Probability Modeling

Lecture 1

Spring Quarter 2002

Experiments

The first element of our probability model is the *experiment*. An experiment is an exercise that produces an *outcome*. The set of possible outcomes is $\mathcal{U} = \{e_1, e_2, \ldots, e_L\}$.

Each outcome has a probability $P(e_i) = p_i$ such that

$$p_i \geq 0 \quad \text{for} \quad 1 \leq i \leq L$$

$$\sum_{i=1}^{L} p_i = 1$$

Repeated Experiments

Suppose that a large number N trials of an experiment is conducted. Let n_i be the number of times outcome e_i occurs in N trials. Then

$$n_1 + n_2 + n_3 + \cdots + n_L = N.$$

$$\frac{n_1}{N} + \frac{n_2}{N} + \frac{n_3}{N} + \cdots + \frac{n_L}{N} = 1$$

Let $f_i = n_i/N$ be the fraction of the times that e_i occurs.

$$f_1 + f_2 + f_3 + \cdots + f_L = 1$$

We expect that the fractions f_i are in some sense predictable when N is large enough. The number that we expect the fraction to approach for large N is called the *probability*, p_i

Events

An *event* $\mathcal{A}$ is a set of possible outcomes.

$$\mathcal{A} \subset \mathcal{U}$$

The event $\mathcal{A}$ is said to occur if *any* of its outcomes occurs.

$$P(\mathcal{A}) = \sum_{e_k \in \mathcal{A}} P(e_k)$$

The probability $P(\mathcal{A})$ is well-defined.

Example

Let $\mathcal{U} = \{e_1, e_2 \ldots e_6\}$ be the outcomes on the roll of a fair die. Let $\mathcal{A}_o$ be the event that an odd face is rolled. Then $\mathcal{A}_o = \{e_1, e_3, e_5\}$ and

$$\begin{aligned} P(\mathcal{A}_o) &= P(e_1) + P(e_3) + P(e_5) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \end{aligned}$$

Number of Possible Events

How many unique events can be defined for an experiment with L outcomes?

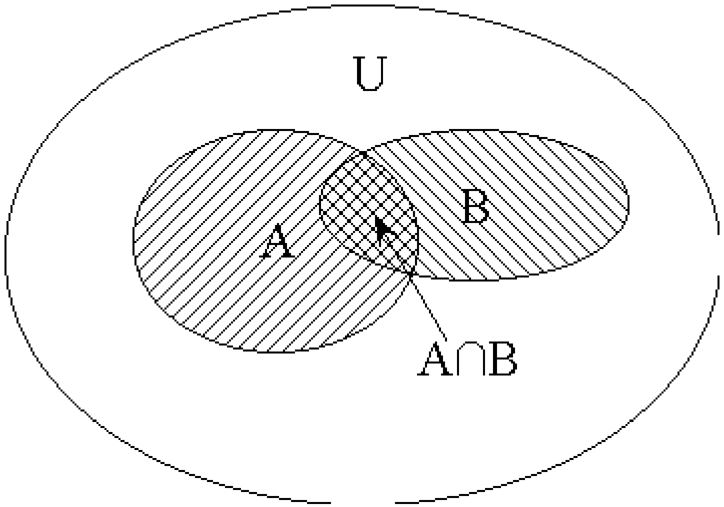
Event code: Consider an L-digit binary number $r = (a_1 a_2 \ldots a_L)$ where $a_k = 1$ if e_k is included in the event and $a_k = 0$ if not. There are 2^L unique values for r, from $r = 0$ to $r = 2^L - 1$.

Each $\mathcal{A}_r$ is a unique event and the set of all such events exhausts the possibilities.

Combining Events

Let $\mathcal{A}$ or $\mathcal{B}$ be events associated with an experiment.

Compound Event	**Expression**
either $\mathcal{A}$ or $\mathcal{B}$ occurred	$\mathcal{A} \cup \mathcal{B}$
$\mathcal{A}$ and $\mathcal{B}$ both occurred	$\mathcal{A} \cap \mathcal{B}$
$\mathcal{B}$ did not occur	$\mathcal{B}^c$



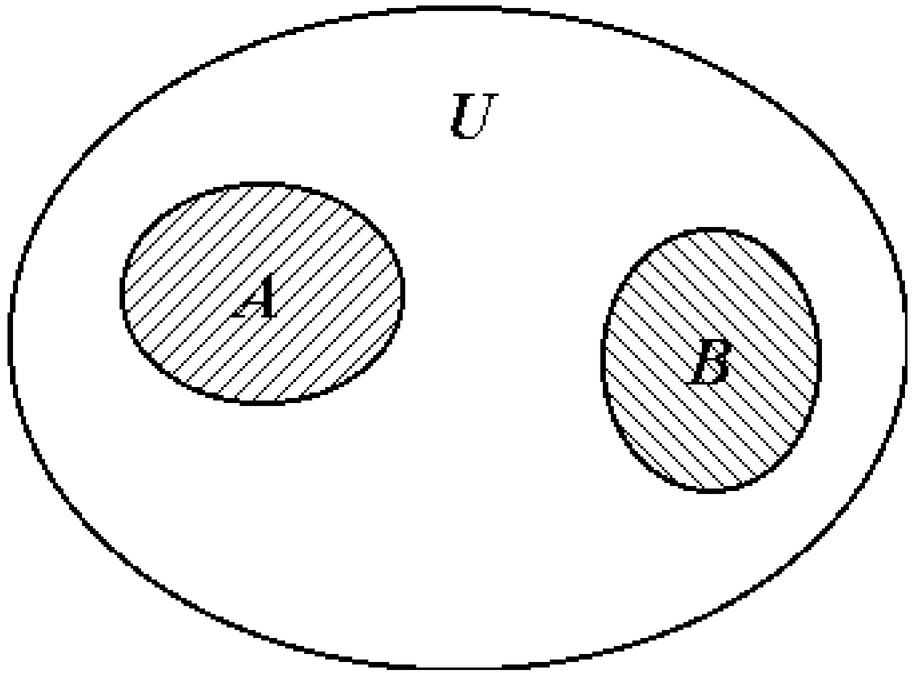
Mutually Exclusive Events

In general

$$P(\mathcal{A} \cup \mathcal{B}) = P(\mathcal{A} + P(\mathcal{B}) - P(\mathcal{A} \cap \mathcal{B})$$

If the events are *mutually exclusive* then

$$P(\mathcal{A} \cap \mathcal{B} = 0 \quad \text{so that} \quad P(\mathcal{A} \cup \mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B})$$



Joint Probability

The *joint probability* of two events is

$$P(\mathcal{A} \cap \mathcal{B}) = P(\mathcal{A})P(\mathcal{B}|\mathcal{A})$$

where $P(\mathcal{B}|\mathcal{A})$ is the probability of observing event $\mathcal{B}$ if you have already observed $\mathcal{A}$.

Since $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$ we must have

$$P(\mathcal{A})P(\mathcal{B}|\mathcal{A}) = P(\mathcal{B})P(\mathcal{A}|\mathcal{B})$$

Statistically Independent Events

If $\mathcal{A}$ and $\mathcal{B}$ are statistically independent then knowledge that one has occurred in an experiment does not give you any additional knowledge about the occurrence of the other. Mathematically,

$$\begin{aligned} P(\mathcal{A}|\mathcal{B}) &= P(\mathcal{A}) \\ P(\mathcal{B}|\mathcal{A}) &= P(\mathcal{B}) \end{aligned}$$

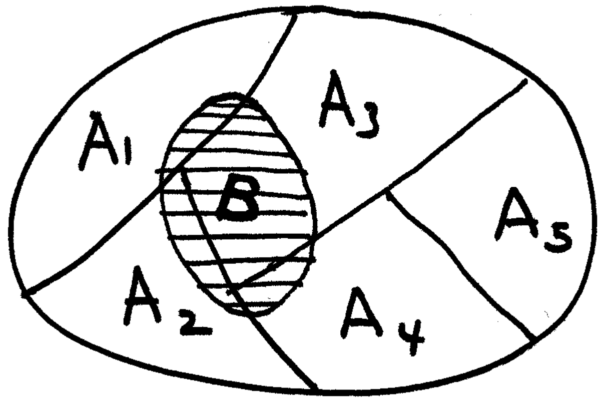
When this is true

$$P(\mathcal{A} \cap \mathcal{B}) = P(\mathcal{A})P(\mathcal{B})$$

Partitioning the Experiment

Suppose events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ partition a sample space $\mathcal{U}$, and that $P(\mathcal{A}_i) > 0$ for all $i = 1, 2, \ldots, n$. Then for any event $\mathcal{B}$ in $\mathcal{U}$

$$P(\mathcal{B}) = \sum_{i=1}^{n} P(\mathcal{B}|\mathcal{A}_i)P(\mathcal{A}_i)$$



Bayes' Rule

Let $\mathcal{B}$ be an event in sample space $\mathcal{U}$. Suppose that the events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ partition $\mathcal{U}$ and that $P(\mathcal{A}_i) > 0$ for all $i = 1, 2, \ldots, n$. Then

$$P(\mathcal{A}_j|\mathcal{B}) = \frac{P(\mathcal{B}|\mathcal{A}_j)P(\mathcal{A}_j)}{\sum_{i=1}^{n} P(\mathcal{B}|\mathcal{A}_i)P(\mathcal{A}_i)}$$