SIMG-261 – Linear Mathematics for Imaging

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SIMG-261 – Linear Mathematics for Imaging
(listed schedule is prediction only and subject to change)

Week 1:

I. Introduction, motivation and goal: to construct alternative representations of functions and systems to allow solution of some imaging tasks, will require aspects of calculus, linear algebra (vectors and matrices), and complex numbers
   A. Imaging "Chain"
   B. Imaging systems: input "object" f, output "image" g, system **O:** $\mathbf{O}\{f[x,y,...]\} = g[x,y,...]$
   C. The three imaging "tasks"
      1. direct problem: find g from **O** and f
      2. inverse problem: find f from **O** and g
      3. system analysis/synthesis: find **O** from g and f
   D. Examples of imaging systems and mathematical models
      1. "imaging" of optical rays
         a. image formed without a lens
         b. "imaging" of optical rays by selection with pinhole
         c. "imaging" of optical rays by selection with multiple pinholes
      2. "redirection" of rays by mirror or lens
         a. concept of "focusing" light
      3. Examples of "imaging tasks" in medicine
         a. Gamma-ray imaging
         b. Radiography
         c. Computed Tomographic Radiography (CT)
      4. Image "quality"
   E. Necessity to constrain mathematical action of system to have mathematically solvable description

II. Functions
   A. Continuous and DiscreteDomains
   B. Continuous and Discrete Ranges
   C. Discrete Domain and Range, "Digital" Functions
   D. Periodic, Aperiodic, and Harmonic Functions
   E. Symmetry Properties of Functions

Week 2:

III. Vector and Matrix Concepts
   A. Scalars and vectors with real-valued components
      1. vector addition
      2. scalar multiplication
      3. triangle inequality
      4. scalar (dot) product
         a. length (norm)
         b. projection of vector onto a "reference" vector (KEY CONCEPT)
         c. Cauchy-Schwarz inequality
      5. Matrices as multiple scalar products
         a. matrix-vector product: projection of one input vector onto multiple reference vectors

   b. matrix-matrix product: projection of multiple input vectors onto multiple reference vectors

Week 3:

6. Types of matrices
   a. square matrices
   b. diagonal matrices
      1) identity matrix (all diagonal elements equal to unity, all others to zero)
      2) solution of inverse problem for diagonal system matrix
7. vector spaces
   a. criteria: closure, null vector, unit vector
8. basis vectors
   a. Constructing different sets of basis vectors
   b. Gram-Schmidt orthogonalization
   c. rotation of vectors

Week 4:

IV. Complex numbers $i \equiv \sqrt{-1}$, $z = a + ib$
   A. representations
      1. as real-valued vectors
      2. as real/imaginary parts
      3. as magnitude/“phase”
   B. Graphical representation on phasor (“Argand”) diagram
   C. complex arithmetic
   D. Euler relation: $cos[\theta] + i\, sin[\theta] = exp[+i\theta]$
   E. roots of complex numbers by deMoivre's theorem
   F. complex functions of a real variable, e.g., $f[x]$
   G. complex functions of a complex variable, e.g., $w[z]$
      1. examples, e.g., z^{+1}, z^{*}, z^{-1}
      2. (VERY brief) introduction to path integrals

Week 5:

V. Vectors with complex-valued components
   A. inner product
      1. norm (length) of a vector
      2. projections of one complex vector onto another
   B. Operators on vectors, matrices
      1. matrix-vector multiplication as multiple scalar products or as simultaneous linear equations
      2. projection of arbitrary vector onto different basis sets, representations of vectors
      3. imaging problems in matrix-vector form
         a. matrix inverses
         b. pseudoinverses
         c. shift invariance, *circulant* matrices

Week 6: First Exam

VI. Eigenvectors and Eigenvalues
   A. diagonal forms of matrix operator
   B. diagonalization operators

C. diagonalization of circulant matrix
D. discrete Fourier transform (DFT)

Week 7:

VII. Matrix-Vector Formulations of the Imaging "Tasks"
  A. Inverse Task
    1. matrix inverse
    2. Solution of inverse problem by diagonalization
  B. Matrix-vector formulation of system analysis
  C. Alternative representation of shift invariant system, rotation of basis vectors

Week 8:

VIII. Functions of continuous variables $f[x]$
  A. Classification
    1. domain and range (real/complex, continuous/discrete)
    2. form (linear/nonlinear, periodic, harmonic)
    3. symmetry (even/odd)
  B. Representations obtained by projecting onto different sets of basis functions
    1. inner product, relation to scalar product
    2. orthogonal/orthonormal sets of functions
    3. power series representations, Taylor series
  C. Representations of functions
    1. real/imaginary parts
    2. magnitude (modulus)/phase
    3. Argand-phasor diagram (Lissajou figures)

IX. Continuous-domain analogues of vectors and operations
  A. one-dimensional function $f[x]$ as infinite-dimensional vector
  B. Analogue of inner product of continuous functions
  C. Projection of one continuous function onto another

Week 8:

X. Definitions of "special" functions
      a. 1-D real-valued functions (constant, *RECT, TRI, SGN, COS, CHIRP, GAUS*)
      b. 1-D Dirac delta function (impulse) and related functions
      c. 1-D complex-valued sinusoid

XI. Mathematical representation of systems, operators
  A. "Linear" systems
  B. "Shift (space-, time-) invariant" systems
    1. all matrix-vector systems are linear
    2. circulant matrix systems are linear and shift invariant

Week 9:

  C. Linear and shift-invariant (LSI) systems
    1. action of system is *convolution* ("filtering")
    2. Classes of filters
      a. Lowpass filters, local averagers
      b. Highpass filters, local differencers
      c. Bandpass filters, differences of local averages
      d. Allpass filters, phase filters

Week 10: Second Exam

D. Representations of systems
   1. linear and discrete (matrix-vector multiplication)
   2. linear and continuous (superposition integral)
   3. LSI and discrete (circulant matrix, diagonalizing transformation)
   4. LSI and continuous (convolution integral)
   5. impulse response/point-spread function as descriptor of system

E. Crosscorrelation and autocorrelation, relate to convolution

Week 11:

XII. Fourier transforms of 1-D continuous functions

A. Properties
   1. linearity
   2. shift variance
   3. "spectrum" of function, analogy to spectrum of light

B. evaluation by direct integration

C. Fourier transforms of special functions

Week 12:

D. theorems of the Fourier transform
   1. multiplication by constant
   2. addition theorem (linearity)
   3. "Fourier transform of a Fourier transform"
   4. central ordinate theorem, areas in the two domains
   5. scaling theorem
   6. shift theorem
   7. filter theorem
   8. modulation theorem
   9. derivative theorem
   10. transform of complex conjugate
   11. transform of crosscorrelation
   12. tranforms of autocorrelation, Wiener-Khintchin theorem
   13. Rayleigh's theorem, energy conservation in the two domains

Week 13:

XIII. Sampling

A. Ideal Sampling of Special Functions

B. Interpolation of Sampled Functions

C. Whittaker-Shannon Sampling Theorem

D. Aliasing of sampled functions
   1. aliased frequencies
   2. "prefiltering" (antialiasing)

Week 14:

XIV. Fourier Transforms of Discrete Functions

A. Discrete Fourier Transform (DFT)

B. DFT over a finite interval

C. Efficient evaluation of the DFT, the Fast Fourier Transform (FFT)

D. "Leakage"
   1. "windowing" to diminish leakage

Week 15:

XV. Applications
   A. digital image processing
   B. optical imaging (if time allows)

Final Exam (as scheduled), I shall request a three-hour block