Joint Probability

Lecture 3

Spring 2002

Joint Probability Distribution Function

$$F_{X_1X_2}(x_1, x_2) = P[(X_1 \le x_1) \cap (X_2 \le x_2)]$$

1. $F_{X_1X_2}(-\infty, -\infty) = 0$
2. $F_{X_1X_2}(-\infty, x_2) = 0$ for any x_2
3. $F_{X_1X_2}(x_1, -\infty) = 0$ for any x_1
4. $F_{X_1X_2}(+\infty, +\infty) = 1$
5. $F_{X_1X_2}(+\infty, x_2) = F_{X_2}(x_2)$ for any x_2
6. $F_{X_1X_2}(x_1, +\infty) = F_{X_1}(x_1)$ for any x_1

Joint Probability Distribution Function

The probability that an experiment produces a pair (X_1, X_2) that falls in a rectangular region with lower left corner (a, c) and upper right corner (b, d) is

$$\begin{aligned} P[(a < X_1 \le b) \cap (c < X_2 \le d)] &= F_{X_1X_2}(b, d) - F_{X_1X_2}(a, d) - \\ &\quad F_{X_1X_2}(b, c) + F_{X_1X_2}(a, c) \end{aligned}$$

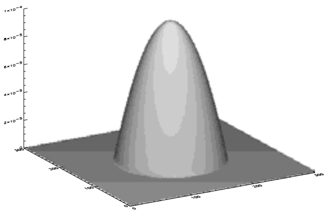
Joint Probability Density Function

$$f_{X_1X_2}(x_1, x_2) = \frac{\partial^2 F_{X_1X_2}(x_1, x_2)}{\partial x_1 \partial x_2}$$

$$\begin{aligned} f_{U,V}(u, v) &\ge 0 \\ F_{U,V}(u, v) &= \int_{-\infty}^{u} \int_{-\infty}^{v} f_{U,V}(\xi, \eta) d\xi d\eta \\ \int_{-\infty}^{\infty} f_{U,V}(\xi, \eta) d\xi d\eta &= 1 \\ F_U(u) &= \int_{-\infty}^{u} \int_{-\infty}^{\infty} f_{U,V}(\xi, \eta) d\xi d\eta \\ F_V(v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{v} f_{U,V}(\xi, \eta) d\xi d\eta \\ f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u, \eta) d\eta \\ f_V(v) &= \int_{-\infty}^{\infty} f_{U,V}(\xi, v) d\xi \end{aligned}$$

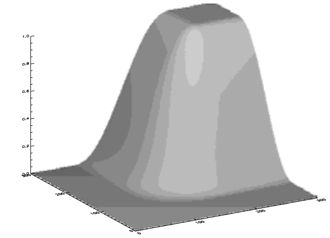
Example

```
;CONSTRUCT X AND Y ARRAYS FOR THE f(X,Y) pdf
t=FindGen(301)
Meshdom,t,t,X,Y
;CONSTRUCT AND DISPLAY THE PDF
f=6400-((x-150)^2+(y-150)^2 < 6400)
f=f/Total(f) Window,/Free
Shade_Surf,f
Wshow
```



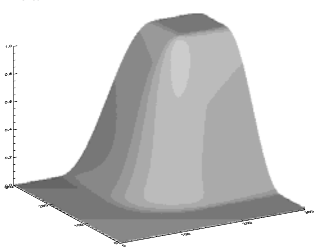
Example (continued)

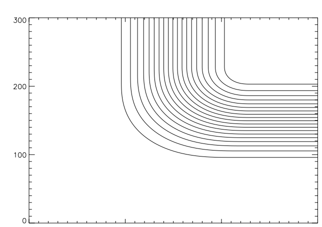
```
;CALCULATE AND DISPLAY THE CDF
FC=FltArr(301,301)
FC[*,0]=CumSum(f[*,0])
FC[0,*]=CumSum(f[0,*])

For ky=1,300 DO $
    For kx=1,300 DO $
      FC[kx,ky]=FC[kx-1,ky]+ $
      FC[kx,ky-1]-FC[kx-1,ky-1]+f[kx,ky]
Window,/Free Shade_Surf,FC Wshow
```



Example (continued)

```
;DISPLAY A CONTOUR PLOT OF THE CDF
Window,/Free
Contour,FC,Levels=Findgen(21)/20
Wshow
```





Conditional Probabilities

$$P[U \in \mathcal{A}, V \in \mathcal{B}] = \int_{\xi \in \mathcal{A}} \int_{\eta \in \mathcal{B}} f_{U,V}(\xi, \eta) d\xi d\eta$$

The probability that $V \in \mathcal{B}$ without regard to the value of U is given by

$$P[V \in \mathcal{B}] = P[U \in \mathcal{U}, V \in \mathcal{B}] = \int_{\xi=-\infty}^{\infty} \int_{\eta \in \mathcal{B}} f_{U,V}(\xi, \eta) d\xi d\eta$$

Conditional Probabilities

$$P[U \in \mathcal{A}|V \in \mathcal{B}] \ = \frac{P[U \in \mathcal{A}, V \in \mathcal{B}]}{P[V \in \mathcal{B}]} \ = \frac{\int_{\xi \in \mathcal{A}} \int_{\eta \in \mathcal{B}} f_{U,V}(\xi,\eta) d\xi d\eta}{\int_{\xi=-\infty}^{\infty} \int_{\eta \in \mathcal{B}} f_{U,V}(\xi,\eta) d\xi d\eta}$$

provided $P[V \in \mathcal{B}] > 0$.

Conditional Probability Distribution Function

$$F_U(u|V \in \mathcal{B}) = P[U \leq u|V \in \mathcal{B}]$$

whenever $P[V \in \mathcal{B}] > 0$.

The conditional probability distribution function has all of the properties of an ordinary one-dimensional probability distribution function. That is, it is a nondecreasing function with $F_U(-\infty|V \in \mathcal{B}) = 0$ and $F_U(\infty|V \in \mathcal{B}) = 1$.

Conditional Probability Density Function

$$f_U(u|V \in \mathcal{B}) = \frac{dF_U(u|V \in \mathcal{B})}{du}$$

wherever the derivative exists.

Statistically Independent Random Variables

Two random variables, U and V are *statistically independent random variables* if and only if

$$P[U \in \mathcal{A}, V \in \mathcal{B}] = P[U \in \mathcal{A}]P[V \in \mathcal{B}]$$

for *every possible choice* of A and B.

Statistically Independent Random Variables

If U and V are statistically independent random variables then

$$P[U \leq u, V \leq v] = P[U \leq u]P[V \leq v]$$

and this is equivalent to the following statement in terms of the distribution functions:

$$F_{U,V}(u,v) = F_U(u)F_V(v)$$

If the distribution functions are differentiable for almost all u and v then we also have the result

$$f_{U,V}(u,v) = f_U(u)f_V(v)$$

Bivariate Normal Distribution

A normal probability density function for two random variables U and V with zero means and unit variance is

$$f(U,V) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left(\frac{U^2-2\rho UV+V^2}{2(1-\rho^2)}\right)}$$

where ρ is the correlation coefficient between U and V. If $\rho = 0$ then

$$f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\left(\frac{u^2+v^2}{2}\right)}$$

Bivariate Normal Distribution

Now,

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)dv = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

which is easily established by using the fact that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-v^2/2} dv = 1$$

By a similar integration, we also find that

$$f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

Therefore

$$f_{U,V}(u,v) = f_U(u)f_V(v)$$

and uncorrelated normal random variables are statistically independent.

Function of a Random Variable

Let U be an random variable and $V = g(U)$. Then V is also a rv.

$$P(V \leq v) = P(U \leq g^{-1}(v))$$

If the inverse function $u = g^{-1}(v)$ is known *and is single-valued* then the probability can be expressed as

$$F_V(v) = P[u \leq g^{-1}(v)] = \int_{-\infty}^{g^{-1}(v)} f_U(u)du$$

$$f_V(v) = \frac{d}{dv} P[u : g(u) \leq v]$$