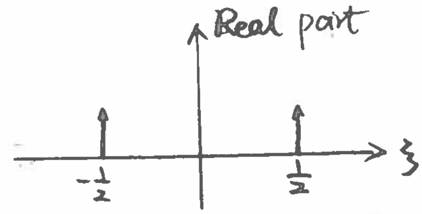
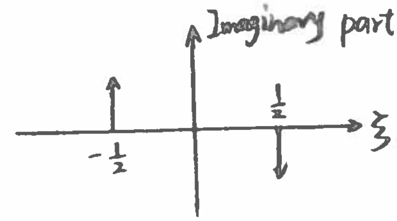
Solution for HW#8 IMGS-261

1\. (a) $\mathcal{F}_1\{\cos(2\pi\frac{x}{2})\} = \frac{1}{2}\delta[\xi+\frac{1}{2}] + \frac{1}{2}\delta[\xi-\frac{1}{2}]$



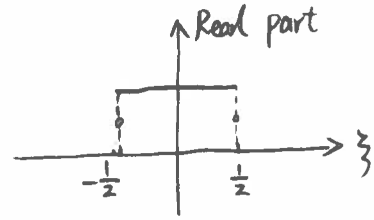
$\mathcal{F}_1\{\sin(2\pi\frac{x}{2})\} = \frac{i}{2}\left(\delta[\xi+\frac{1}{2}] - \delta[\xi-\frac{1}{2}]\right)$

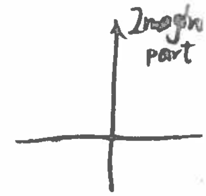


$\therefore \mathcal{F}_1\{\cos(2\pi\frac{x}{2}) + \sin(2\pi\frac{x}{2})\} = \frac{1+i}{2}\delta[\xi+\frac{1}{2}] + \frac{1-i}{2}\delta[\xi-\frac{1}{2}]$

(b) $\because \mathcal{F}_1\{\mathrm{RECT}[x]\} = \mathrm{SINC}[\xi]$

$\therefore \mathcal{F}_1\{\mathrm{SINC}[x]\} = \mathrm{RECT}[-\xi] = \mathrm{RECT}[\xi]$

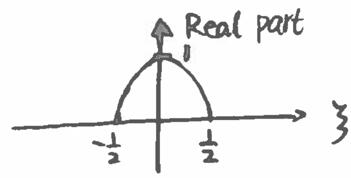


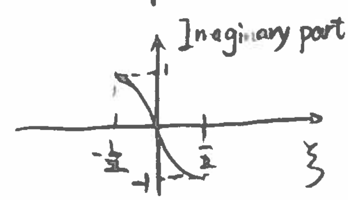


(c) $\mathcal{F}_1\{\mathrm{SINC}[x-\frac{1}{2}]\} = e^{-2\pi i\xi\frac{1}{2}}\mathcal{F}_1\{\mathrm{SINC}[x]\}$

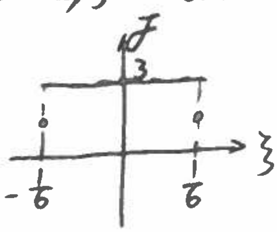
$= e^{-2\pi i\xi\cdot\frac{1}{2}}\,\mathrm{RECT}[\xi]$

$= \left(\cos[2\pi\cdot\frac{1}{2}\xi] - i\sin[2\pi\cdot\frac{1}{2}\xi]\right)\mathrm{RECT}[\xi]$



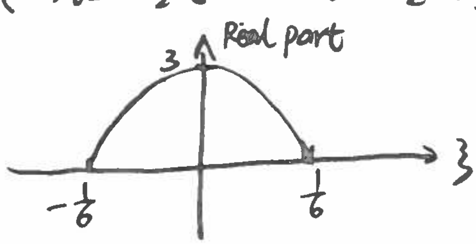


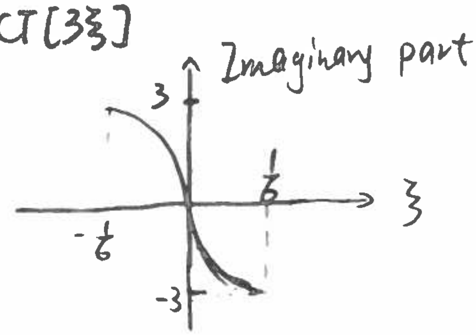
(d) $\mathcal{F}_1\{\mathrm{SINC}[\frac{x}{3}]\} = 3\,\mathrm{RECT}[3\xi]$



(e) $\mathcal{F}_1\{\mathrm{SINC}[\frac{x}{3}-\frac{1}{2}]\} = \mathcal{F}_1\{\mathrm{SINC}[\frac{x-\frac{3}{2}}{3}]\} = e^{-2\pi i\xi\cdot\frac{3}{2}}\cdot 3\mathrm{RECT}[3\xi]$

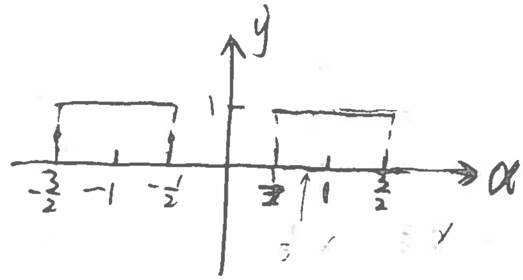
$= \left(\cos[2\pi\cdot\frac{3}{2}\xi] - i\sin[2\pi\cdot\frac{3}{2}\xi]\right)3\mathrm{RECT}[3\xi]$





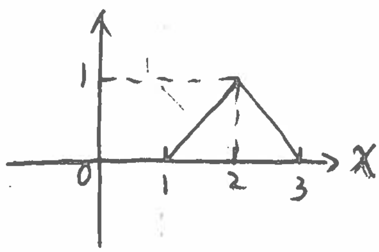
2. $\mathrm{RECT}[x-1] \star \mathrm{RECT}[x+1] = \mathrm{RECT}[x-1] * \left(\mathrm{RECT}[-x+1]\right)^*$

$= \mathrm{RECT}[x-1] * \mathrm{RECT}[-x+1]$



$$= \int_{-\infty}^{\infty} \mathrm{RECT}[\alpha-1]\cdot \mathrm{RECT}[-(x-\alpha)+1]\, d\alpha$$

$$= \begin{cases} \int_{\frac{1}{2}}^{-\frac{1}{2}+x} 1\, d\alpha = x-1 & 1<x\leq 2 \\ \int_{-\frac{3}{2}+x}^{\frac{3}{2}} 1\, d\alpha = 3-x & 2<x<3 \\ 0 & \text{others} \end{cases}$$



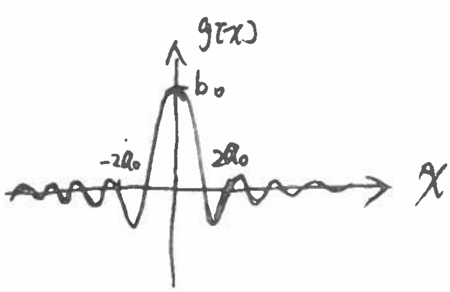
3. $G[\xi] = \mathcal{F}_1\{g[x]\} = \mathcal{F}_1\{\mathrm{SINC}[\frac{x}{a_0}]\} \cdot \mathcal{F}_1\{\mathrm{SINC}[\frac{x}{b_0}]\}$

$= a_0 \mathrm{RECT}[a_0\xi] \cdot b_0 \mathrm{RECT}[b_0\xi]$

(assume $a_0 > b_0$) $= a_0 b_0\, \mathrm{RECT}[a_0\xi]$

$g[x] = \mathcal{F}_1^{-1}\{G[\xi]\} = \mathcal{F}_1^{-1}\{a_0 b_0 \mathrm{RECT}[a_0\xi]\} = b_0 \mathcal{F}_1^{-1}\{a_0 \mathrm{RECT}[a_0\xi]\}$

$= b_0 \cdot \mathrm{SINC}[\frac{x}{a_0}]$



4. (a) $g[x] = \cos[2\pi \frac{x}{4}] * RECT[x]$

$$G[\xi] = \mathcal{F}_1\{g[x]\} = \mathcal{F}_1\{\cos[2\pi \tfrac{x}{4}]\} \cdot \mathcal{F}_1\{RECT[x]\}$$

$$= \frac{1}{2}\left(\delta[\xi+\tfrac{1}{4}] + \delta[\xi-\tfrac{1}{4}]\right) \cdot SINC[\xi]$$

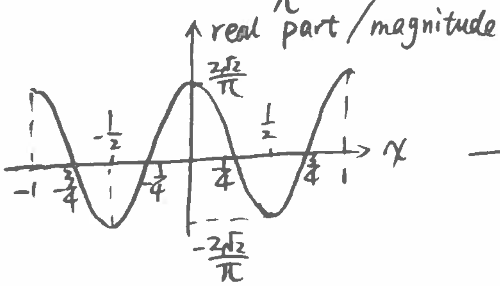
$$= \frac{1}{2} SINC[\tfrac{1}{4}]\left(\delta[\xi+\tfrac{1}{4}] + \delta[\xi-\tfrac{1}{4}]\right)$$

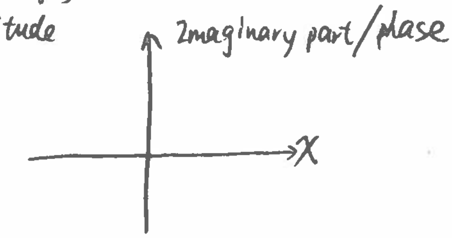
$$= \frac{1}{2} \cdot \frac{\sin\frac{\pi}{4}}{\pi \cdot \frac{1}{4}}\left(\delta[\xi+\tfrac{1}{4}] + \delta[\xi-\tfrac{1}{4}]\right)$$

$$= \frac{\sqrt{2}}{\pi}\left(\delta[\xi+\tfrac{1}{4}] + \delta[\xi-\tfrac{1}{4}]\right)$$

$$g[x] = \mathcal{F}_1^{-1}\{G[\xi]\} = \mathcal{F}_1^{-1}\left\{\frac{2\sqrt{2}}{\pi} \cdot \frac{1}{2}\left(\delta[\xi+\tfrac{1}{4}] + \delta[\xi-\tfrac{1}{4}]\right)\right\}$$

$$= \frac{2\sqrt{2}}{\pi} \cdot \cos[2\pi \tfrac{x}{4}]$$





(b) $g[x] = \cos[2\pi \frac{x}{2}] * RECT[x]$

$$G[\xi] = \frac{1}{2}\left(\delta[\xi+\tfrac{1}{2}] + \delta[\xi-\tfrac{1}{2}]\right) \cdot SINC[\xi]$$

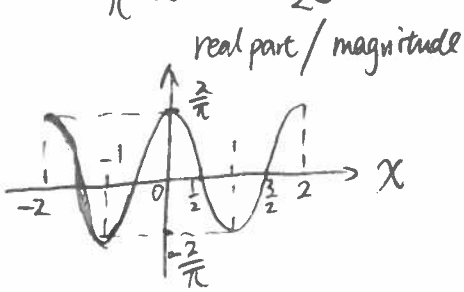
$$= \frac{1}{2} SINC[\tfrac{1}{2}]\left(\delta[\xi+\tfrac{1}{2}] + \delta[\xi-\tfrac{1}{2}]\right)$$

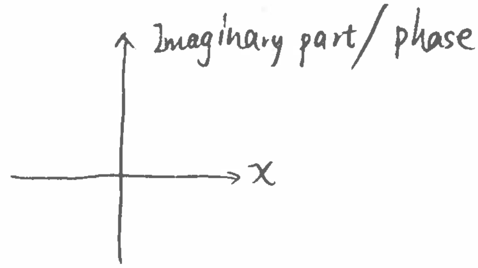
$SINC[\frac{1}{2}] = \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} = \frac{2}{\pi}$

$$= \frac{1}{\pi}\left(\delta[\xi+\tfrac{1}{2}] + \delta[\xi-\tfrac{1}{2}]\right)$$

$$g[x] = \mathcal{F}_1^{-1}\{G[\xi]\} = \frac{2}{\pi}\mathcal{F}_1^{-1}\left\{\frac{1}{2}\left(\delta[\xi+\tfrac{1}{2}] + \delta[\xi-\tfrac{1}{2}]\right)\right\}$$

$$= \frac{2}{\pi}\cos[2\pi \tfrac{x}{2}]$$





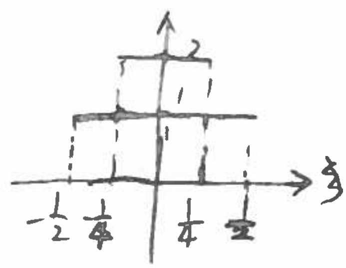
(c) $g[x] = \cos[2\pi \frac{x}{T}] * RECT[x]$

$G[\xi] = \frac{1}{2}(\delta[x+1] - \delta[x-1]) \cdot SINC[x]$

$= \underbrace{\frac{1}{2} SINC[1]}_{0} \cdot \left(\frac{1}{2}\delta[x+1] - \delta[x-1]\right) = 0$

$\therefore g[x] = \mathcal{F}^{-1}\{0\} = 0$

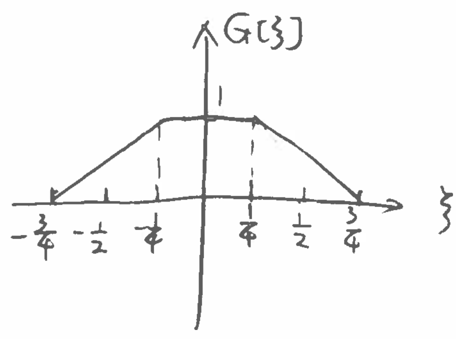
5. (a) $g[x] = SINC[x] \cdot SINC[\frac{x}{2}]$



$G[\xi] = RECT[\xi] * 2RECT[2\xi]$

$$= \begin{cases} \int_{-\frac{1}{4}+x}^{\frac{1}{4}+x} 2\,d\alpha, & -\frac{1}{4} < x < \frac{1}{4} \\ \int_{-\frac{1}{4}+x}^{\frac{1}{2}} 2\,d\alpha, & \frac{1}{4} < x < \frac{3}{4} \\ \int_{-\frac{1}{2}}^{\frac{1}{4}+x} 2\,d\alpha, & -\frac{3}{4} < x < -\frac{1}{4} \\ 0, & \text{others} \end{cases}$$

$$= \begin{cases} 1, & -\frac{1}{4} < x < \frac{1}{4} \\ \frac{3}{2} - 2x, & \frac{1}{4} < x < \frac{3}{4} \\ \frac{3}{2} + 2x, & -\frac{3}{4} < x < -\frac{1}{4} \\ 0, & \text{others} \end{cases}$$



5 b)

$$\begin{aligned}\mathcal{F}\{cos[2\pi\xi_0 x] \cdot RECT[\frac{x}{b_0}]\} &= \mathcal{F}\{cos[2\pi\xi_0 x]\} * \mathcal{F}\{RECT[\frac{x}{b_0}]\} \\ &= \frac{1}{2}(\delta[\xi+\xi_0] + \delta[\xi-\xi_0]) * |b_0|SINC[b_0\xi] \\ &= \frac{1}{2}|b_0|(SINC[b_0(\xi+\xi_0)] + SINC[b_0(\xi-\xi_0)])\end{aligned}$$

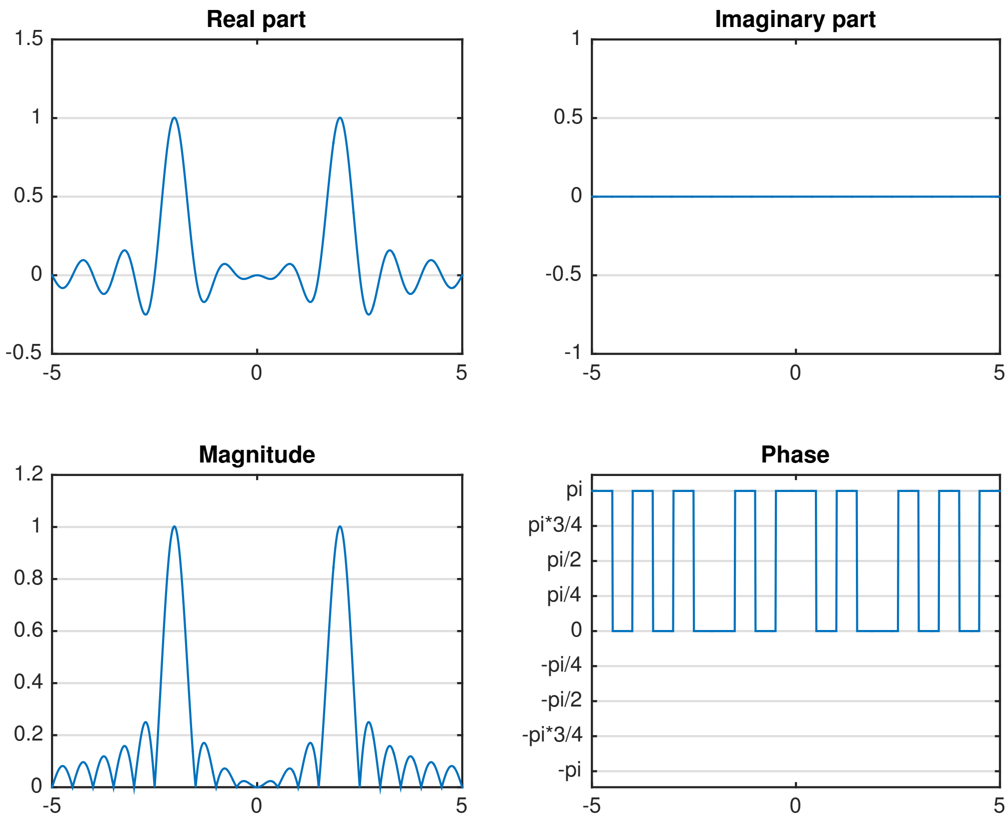


Figure 1: The plot of Fourier transform of $cos[2\pi\xi_0 x] \cdot RECT[\frac{x}{b_0}]$ with $b_0 = 2$ and $\xi_0 = 2$

5 c)

$$\begin{aligned}\mathcal{F}\{sin[2\pi\xi_0 x] \cdot RECT[\frac{x}{b_0}]\} &= \mathcal{F}\{sin[2\pi\xi_0 x]\} * \mathcal{F}\{RECT[\frac{x}{b_0}]\} \\ &= \frac{i}{2}(\delta[\xi+\xi_0] - \delta[\xi-\xi_0]) * |b_0|SINC[b_0\xi] \\ &= \frac{i}{2}|b_0|(SINC[b_0(\xi+\xi_0)] - SINC[b_0(\xi-\xi_0)])\end{aligned}$$

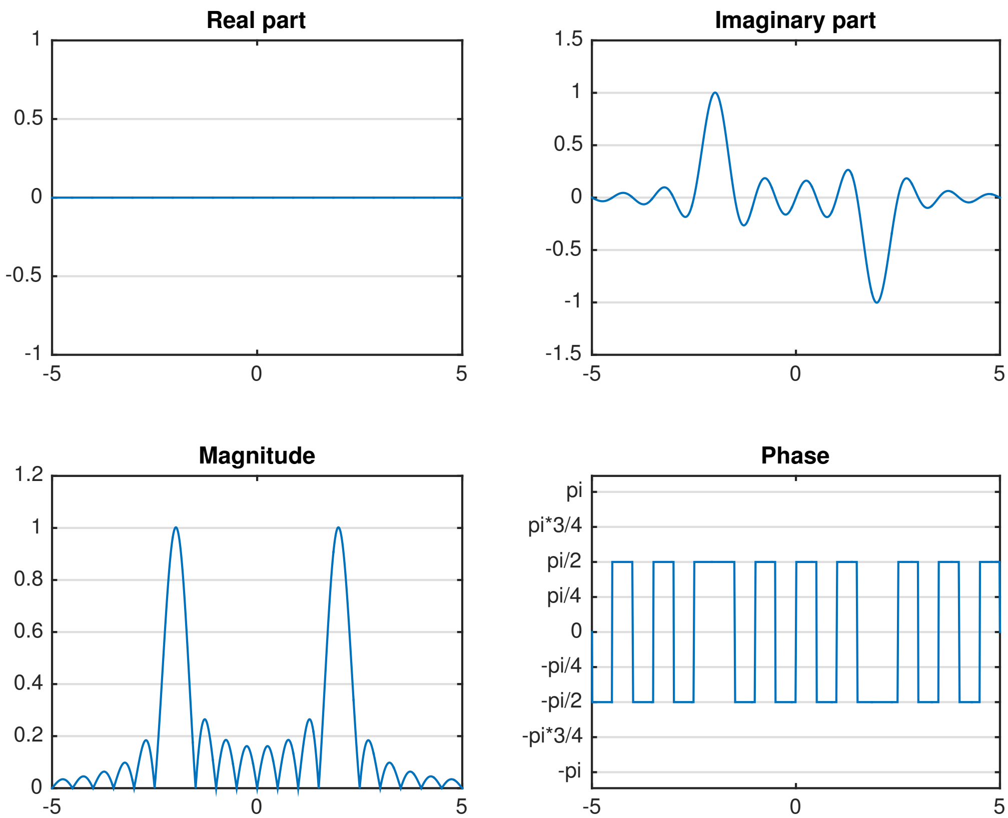


Figure 2: The plot of Fourier transform of $sin[2\pi\xi_0 x] \cdot RECT[\frac{x}{b_0}]$ with $b_0 = 2$ amd $\xi_0 = 2$