} = \frac{1}{n} \sum_{k=1}^{L} n_k u_k = \sum_{k=1}^{L} \frac{n_k}{n} u_k$$

The concept of statistical averages extends from this simple concept